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Original Article

Arithmetic Differential Equations Defined by Generalized Subderivatives

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Abstract - *The arithmetic derivative, introduced by Barbeau (1961), is an integer-valued analogue of the usual derivative. For a nonempty set S of primes, the arithmetic subderivative D_S differentiates an integer only with respect to the primes in S, treating primes outside S as constants. The present work studies arithmetic differential equations on the integers defined by D_S and its iterates. Criteria are obtained for the vanishing of iterates $D_S^k(n)$, in terms of the S-support of intermediate values; in particular, $D_S^2(n) = 0$ occurs exactly when $D_S(n)$ is free of primes from S. Fixed points of D_S are also determined: a positive integer n satisfies $D_S(n) = n$ precisely when $\Sigma_{p\in S} v_p(n)/p = 1$, which forces $n = p^p m$, where $p \in S$ and m has no prime factor in S. Examples are included to illustrate both terminating and non-terminating trajectories for different choices of S.*

Keywords *- Arithmetic derivative, Arithmetic subderivatives, Fixed points, Leibniz rule, Prime factorization.*

1. Introduction

Barbeau introduced the arithmetic derivative D on the positive integers by prescribing $D(p) = 1$ for each prime p and extending the function via the Leibniz rule. This construction yields, for instance, the prime-power formula

$$\mathrm{D}(p^k) = \mathrm{k}p^{k-1} \; (p \text{ prime }, k \geq 1),$$

and the identity

$$D(n) = n \sum_{p\in S} \frac{v_p(n)}{p}$$

in terms of the prime factorization of n. The arithmetic derivative has since been examined from several viewpoints, including structural properties and associated antiderivative problems (see [1–3]).

Differentiation with respect to selected primes leads to operators that interpolate between the full derivative and a partial derivative. The arithmetic partial derivative with respect to a prime p treats p as the "variable" and all other primes as constants [3,7]. Merikoski, Haukkanen, and Tossavainen introduced arithmetic subderivatives relative to subsets S of primes and studied them within the broader class of Leibniz-additive arithmetic functions [4]. Related directions include arithmetic analogues of partial differential equations [5], discontinuity/continuity behavior of subderivatives on the integers [6], and higher-order behavior of arithmetic partial derivations and antidifferentiation [7].

Much of the existing literature emphasizes one-step identities or analytic properties. By contrast, iterating a subderivative produces a discrete dynamical system on the positive integers whose long-term behavior depends on the chosen prime set S. A detailed description of finite termination (i.e., the existence of k such that the k-fold iterate equals 0) and of fixed points for general S appears limited. The present work addresses this gap by giving explicit criteria for when the k-fold iterate vanishes,

$$D_S^k(n) = 0,$$

and by classifying all fixed points of D_S. Dependence on S is also described, and examples highlight both terminating and non-terminating trajectories.

The arithmetic derivative, arithmetic subderivatives, and related operators have been investigated in a variety of directions in the literature (see, for example, [3-7]). The present work focuses specifically on the iterative dynamics of the arithmetic subderivative D_S on $\mathbb{N}$ for an arbitrary nonempty set S of primes. In this setting, the paper (i) gives a general criterion characterizing when an iterate $D_S^k(n)$ becomes zero, namely when $D_S^{(k-1)}(n)$ is free of primes from S (Theorems 3.1-3.2), (ii) derives an explicit necessary restriction on prime exponents in n that rules out termination for many integers (Corollary 3.3), and (iii) provides an explicit description of all fixed points of D_S in terms of the prime factorization of n, together with a comparison result for inclusions of prime sets (Theorem 3.4 and Proposition 3.5). The examples illustrate how these phenomena depend on the choice of S, including terminating trajectories, fixed points, and non-terminating growth.

2. Preliminaries

Let $\mathcal{P}$ denote the set of all prime numbers. Every positive integer n has a unique prime factorization.

$$n = \prod_{p|n} p^{\nu_p(n)}.$$

In this factorization, only finitely many exponents $\nu_p(n)$ are nonzero. The support of n is

$$supp(n) = p \in \mathcal{P}: \nu_p(n) > 0.$$

For a set $S \subseteq \mathcal{P}$, the S-support of n is $S \cap supp(n)$, i.e., the primes from S that divide n.

