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Original Article

New Algebraic Proofs on the Correctness of Collatz Conjecture basing on New Unified Formulas along with Computational Results which Provide Insights on Prime Numbers

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Abstract *- This paper presents new mathematical proofs on the correctness of the Collatz conjecture based on new unified formulas re-expressing this conjecture in a scalable algebraic form. We are proposing new precise formulas that enable expressing all natural numbers in the Collatz conjecture according to a unified methodology that allows scaling calculations on infinite numbers, and then we use these unified formulas to prove that all natural numbers are converging to the number one when we perform Collatz operations on them. In addition, we are presenting computational codes based on these algebraic formulas to provide computational results that guide the development of proofs. As a result, this paper is presenting twenty-seven new theorems along detailed proofs, where the first seven theorems are proposing new unified formulas to re-express the values of numbers according to unified algebraic forms, whereas the other theorems are using these new formulas to demonstrate that all natural numbers are converging to the number "1" when we forward Collatz operations on them; whereas demonstrating that there is no divergency over these operations. We also use these formulas to demonstrate that all natural numbers create loops in infinity that may cause Collatz calculations to circulate in a ring of numbers. Furthermore, we demonstrate that the only ring of natural numbers that creates a loop for the Collatz conjecture is where we go from "1" to "4", then calculations converge back to the number one. All theorems presented in this paper are developed according to an engineering methodology based on structuring unified formulas to re-express Collatz calculations along step-by-step proofs, which allow us to develop a breakthrough demonstrating the correctness of the Collatz conjecture, while providing new insights into the characteristics of prime numbers and their distribution.*

Keywords - *Collatz conjecture, New algebraic proofs, New theorems, New unified formulas, Computational codes, Prime numbers characteristics.*

1. Introduction

In 1930, the Collatz conjecture was introduced by German mathematician Lothar Collatz [1], and it studies natural numbers according to consecutive sequences. This conjecture is also referred to as the "Syracus Problem" [2].

The Collatz conjecture determines two specific operations to conduct on a natural number n_1, depending on the condition, whether it is even or odd, which allows us to calculate a consecutive number n_2.

Basically, the Collatz conjecture states that if the natural number n_1 is even (pair), then the consecutive number n_2 will be equal $\frac{n_1}{2}$, whereas n_2 will be equal $\frac{(3n_1+1)}{2}$ if n_1 is odd (impair) [3].

The Collatz conjecture also states that if we repeat these operations by starting from any natural number, calculations will eventually converge to the number "1", and operations will go from "1" to the number "4", then they will loop back to "1" [4]. In the 20th century, there were many published attempts by mathematicians trying to verify the statements of the Collatz conjecture by conducting its proposed operations on high natural numbers [5], and see if calculations are actually converging to the number "1", especially after the introduction of computation technologies starting from 1946.

Starting from 1946, when the first computer was introduced [6-7], mathematicians had more access to computational resources that allowed them to automate calculations, whereas conducting operations on larger variables containing extensive amounts of digits.

During the second half of the 20th century, between 1946 and 2000, mathematicians were able to use computational resources on the Collatz conjecture to verify its statements, which allowed them to validate that it is keeping its accuracy for (almost) all natural numbers with values less than 2^{32} [8-10].

Starting from the beginning of the 21st century, mathematicians had access to more powerful computers and programming languages that allowed extending operations on extremely large values, which also enabled them to verify that the statements of the Collatz conjecture are actually keeping their accuracy on natural numbers reaching up to 2^{64} [10-11].

These advanced computational resources also allowed us to graphically map trees showing interconnections among natural numbers based on the Collatz conjecture, which links natural numbers according to extensive branches as shown in Figure 1, where all reached natural numbers are converging to the number “1”.

Even though computation is allowing to verify that the given statements in Collatz conjecture are continuing to be true on extensive large numbers [10], there have been no algebraic proof that absolutely demonstrate the convergency of all natural numbers (including infinity numbers) to the number “1” when we keep repeating the given operations in Collatz conjecture whereas starting these operations on any even number or odd number [12-13].

Due to the absence of a conclusive algebraic proof before publishing this paper, the Collatz conjecture has been considered as an unsolved problem in number theory [14-15].

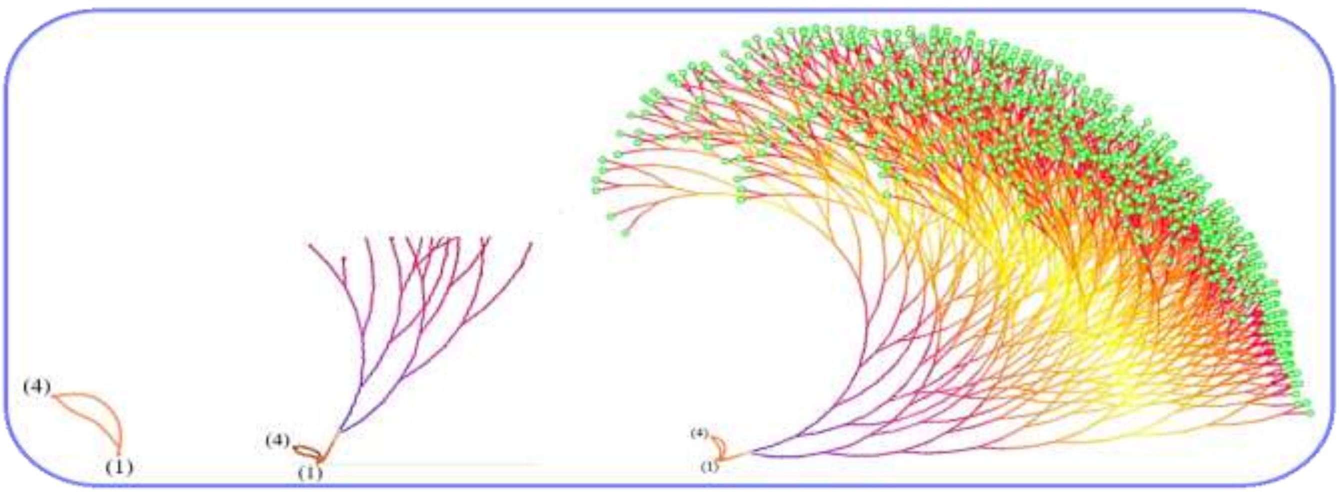

Fig. 1 The tree and loop of natural numbers according to conducted calculations in Collatz conjecture

The shown tree maps in Figure 1 present branching patterns among natural numbers where all Collatz operations eventually converge to the number “1”, then creating a loop between the numbers “1” and “4”. These converging patterns are inspiring us to develop unified formulas re-expressing the Collatz conjecture according to algebraic forms that we can use to structure algebraic proofs on the convergence of infinite natural numbers toward “1”.

In addition, the shown relations among numbers in Figure 1 are inspiring to analyze the distribution of prime numbers on these tree branches, while identifying common criteria among them according to the Collatz conjecture.

Furthermore, the Collatz conjecture and the shown branches in Figure 1 are encouraging to re-express natural numbers according to a distributed architecture presented as a sum of terms, then using this architecture to analyze odd numbers, especially prime numbers, instead of relying only on factorization for number analysis.

Developing a distributed architecture to represent natural numbers is reflected in the published work in [16], [17], and [18]. These published papers propose distributed structures to represent all roots of fourth-degree, fifth-degree, and sixth-degree polynomials, which allow for determining some common criteria among distributed terms, like being able to be neutralized and simplified when they are multiplied by each other.

Furthermore, the published work in [16], [17], and [18] introduces the axis of patterns revealing and calculations converging toward building algebraic proofs based on precise unified formulas.

Engineering a distributed architecture to represent natural numbers, especially odd numbers and primes, will allow analyzing these numbers from different views, such as the number of distributed terms in each natural number and the distance between every two consecutive terms. In addition, this distributed architecture will allow analyzing odd numbers and prime numbers according to an exponential plan based on the narrowed group $\{2^k, 3^j\}$Instead of conducting analysis in a linear plan built on identifying possible factors of each odd number to determine whether it is prime or not.

Furthermore, engineering a distributed architecture to represent natural numbers, whereas deducing common characteristics among primes will open the way to verify prospective infinity prime numbers by using parallel computation to calculate all included terms in parallel, which can minimize the computation time exponentially.

The strategy of the presented work in this paper is to use an engineering methodology to develop the proofs on the Collatz conjecture from scratch by building unified formulas and theorems, which will allow the logic and formulas in this paper to be solidly built on each other step-by-step according to a scalable engineered structure.

Therefore, this paper presents twenty-seven theorems based on new unified formulas re-expressing the statements of the Collatz conjecture according to distributed algebraic forms, which allow proving the correctness of the Collatz conjecture, while providing new insights and analytic methodologies on odd numbers and prime numbers.

The first seven theorems in this paper present new unified formulas along with detailed proofs, which allow re-expressing the statements of the Collatz conjecture according to scalable algebraic forms based on distributed architectures of terms. These terms rely only on the group $\{2^k, 3^j\}$ to build distributed architectures, which are to be presented as sums of exponential terms. The eighth and ninth theorems in this paper demonstrate the reciprocity of the proposed formulas, which allow us to prove that any natural number n_s expressed according to these proposed formulas, this number will converge to "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_s.

The tenth theorem in this paper is relying on the proposed unified formulas in precedent theorems in order to demonstrate along a detailed proof that the Collatz loop where we go from the number "1" to the number "4" and then going back to the number "1", is the only Collatz loop of its kind in the group of natural numbers $\mathbb{N}$.

The presented theorems in this paper, from Theorem 11 up to Theorem 18, use previous unified formulas to demonstrate step-by-step that there is no Collatz loop composed of "m" elements in the group of natural numbers $\mathbb{N}$, except the demonstrated loop in Theorem 10.

The presented theorems in this paper, from Theorem 20 up to Theorem 27, use the previously unified formulas to demonstrate that every odd number and every even number, except zero, eventually converge to the number "1" when we keep repeating the operations of the Collatz conjecture. Furthermore, these theorems rely on this convergence to present all odd numbers and even numbers according to unified algebraic formulas.

In the final sections, this paper provides new insights into odd numbers and prime characteristics according to proposed unified formulas, while revealing patterns of prime distribution, which forward the results of this paper to reach the harmonics of Riemann, the zeta function, and the Riemann hypothesis.