Definition 2.1. Let $S \subseteq \mathcal{P}$ be a nonempty set of primes. For $n \in \mathbb{N}$ (positive integers), the arithmetic subderivative with respect to S, denoted $D_S(n)$, is defined by

$$D_S(n) = n \sum_{p \in S,\ p|n} \frac{\nu_p(n)}{p}$$

The sum is taken over primes $p \in S$ that divide n; if $S \cap supp(n) = \emptyset$, then the sum is empty and $D_S(n) = 0$. Primes outside S are treated as constants, so $D_S(p) = 0$ for primes $p \notin S$, while $D_S(p) = 1$ for primes $p \in S$. In particular, for a prime power p^e one has $D_S(p^e) = ep^{e-1}$ when $p \in S$ and $D_S(p^e) = 0$ when $p \notin S$.

The arithmetic subderivative satisfies the usual product rule. For positive integers m and n, one has

$$D_S(mn) = mD_S(n) + nD_S(m).$$

This rule follows immediately from the valuation identity $\nu_p(mn) = \nu_p(m) + \nu_p(n)$. It also places D_S in the class of Leibniz-additive arithmetic functions studied in [4]. In particular, D_S is completely determined by its values on prime powers. For later reference, an explicit formula for $D_S(n)$ is recorded in terms of the prime factorization of n. If

$$n = \prod_{(p|n)} p^{(e_p)} \text{ with } e_p = \nu_p(n)$$

then

$$D_S(n) = \sum_{p \in S} e_p \frac{n}{p}$$

Indeed, for each p ∈ S, differentiating p^{e_p} contributes $e_p p^{e_p - 1}$, and multiplying by the other factors of n gives $e_p(n/p)$.
Two extreme special cases are worth noting:

• If $S = \mathcal{P}$ (the set of all primes), then $D_{\mathcal{P}}(n)$ coincides with the classical arithmetic derivative $D(n)$ (as defined by Barbeau [1]). For example, $D(6) = 2D(3) + 3D(2) = 2 \cdot 1 + 3 \cdot 1 = 5$. In general,

$$D(n) = \sum_{p|n} \nu_p(n) \cdot (n/p)$$

• If $S = p$ is a singleton, then $D_p(n)$ is the arithmetic partial derivative with respect to p [3]. In this case,

$$D_p(n) = n \cdot \left(\nu_p(n)/p\right).$$

For instance, for $n = 12$ one has $D_2(12) = 12 \cdot (2/2) = 12$ (since $\nu_2(12) = 2$), whereas $D_3(12) = 12 \cdot (1/3) = 4$. Note that the full derivative $D(12) = 16$ is different, illustrating that partial subderivatives capture growth from one prime at a time.

If desired, the definition can be extended to 0 by setting $D_S(0) = 0$ (consistent with the product rule), and to negative integers by $D_S(-n) = -D_S(n)$. Unless stated otherwise, attention is restricted to $n \in \mathbb{N}$.

3. Main Results

This section collects the main results. First, criteria are given for higher-order vanishing under iteration of D_S (that is, when repeated application yields 0). Second, fixed points of D_S are classified. Proofs are supplied for each theorem, and illustrative examples appear in the subsequent section.

For an integer n, the S-support of n means $S \cap supp(n)$, i.e., the set of prime divisors of n that lie in S. Note that $D_S(n) = 0$ if and only if the S-support of n is empty. For $k \geq 0, D_S^k(n)$ denotes the k-fold iterate of D_S, with $D_S^0(n) = n$ and $D_S^k(n) = D_S\left(D_S^{k-1}(n)\right)$ for $k \geq 1$.

Theorem 3.1. *For any positive integer* n *and any nonempty set* $S \subseteq \mathcal{P}$ *of primes,*

$$D_S^2(n) = 0 \Leftrightarrow D_S(n)$$

has no prime factor in S.
Equivalently, $D_S^2(n) = 0$ *if and only if the S-support of* $D_S(n)$ *is empty.*

Proof. By definition, $D_S(x) = x \cdot \Sigma_{p \in S} \nu_p(x)/p$ for $x \in \mathbb{N}$. Hence $D_S(x) = 0$ holds if and only if $\nu_p(x) = 0$ for all $p \in S$, i.e., if and only if x has no prime factor from S. Applying this observation to $x = D_S(n)$ gives $D_S^2(n) = D_S\left(D_S(n)\right) = 0$ exactly when $D_S(n)$has empty S-support.

Remark
Theorem 3.1 provides a simple condition: the second iterate vanishes exactly when $D_S(n)$ *itself has no prime in S. The argument extends naturally to higher iterates:*

Theorem 3.2. Let $k \geq 1$ be a fixed positive integer and let $n \in \mathbb{N}$. The following are equivalent:
1. $D_S^k(n) = 0$.
2. $D_S^{k-1}(n)$ has no prime factor in S.
3. The S-support of $D_S^{k-1}(n)$ is empty; equivalently, $S \cap supp\left(D_S^{k-1}(n)\right) = \emptyset$.

In particular, $D_S^k(n) = 0$ if and only if after $k - 1$ iterations the subderivative produces an integer free of primes from S. Moreover, if $D_S^j(n) = 0$ for some $j \geq 1$, then $D_S^m(n) = 0$ for every integer $m \geq j$.

Proof. The argument proceeds by induction on k. Recall that for $n \in \mathbb{N}$,

$$D_S(x) = x \cdot \textstyle\sum_{p \in S,\ p|x} \left(\nu_p(x)/p\right).$$

Hence $D_S(x) = 0$ if and only if the sum is empty, i.e., no prime $p \in S$ divides x; equivalently, x is S-free (its S-support is empty).

Base case ($k = 1$). Since $D_S^1(n) = D_S(n)$ and $D_S^0(n) = n$, it follows that

$$D_S^1(n) = 0 \Leftrightarrow n \text{ has no prime factor in } S,$$

which is exactly statement (2) for $k = 1$.

Induction step. Assume the equivalence holds for a fixed $k \geq 1$. Consider $k + 1$. By the definition of iterates,

$$D_S^{k+1}(n) = D_S\left(D_S^k(n)\right).$$

Applying the base-case criterion to the integer $x = D_S^k(n)$, it yields

$$D_S^{k+1}(n) = 0 \Leftrightarrow D_S^k(n) \text{ has no prime factor in } S,$$

which is precisely statement (2) with k replaced by $k + 1$. Hence, the equivalence holds for all $k \geq 1$. The equivalence between (2) and (3) is immediate from the definition of S-support.

Corollary 3.3. *If $D_S^k(n) = 0$ for some finite k, then for every prime $p \in S$ dividing n, the exponent of p in n satisfies $v_p(n) < p$. Equivalently, no prime in S divides n to a power $\geq p$.*

Proof. Suppose $p \in S$ divides n with $a = v_p(n) \geq p$. Write $n = p^a m$ with $p \nmid m$.

In the definition $D_S(n) = n\sum_{q\in S, q|n} v_q(n)/q$, the term with $q = p$ contributes $a \cdot p^{a-1}m$, while each term with $q \neq p$ is divisible by p^a(since it equals $p^a \cdot (m/q) \cdot v_q(n)$ for some $q|m$). Hence $D_S(n)$ is divisible by p^{a-1}. If $a > p$ then $a - 1 \geq p$; if $a = p$, then the coefficient a supplies an additional factor p. In either case $v_p\big(D_S(n)\big) \geq p$. Repeating this argument shows $v_p\left(D_S^j(n)\right) \geq p$ for all $j \geq 1$, so no iterate becomes S-free. By Theorem 3.2, the iterates therefore cannot terminate at 0. The contrapositive gives the stated necessary condition.

This gives a quick check to rule out many integers from ever having their subderivative sequence terminate.