This paper also presents programmed codes in Python and Java along with their outputs, which we developed to compute the proposed formulas in this paper and to visualize computational results, in order to be used in the presented proofs and to guide the development of theorems.

As a result, the contents of this paper are structured as follows: Section 2 presents the used methods and analytic logic of development, starting from re-expressing the statements of the Collatz conjecture toward providing algebraic proofs about their correctness. Section 3, presenting seven theorems proposing unified formulas re-expressing the statements of the Collatz conjecture along detailed proofs. Section 4 presents new theorems demonstrating the reciprocity of proposed formulas and the reciprocity of Theorem 6 and Theorem 7. Section 5, presenting nine new theorems demonstrating that there is no loop that can contradict the Collatz conjecture. Section 6, presenting four theorems demonstrating the convergence of odd numbers and even numbers toward "1" when we keep repeating the operations of the Collatz conjecture. Section 7, presenting three theorems proving the correctness of the Collatz conjecture, which allow re-expressing all odd numbers and even numbers, except zero, according to unified formulas. Section 8 presents new insights about characteristics of prime numbers and how to verify them, while highlighting common patterns of prime distribution based on the proposed unified formulas in this paper. Finally, section 9 is for the conclusion.

2. Methods

Instead of using only a research methodology and relying on forward calculations, we are building the results of this paper according to an engineering methodology where we develop formulas, theorems, and proofs according to a scalable logic step-by-step. Therefore, we architect the structure of new unified formulas that will allow us to build algebraic proofs and concretize a general logic considering all possible scenarios and all expected inputs and parameters, which will allow us to build the proofs in precise details step-by-step.

Considering that Collatz conjecture is stating two different operations conducted on natural numbers depending on their nature whether they are even or odd, we need to re-express these two operations according to one unified form that we can scale on all natural numbers and that link between two consecutive odd numbers, in order to minimize the size of numbers group and to zoom on the characteristics of prime numbers.

After determining one unified form to re-express the Collatz conjecture in order to link two consecutive odd numbers, we need to scale this form on tree-branches containing odd numbers, where we can express each odd number by using a different odd number in the same tree-branch.

In order to demonstrate algebraically that the statements of the Collatz conjecture are correct, we need to prove that there is no loop among natural numbers where Collatz operations can circulate infinitely, except the loop where we go from the number "1" to the number "4" and then go back to "1". In addition, we need to prove that when we commence using the operations of the Collatz conjecture on any natural number *S*, calculations will not diverge to infinity and they will actually converge to a value *T* inferior to *S*.

Furthermore, we need to use the new unified formulas in this paper to demonstrate that every natural number *S* can be expressed as a function of a number *T* on the same tree-branch where the value of *T* is inferior to *S*. Then, proving that when we keep repeating the operations of the Collatz conjecture, starting these operations on any natural number *T*, calculations will eventually converge to the number "1".

Since the proposed unified formulas in this paper are based on distributed architectures of exponential terms, we will use them to analyze the characteristics of prime numbers according to the number of distributed terms in them and according to the exponential distances among contained terms. In addition, we will use these new formulas to study the distribution of prime numbers according to trees of numbers, which will highlight new insights away from relying on factorization.

The proposed formulas in this paper will allow proving the correctness of the Collatz conjecture, and also will allow highlighting specific criteria deduced from the structure of these formulas to be used in identifying whether an odd number is prime or not, while guiding statistics analysis on prime distribution.

As a result, the engineered logic in this paper is as follows:

1. Developing a unified formula re-expressing the operations of the Collatz conjecture on consecutive numbers.
2. Developing a unified formula expressing a tree-branch of consecutive numbers linked by Collatz operations.
3. Developing a unified formula expressing potential loops of Collatz operations and allowing for the calculation of the value of the first number in any potential loop.
4. Developing a fourth formula generalizing the third formula by expressing the value of any natural number inside a potential loop created by Collatz operations.
5. Developing a fifth formula expressing an odd number *S* as a function of another odd number *T*, where starting the operations of the Collatz conjecture on *S* eventually converges to *T* after repeating these operations.
6. Developing a sixth formula scaling the fifth one by expressing an odd number *S* in function of the odd number *"T=1"*, where starting the operations of the Collatz conjecture on *S* eventually converging to the number "1".

7. Developing a seventh formula generalizing the sixth formula by expressing neighbors of the odd number *S*, whether these neighbors are even or odd.
8. Developing reciprocal theorems based on the sixth formula and the seventh formula, in order to be used in the demonstration of the convergence of the operations of the Collatz conjecture to the number "1".
9. Using the third formula to demonstrate that there is only one Collatz loop starting from a number n_1 and going back to the same number n_1, and this loop is concretized when $n_1 = 1$.
10. Using the third and fourth proposed formulas to demonstrate that there is no Collatz loop consisting of "m" elements where Collatz operations can circulate among them, except the loop where calculations start from "$n_1 = 1$" and then going back to n_1.
11. Using the eighth and ninth proposed formulas to demonstrate that when we start Collatz operations on any odd number *S*, calculations will not diverge to infinity, and they will actually converge to a value *T* inferior to *S*, which allows us to express all odd numbers according to a unified formula.
12. Using the eighth and ninth proposed formulas to demonstrate that when we start Collatz operations on any even number *S*, calculations will not diverge to infinity, and they will actually converge to a value *T* inferior to *S*. Therefore, we will be able to express all even numbers according to a unified formula.
13. Using the eighth and ninth proposed formulas to demonstrate that when starting the operations of the Collatz conjecture on any natural number S different from zero, these operations will eventually converge to the number "1", which allows us to express all natural numbers according to one unified formula.
14. Using the distributed architecture of the proposed formulas and their included parts to identify whether an odd number is prime or not.
15. Using the distributed architecture of proposed formulas to highlight specific patterns that will guide statistics analysis on the distribution of prime numbers in the tree of natural numbers (Figure 1) after interconnecting them by Collatz operations, in order to deduce new approximation formulas counting prime numbers with high precision.
16. Using Python and Java programming languages to develop codes based on proposed formulas, in order to generate visualized graphics of computational results, which will guide the development of theorems and proofs.

3. New Unified Formulas Re-expressing the Collatz Conjecture

In this section, we are presenting seven new theorems proposing new unified formulas along detailed proofs step-by-step, in order to re-express the given statements and operations in the Collatz conjecture according to a scalable algebraic logic.

3.1. First Unified Formula

This subsection presents a new theorem proposing the first unified formula re-expressing the statements and operations of the Collatz conjecture to interconnect only odd numbers, since when we have an even number, we keep dividing by "2" until obtaining an odd number.

Theorem 1

We can re-express the operations of the Collatz conjecture by applying them only to the group of odd numbers, in order to connect each pair of consecutive odd numbers n_1 and n_2 in a Collatz branch as shown in **(**Equation 1**)**.

$$n_2 = \frac{3n_1+1}{2^k} \mid k \in [\![1, +\infty[\![\; and \; n_i \in \{\mathbb{N} - \{0\}\} \; and \; n_i \; MOD[2] = 1 \quad (1)$$

Proof of Theorem 1

The Collatz conjecture states two different algebraic operations to be conducted on any natural number $(n_1 \neq 0)$ depending on its nature, whether it is odd or even.

If a natural number $(n_1 \neq 0)$ It is odd, we multiply it by three, then we add one to the output as shown in Figure 2, in order to obtain a resulting even number "M" where $M = 3n_1 + 1$.

If a natural number "M" is even, we divide it by "2" as shown in Figure 2, and if the output of the division is also even, we divide it again by "2". As a result, we keep dividing the output by "2" until we obtain the output as an odd number n_2, which allows expressing the even number "M" as follows: $M = 2^k n_2 \mid k \in [\![1, +\infty[\![$.

Therefore, we deduce the possibility of expressing every two consecutive odd numbers n_1 and n_2, in the Collatz conjecture as follows: $3n_1 + 1 = 2^k n_2$. As a result, the proposed statements and the proposed formula (Equation 1) in Theorem 1 are correct.

3.2. Second Unified Formula

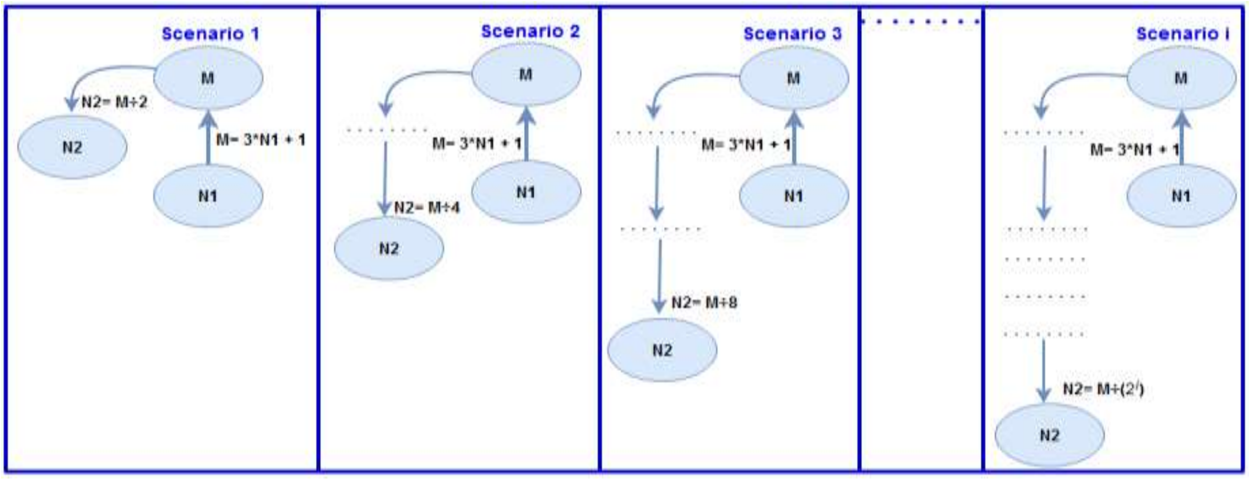

Fig. 2 Illustration of re-expressing the operations of Collatz conjecture to interconnect two different odd numbers

This subsection presents a new theorem proposing a second unified formula re-expressing the statements of the Collatz conjecture by expressing only sequences of consecutive odd numbers in a Collatz branch, which is based on extending the first proposed theorem in this paper.