For example, if $n = 2^4 = 16$ and $S = 2$, then $v_2(n) = 4 \nless 2$ and indeed one computes $D_2(16) = 32, D_2(32) = 80$, etc., a sequence that grows and never reaches zero.

On the other hand, if $n = 9 = 3^2$ and $S = 3$, then $v_3(9) = 2 < 3$, and one checks $D_3(9) = 6, D_3(6) = 2$, and $D_3(2) = 0$. Thus $D_3^3(9) = 0$.

Equivalently, for every prime $p \in S$ that divides n, the exponent $v_p(n)$ must satisfy $v_p(n) < p$. In plain terms, no prime from S should appear in n raised to a power greater than or equal to itself, or else the iterates of D_S will never reach zero.

Next, the fixed points of D_S are classified. By a fixed point, one means a positive integer n such that $D_S(n) = n$. Solving $D_S(n) = n$ is akin to finding eigenfunctions of the subderivative operator with eigenvalue 1. In the arithmetic setting, the set of fixed points is infinite and is described explicitly in terms of the prime factorization of n.

Theorem 3.4. *Let $S \subseteq \mathcal{P}$ be a fixed nonempty set of primes. A positive integer n satisfies $D_S(n) = n$ if and only if*

$$\sum_{p\in S}\frac{v_p(n)}{p} = 1.$$

In particular, any fixed point n must be of the form,

$$n = p^p \cdot m,$$

where $p \in S$ is some prime, and m is an arbitrary positive integer whose prime factors lie outside S. Moreover, for each prime $p \in S$, the pure power p^p (with no other factors) is itself a fixed point of D_S. These are the minimal fixed-point solutions corresponding to each prime in S.

Proof. Let $n \in \mathbb{N}$ satisfy $D_S(n) = n$. Since $D_S(n) = n \cdot \Sigma_{p\in S} v_p(n)/p$, cancellation of n gives $\Sigma_{p\in S} v_p(n)/p = 1$. Let $T = S \cap supp(n)$ and set $P_T = \prod_{p\in T} p$. Multiplying by P_T yields $\Sigma_{p\in T} v_p(n) \cdot (P_T/p) = P_T$. Fix $r \in T$ and reduce this identity modulo r. Every term with $p \neq r$ is divisible by r, and $P_T \equiv 0 (mod r)$, so $v_r(n) \cdot (P_T/r) \equiv 0 (mod r)$. Because $gcd(P_T/r, r) = 1$, it follows that $r|v_r(n)$. Hence each $v_r(n)/r$ is a nonnegative integer, and the sum $\Sigma_{p\in T} v_p(n)/p$ equals 1 only if exactly one term equals 1 and the rest are 0. Therefore, there exists $p \in S$ with $v_p(n) = p$ and $v_q(n) = 0$ for all q $\in S\backslash\{p\}$, which is equivalent to $n = p^p m$ with m having no prime factor in S. Conversely, if $n = p^p m$ with $p \in S$ and m S-free, then $\Sigma_{q\in S} v_q(n)/q = p/p = 1$ and hence $D_S(n) = n$, as required.

Remark. The fixed points of D_S on $\mathbb{N}$ are precisely those integers of the form $p^p \cdot m$, where $p \in S$ and m is an S-free positive integer. These are the minimal fixed-point solutions corresponding to each prime in S. In the special case $S = \mathcal{P}$ (the set of all primes), this reduces to the classical fixed-point family $\{p^p\colon p \text{ prime}\}$; for example, $2^2 = 4, 3^3 = 27$, and $5^5 = 3125$ are all fixed under the full arithmetic derivative D, while primes themselves $(e.g.., 2, 3, 5)$ are not fixed points, since $D(p) = 1 \neq p$.

Proposition 3.5. Let $T \subseteq S$ be nonempty. If $n \in \mathbb{N}$ is a fixed point of D_S, then either (i) n is also a fixed point of D_T, or (ii) $D_T(n) = 0$. More precisely, writing the fixed point as $n = p^p m$ (Theorem 3.4), one has $D_T(n) = n$ when $p \in T$ and $D_T(n) = 0$ when $p \notin T$. Conversely, a fixed point of D_T need not be a fixed point of D_S.