Theorem 2

Supposing a sequence of odd numbers $\{n_1, n_2, n_3, n_4 \ldots, n_m\}$ that create a Collatz branch, where the numerical process to pass from one odd number of this sequence to the following one is as stated by the operations of the Collatz conjecture, which are re-expressed in Theorem 1. The unified mathematical formula to calculate any number n_s of this sequence in function of n_1 where $(2 \leq s)$ is as shown in **(**Equation 2**)**.

$$n_s = \frac{n_1 * 3^{s-1} + \sum_{i=0}^{i=s-2}\left[3^i * 2^{\sum_{j=0}^{j=s-i-2} K_{(j)}}\right]}{2^{\sum_{l=0}^{l=s-1} K_{(l)}}} \mid K_0 = 0 \ and \ K_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \quad (2)$$

Proof of Theorem 2

In the case of having a sequence of two odd numbers $\{n_1, n_2\}$ that create a Collatz branch, where the numerical process to pass from one odd number of this sequence to the following one is as described in (Equation 1) (Theorem 1); we express the value of n_2 as follows: $3n_1 + 1 = 2^{k_1} n_2$. Therefore, we can express this value of n_2 as shown in (Equation 3).

$$n_2 = \frac{3^{(2-1)} n_1 + \sum_{i=0}^{i=2-2}\left[3^i * 2^{\sum_{j=0}^{j=2-i-2} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=2-1} k_{(l)}}} = \frac{(3^1 n_1 + 2^{k_0})}{2^{k_0 + k_1}} = \frac{(3n_1 + 2^0)}{2^{0+k_1}} = \frac{(3n_1 + 1)}{2^{k_1}} \quad (3)$$

As a result, the proposed formula in **(**Equation 2**)** (Theorem 2) is correct for the case of having a sequence of two consecutive odd numbers $\{n_1, n_2\}$ in a Collatz branch where $k_0 = 0$.

In the case of having a sequence of consecutive three odd numbers $\{n_1, n_2, n_3\}$ that create a Collatz branch, where the numerical process to pass from one number of this sequence to the following one is as described in **(**Equation 1) (Theorem 1); we express the values of n_2 and n_3, as follows: $3n_1 + 1 = 2^{k_1} n_2 \ and \ 3n_2 + 1 = 2^{k_2} n_3$.

After replacing the value of n_2 by $\left(n_2 = \frac{3n_1+1}{2^{k_1}}\right)$ in order to identify the expression of $\left(n_3 = \frac{3n_2+1}{2^{k_2}}\right)$ in terms of n_1, we obtain the following result: $n_3 = \frac{9n_1+3+2^{k_1}}{2^{(k_1+k_2)}}$. Therefore, we can express this value of n_3 as shown in (Equation 4).

$$n_3 = \frac{3^{(3-1)}n_1 + \sum_{i=0}^{i=3-2}\left[3^i * 2^{\sum_{j=0}^{j=3-i-2} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=3-1} k_{(l+1)}}} = \frac{3^2 n_1 + 3^1 * 2^{k_0} + 3^0 * 2^{k_0+k_1}}{2^{(k_0+k_1+k_2)}} \quad (4)$$

As a result, the proposed formula in (Equation 2) (Theorem 2) is correct for the case of having a sequence of three odd numbers $\{n_1, n_2, n_3\}$ creating a Collatz branch where $k_0 = 0$.

In the case of having a sequence of four odd numbers $\{n_1, n_2, n_3, n_4\}$ that create a Collatz branch, where the numerical process to pass from one number of this sequence to the following one is as described in (Equation 1) (Theorem 1); we express the values of n_2, n_3 and n_4, as follows: $3n_1 + 1 = 2^{k_1}n_2$ *and* $3n_2 + 1 = 2^{k_2}n_3$ *and* $3n_3 + 1 = 2^{k_3}n_4$.

After replacing the value of n_2 by $\left(n_2 = \frac{3n_1+1}{2^{k_1}}\right)$ in order to identify the expression of $\left(n_3 = \frac{3n_2+1}{2^{k_2}}\right)$ in terms of n_1, we obtain the result: $n_3 = \frac{9n_1+3+2^{k_1}}{2^{(k_1+k_2)}}$. Then, we replace the value of n_3 by $\left(n_3 = \frac{9n_1+3+2^{k_1}}{2^{(k_1+k_2)}}\right)$ in order to identify the expression of $\left(n_4 = \frac{3n_3+1}{2^{k_3}}\right)$ in terms of n_1, which allows us to determine the expression of n_4 as follows: $n_4 = \frac{27n_1+9+3*2^{k_1}+2^{(k_1+k_2)}}{2^{(k_1+k_2+k_3)}}$. Therefore, we can express this value of n_4 as shown in (Equation 5), where $k_0 = 0$.

$$n_4 = \frac{3^{(4-1)}n_1 + \sum_{i=0}^{i=4-2}\left[3^i * 2^{\sum_{j=0}^{j=4-i-2} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=4-1} k_{(l)}}} = \frac{3^3 n_1 + 3^0 * 2^{(k_0+k_1+k_2)} + 3^1 * 2^{k_0+k_1} + 3^2 * 2^{k_0}}{2^{(k_0+k_1+k_2+k_3)}} \quad (5)$$

As a result, the proposed formula in (Equation 2) (Theorem 2) is correct for the case of having a sequence of four odd numbers $\{n_1, n_2, n_3, n_4\}$ creating a Collatz branch.

The next step in this proof is using recurrence (induction). Therefore, we suppose that the proposed formula in (Equation 2) (Theorem 2) is correct for a sequence of "m" odd numbers $\{n_1, n_2, n_3, n_4 \dots, n_m\}$ that create a Collatz branch, where the numerical process to pass from one number of this sequence to the following one is as described in (Equation 1) (Theorem 1). Then, we add another odd number $n_{(m+1)}$ to this sequence, and we try to verify whether the proposed formula in (Equation 2) (Theorem 2) does extend itself correctly by expressing the value of $n_{(m+1)}$ in terms of n_1.

Let us suppose that we have a sequence of $(m+1)$ odd numbers $\{n_1, n_2, n_3, n_4 \dots, n_m, n_{(m+1)}\}$ that creates a Collatz branch as shown in Figure 3, where the numerical process to pass from one odd number of this sequence to the following one is as described in (Equation 1) (Theorem 1). Then, we suppose that the proposed formula in (Equation 2) is correct for each element n_s from this sequence where $(2 \le s \le m)$. Therefore, we suppose that the calculated expression in (Equation 6) for $n_{m,}$ by using the proposed formula in (Equation 2) (Theorem 2) is correct.

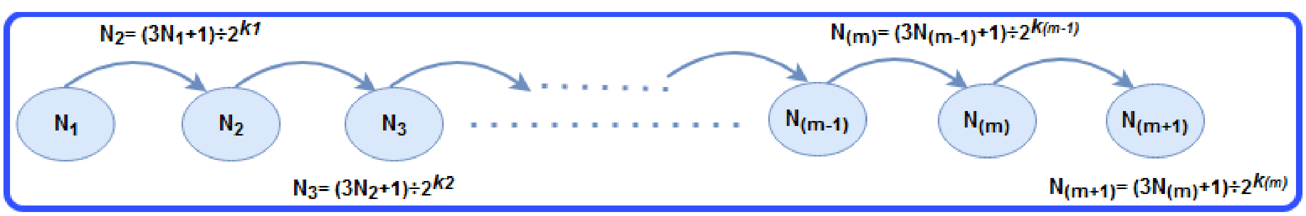

Fig. 3 Illustration of Collatz branch consisting of (m+1) odd numbers interconnected according to the re-expressed operations of Clollatz conjecture

$$n_m = \frac{n_1 * 3^{m-1} + \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{2^{\sum_{l=0}^{l=m-1} K_{(l)}}} \mid K_0 = 0 \text{ and } K_{(i\ge1)} \in \{\mathbb{N} - \{0\}\} \quad (6)$$

Finally, we replace n_m in the equation $\left(n_{(m+1)} = \frac{3n_m+1}{2^{K_m}}\right)$ by its shown expression in (Equation 6), in order to express the odd number $n_{(m+1)}$ in function of n_1, which gives us the shown result in (Equation 7).

$$n_{(m+1)} = \frac{3 * 3^{m-1} n_1 + 3 * \sum_{i=0}^{i=m-2} \left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}} \right] + 2^{\sum_{l=0}^{l=m-1} K_{(l)}}}{2^{K_{(m)}} * 2^{\sum_{l=0}^{l=m-1} K_{(l)}}}$$

$$\Rightarrow n_{(m+1)} = \frac{3^m n_1 + \sum_{i=0}^{i=m-1} \left[3^i * 2^{\sum_{j=0}^{j=m-i-1} K_{(j)}} \right]}{2^{\sum_{l=0}^{l=m} K_{(l)}}} \mid K_0 = 0 \; and \; K_{(i \geq 1)} \in \{ \mathbb{N} - \{0\} \} \quad (7)$$

As a result, we deduce that Theorem 2 is true, because its proposed statements and proposed formula in **(**Equation 2**)** are correct by recurrence (induction).

3.3. Third Unified Formula

This subsection presents a new theorem proposing a third unified formula re-expressing the statements of the Collatz conjecture by expressing only sequences of consecutive odd numbers that can create a loop forwarding calculations back to the starting number, which we describe as the Collatz loop. This subsection is based on the proposed formulas in Theorem 1 and Theorem 2.

A Collatz loop is a sequence of integer numbers where Collatz operations circulate from one to another of these numbers without breaking out of the loop.

In order to define a new unified formula for calculating the first number n_1 of any potential sequence of numbers $\{n_1, n_2, n_3, n_4 \dots, n_m\}$ that can create a Collatz loop, we need first to identify a common structure of calculation among small sequences of Collatz loops that may consist of two, three, or four numbers. Then, generalizing this structure on larger sequences of numbers that may build extensive Collatz loops, which can be proven afterward by induction (recurrence).