Proof. By Theorem 3.4, any positive fixed point of D_S has the form $n = p^p m$ with $p \in S$ and m S-free. If $p \in T$, then $\Sigma_{q\in T} v_q(n)/q = p/p = 1$ and therefore $D_T(n) = n$. If $p \notin T$, then $v_q(n) = 0$ for all $q \in T$ and hence $D_T(n) = 0$. For the converse, choose $p \in T$ and a prime $q \in S\backslash T$ and set $n = p^p q$. Then n is fixed for D_T (only p contributes), but $D_S(n) = n \cdot (1 + 1/q) > n$, so n is not fixed for D_S.

4. Examples

Several examples illustrate Theorems 3.1–3.2, Corollary 3.3, and Theorem 3.4

Example 4.1 Let $S = 3$. The integer $27 = 3^3$ satisfies $D_S(27) = 27$, so it is a fixed point, and its trajectory under iteration is constant.

$$D_{\{3\}}(27) = 27 \cdot \frac{3}{3} = 27.$$

Hence $D_S^k(27) = 27$ for all $k \geq 1$, in agreement with the fixed-point description in Theorem 3.4.

For $n = 9 = 3^2$, the trajectory terminates after three steps:

$$D_{\{3\}}(9) = 9 \cdot \frac{2}{3} = 6, \quad D_{\{3\}}(6) = 6 \cdot \frac{1}{3} = 2, \quad D_{\{3\}}(2) = 2 \cdot 0 = 0$$

Thus $D_S^3(9) = 0$. Note that $v_3(9) = 2 < 3$, consistent with Corollary 3.3.
Similarly, for $n = 18 = 2 \cdot 3^2$ one obtains:

$$D_{\{3\}}(18) = 18 \cdot \frac{2}{3} = 12, \quad D_{\{3\}}(12) = 12 \cdot \frac{1}{3} = 4, \quad D_{\{3\}}(4) = 0.$$

so again $D_S^3(18) = 0$. In these terminating cases, the prime 3 appears with exponent 2, whereas exponent 3 produces a fixed point.

Example 4.2. Let $S = \{2,3\}$. For $n = 210 = 2 \cdot 3 \cdot 5 \cdot 7$, one has $D_S(210) = 175$ and then $D_S(175) = 0$, so $D_S^2(210) = 0$. Indeed,

$$D_{2,3}(210) = 210(1/2 + 1/3) = 210 \cdot (5/6) = 175.$$

Here $175 = 5^2 \cdot 7$ is S-free, so the second step vanishes as predicted by Theorem 3.2. The exponents $v_2(210) = v_3(210) = 1$ also satisfies Corollary 3.3.
By contrast, starting from $n = 32 = 2^5$, the iterates do not terminate; the first few values are:
$D_{2,3}(32) = 32 \cdot (5/2) = 80$,
$D_{2,3}(80) = 80 \cdot (2/1) = 160$,
$D_{2,3}(160) = 160 \cdot (5/2) = 400$.
In fact, one checks inductively that for all $k \geq 0$,
$D_S^{2k}(32) = 32 \cdot 5^k$ and $D_S^{2k+1}(32) = 80 \cdot 5^k$.
In particular, every iterate remains divisible by 2, so no iterate becomes S-free and hence. $D_S^k(32) \neq 0$ for all k.

Example 4.3 Theorem 3.4 is illustrated in the following cases:

- If $S = \{2\}$, the fixed point condition is $\frac{\nu_2(n)}{2} = 1$. This forces $\nu_2(n) = 2$, and no factor of 2 appears elsewhere. Thus, the 2-part of n is $2^2 = 4$, and n can be $4m$ where m is odd. So the fixed points of $D_{\{2\}}$ are exactly the integers of the form $4m$ with m odd. Indeed, $D_{\{2\}}(4m) = 4m \cdot \frac{2}{2} = 4m$. For example, $4, 12, 20, 28, 36, \ldots$ are all fixed under $D_{\{2\}}$.
- If $S = \{3\}$, the fixed point condition is $\frac{\nu_3(n)}{3} = 1$, so $\nu_3(n) = 3$. Thus, the 3-part of n is $3^3 = 27$, and any integer m coprime to 3 can multiply it. So the fixed points for $D_{\{3\}}$ are $27m$ with $\gcd(m, 3) = 1$. For instance, $27, 54, 81, 135, 162, \ldots$ are fixed by $D_{\{3\}}$. (E.g. $D_{\{3\}}(54) = 54 \cdot \frac{3}{3} = 54$ since $\nu_3(54) = 3$.)
- If $S = \{2,3\}$, a fixed point must satisfy $\frac{\nu_2(n)}{2} + \frac{\nu_3(n)}{3} = 1$. The only solutions in nonnegative integers are either $\nu_2(n) = 2, \nu_3(n) = 0$ *or* $\nu_2(n) = 0, \nu_3(n) = 3$. Thus n is either $2^2 \cdot m$ (with m having no factor 3) or $3^3 \cdot m$ (with m having no factor 2). For example, $n = 20 = 2^2 \cdot 5$ is fixed since $D_{\{2,3\}}(20) = 20\left(\frac{2}{2} + 0\right) = 20$, and $n = 135 = 3^3 \cdot 5$ is fixed since $D_{\{2,3\}}(135) = 135\left(0 + \frac{3}{3}\right) = 135$. On the other hand, 12 is not fixed because $\frac{\nu_2(12)}{2} + \frac{\nu_3(12)}{3} = 1 + \frac{1}{3} \neq 1$, and indeed $D_{\{2,3\}}(12) = 16 \neq 12$.
- If $S = \{2,5\}$, then $\frac{\nu_2(n)}{2} + \frac{\nu_5(n)}{5} = 1$. This yields either $\nu_2(n) = 2$ or $\nu_5(n) = 5$. Hence, fixed points are of the form $4m$ (with no factor 5 in m) or $5^5 m'$ (with no factor 2 in m'). For example, $n = 12 = 4 \cdot 3$ is fixed for $S = \{2,5\}$ (since $D_{\{2,5\}}(12) = 12$), and $n = 3125 = 5^5$ is fixed (indeed $D_{\{2,5\}}(3125) = 3125 \cdot \frac{5}{5} = 3125$).

As another observation, if $n = p^p m$ is a fixed point of D_S (Theorem 3.4) and $q \in S$ is a different prime, then $D_S(nq) = nq \cdot (1 + 1/q) > nq$. Thus, introducing additional primes from S strictly increases the value of D_S, and equality $D_S(n) = n$ occurs only in the fixed-point families described above.

5. Discussion

The termination and fixed-point results obtained here place the iteration of D_S on $\mathbb{N}$ into a dynamical framework: Theorem 3.2 gives an exact termination criterion, Corollary 3.3 provides a practical obstruction, and Theorem 3.4 together with Proposition 3.5 classifies fixed points and their dependence on S. Unlike work emphasizing primarily one-step identities or operator-level properties of subderivatives (see [3–7]), the present results are formulated explicitly for iterates D_S^k. The framework uniformly includes the classical arithmetic derivative ($S = \mathcal{P}$) and arithmetic partial derivatives ($|S| = 1$). The examples show that trajectories can terminate, stabilize, or grow depending on S. Future directions include periodic points, quantitative growth for non-terminating trajectories, and finer comparisons across different prime sets.

6. Conclusion

This work studied arithmetic differential equations on the integers induced by the arithmetic subderivative D_S. A criterion for the vanishing of iterates $D_S^k(n)$ was established in terms of the S-support of intermediate values, and a necessary condition on prime exponents was recorded (Corollary 3.3). Fixed points of D_S were classified completely, leading to the explicit families $n = p^p m$ with an S-free cofactor. The examples show that iteration can lead to termination, fixed points, or persistent growth depending on S and on the initial integer. Further directions include periodic points ($D_S^k(n) = n$ for $k > 1$) and quantitative growth estimates for non-terminating trajectories.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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