Theorem 3

Supposing a sequence of consecutive odd numbers $\{n_1, n_2, n_3, n_4 \dots, n_m\}$ that creates a Collatz loop, where the numerical process to pass from one odd number of this sequence to the following one is as described in (Equation 8); the unified mathematical formula to calculate the first number n_1, sequence of this is as shown in (Equation 9).

$$3n_1 + 1 = 2^{k_1} n_2;\; 3n_2 + 1 = 2^{k_2} n_3;\; 3n_3 + 1 = 2^{k_3} n_4; \dots;\; 3n_{m-1} + 1 = 2^{k_{m-1}} n_m;\; 3n_m + 1 = 2^{k_m} n_1 \quad (8)$$

$$n_1 = \frac{\sum_{i=0}^{i=m-1} \left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}} \right]}{2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m} \mid k_0 = 0 \; and \; k_{(i \geq 1)} \in \{ \mathbb{N} - \{0\} \} \quad (9)$$

Proof of Theorem 3

We are using the proof by recurrence (induction). Therefore, the first stage is to treat the second smallest Collatz loop, which consisted of two odd numbers $\{n_1, n_2\}$, where we can pass from one to another between them by using the operations shown in (Equation 10).

$$3n_1 + 1 = 2^{k_1} n_2;\; 3n_2 + 1 = 2^{k_2} n_1 \quad (10)$$

The first step of this stage is calculating n_2 in function of n_1 at the second iteration, where $3n_2 + 1 = 2^{k_2} n_1$. The resulting expression at this first step is as shown in (Equation 11).

$$n_2 = \frac{(2^{k_2} n_1 - 1)}{3} \quad (11)$$

The second step is replacing n_2 by the shown expression in (Equation 11), in order to determine the value of n_1 in the first iteration, where $3n_1 + 1 = 2^{k_1} n_2$. The resulting expression at this second step is as shown in (Equation 12).

$$n_1 = \frac{(n_1 * 2^{k_1 + k_2} - 2^{k_1} - 3)}{9} \quad (12)$$

The third step of this stage is calculating the general value of n_1 by using (Equation 12). As a result, the final expression of $n_{1,}$ for any Collatz loop that may consist of only two odd numbers is as shown in (Equation 13), which respects the proposed unified formula for Collatz loops in (Equation 9) (Theorem 3).

$$n_1 = \frac{(2^{k_0+k_1}+3)}{2^{k_0+k_1+k_2}-9} \mid k_0 = 0 \; and \; k_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \tag{13}$$

The second stage is to treat the third smallest Collatz loop, which consists of three odd numbers $\{n_1, n_2, n_3\}$. We can pass from one to another among these three odd numbers by using the described operations in (Equation 14).

$$3n_1 + 1 = 2^{k_1} n_2;\; 3n_2 + 1 = 2^{k_2} n_3;\; 3n_3 + 1 = 2^{k_3} n_1 \tag{14}$$

The first step of this second stage is calculating n_3 in function of n_1 at the third iteration, where $3n_3 + 1 = 2^{k_3} n_1$. The resulting expression at this first step of stage 2 is as shown in (Equation 15).

$$n_3 = \frac{(2^{k_3} n_1 - 1)}{3} \tag{15}$$

The second step at this stage is calculating n_2 in function of n_1 at the second iteration, where $(3n_2 + 1 = 2^{k_2} n_3)$ by replacing n_3 with its shown value in (Equation 15). The resulting expression at this second step (stage 2) is as shown in (Equation 16).

$$n_2 = \frac{(2^{k_3+k_2} n_1 - 2^{k_2} - 3)}{9} \tag{16}$$

The third step (second stage) is replacing n_2 by the shown expression in (Equation 16) in order to determine the value of n_1 in the first iteration, where $(3n_1 + 1 = 2^{k_1} n_2)$. The resulting expression at this third step (stage 2) is as shown in (Equation 17).

$$n_1 = \frac{(2^{k_3+k_2+k_1} n_1 - 2^{k_2+k_1} - 3*2^{k_1} - 9)}{27} \tag{17}$$

The fourth step of this second stage is calculating the general value of n_1 by using (Equation 17). As a result, the final expression of n_1 for any possible Collatz loop that may consist of only three odd numbers is as shown in (Equation 18), which respects the proposed unified formula for Collatz loops in (Equation 9) (Theorem 3).

$$n_1 = \frac{(2^{k_2+k_1} + 3*2^{k_1} + 9)}{2^{k_1+k_2+k_3} - 27} = \frac{\sum_{i=0}^{i=2}\left[3^{i} * 2^{\sum_{j=0}^{j=2-i} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=3} k_{(l)}} - 3^3} \mid k_0 = 0 \; and \; k_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \tag{18}$$

The third stage is treating the fourth smallest Collatz loop, which consists of four odd numbers $\{n_1, n_2, n_3, n_4\}$. We can pass from one to another among these four odd numbers by using the described operations in (Equation 19).

$$3n_1 + 1 = 2^{k_1} n_2;\; 3n_2 + 1 = 2^{k_2} n_3;\; 3n_3 + 1 = 2^{k_3} n_4;\; 3n_4 + 1 = 2^{k_4} n_1 \tag{19}$$

The first step of this third stage is calculating n_4 in function of n_1 at the fourth iteration, where $(3n_4 + 1 = 2^{k_4} n_1)$. The resulting expression at this first step (stage 3) is as shown in (Equation 20).

$$n_4 = \frac{(2^{k_4} n_1 - 1)}{3} \tag{20}$$

The second step at this stage is calculating n_3 in function of n_1 at the third iteration, where $(3n_3 + 1 = 2^{k_3} n_4)$ by replacing n_4 with its shown value in (Equation 20). The resulting expression at this second step (stage 3) is as shown in (Equation 21).

$$n_3 = \frac{(2^{k_4+k_3} n_1 - 2^{k_3} - 3)}{9} \tag{21}$$

The third step at this stage is calculating n_2 in function of n_1 at the second iteration, where $(3n_2 + 1 = 2^{k_2} n_3)$ by replacing n_3 with its shown value in (Equation 21). The resulting expression at this third step (stage 3) is as shown in (Equation 22).

$$n_2 = \frac{(2^{k_4+k_3+k_2}n_1 - 2^{k_3+k_2} - 3*2^{k_2} - 9)}{27} \quad (22)$$

The fourth step (third stage) is to replace n_2 by the shown expression in (Equation 22) in order to determine the value of n_1 in the first iteration, where $(3n_1 + 1 = 2^{k_1}n_2)$. The resulting expression at this fourth step is as shown in (Equation 23).

$$n_1 = \frac{(2^{k_4+k_3+k_2+k_1}n_1 - 2^{k_3+k_2+k_1} - 3*2^{k_2+k_1} - 9*2^{k_1} - 27)}{81} \quad (23)$$

The fifth step of this third stage is calculating the general value of n_1 by using (Equation 23). As a result, the final expression of n_1 for any Collatz loop that may consist of exactly four odd numbers is as shown in (Equation 24), which respects our proposed unified formula for Collatz loops in (Equation 9) (Theorem 3).

$$n_1 = \frac{(2^{k_3+k_2+k_1} + 3*2^{k_2+k_1} + 9*2^{k_1} + 27)}{2^{k_1+k_2+k_3+k_4} - 81} = \frac{\sum_{i=0}^{i=3}\left[3^i * 2^{\sum_{j=0}^{j=3-i} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=4} k_{(l)}} - 3^4} \mid k_0 = 0 \ and \ k_{(i\geq1)} \in \{\mathbb{N} - \{0\}\} \quad (24)$$

At the previous stages one, two, and three of this proof, we were able to obtain downsizing expressions from the shown formula in (Equation 9) by scaling calculations on the finite values of $(m = 2)$, $(m = 3)$ and $(m = 4)$ where our proposed unified formula in (Equation 9) (Theorem 3) was correct. Therefore, the next stage, stage four, is about proving that the proposed unified formula in Theorem 3 (Equation 9) is correct for a sequence of $(m + 1)$ odd numbers.

However, we need to rely on a transitional algebraic form of the unified formula shown in (Equation 9) to make calculations extendable and distinguishable from n_m to n_{m+1}. Therefore, we will rely on the shown expression in (Equation 25), which is deduced from Theorem 2.

$$n_m = \frac{3^{m-1}n_1 + \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=m-1} k_{(l)}}} \quad (25)$$

We rely on the proposed expression in (Equation 25) for a sequence of "m" odd numbers. Then, we add another odd number n_{m+1} to this sequence in order to create a Collatz loop consisted of $(m + 1)$ numbers, which give us the final iterations shown in (Equation 26):

$$3n_m + 1 = 2^{k_m} * n_{(m+1)} \ and \ 3 * n_{(m+1)} + 1 = 2^{k_{(m+1)}} * n_1 \quad (26)$$

By using the shown expressions in (Equation 26) and (Equation 25), we deduce the shown result in (Equation 27):

$$n_{(m+1)} = \frac{3^m n_1 + \sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=m} k_{(l)}}} \ and \ 3 * n_{(m+1)} + 1 = 2^{k_{(m+1)}} * n_1 \quad (27)$$

As a final step, we use the shown expressions in (Equation 27) to calculate the value of n_1, which is presented in (Equation 28):

$$n_1 = \frac{3^{m+1}n_1 + \sum_{i=0}^{i=m}\left[3^i * 2^{\sum_{j=0}^{j=m-i} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=m+1} k_{(l)}}} \Rightarrow n_1 = \frac{\sum_{i=0}^{i=m}\left[3^i * 2^{\sum_{j=0}^{j=m-i} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=m+1} k_{(l)}} - 3^{m+1}} \mid k_0 = 0 \ and \ k_{(i\geq1)} \in \{\mathbb{N} - \{0\}\} \quad (28)$$

As a result, the proposed unified formula in (Equation 9) (Theorem 3) to calculate the first number n_1 from a sequence of odd numbers $\{n_1, n_2, n_3, n_4 \dots, n_m\}$ that can create a Collatz loop is correct for $(m \in \{\mathbb{N} - \{0\}\})$.

3.4. Fourth Unified Formula

This subsection presents a new theorem proposing a fourth unified formula scaling the third proposed formula in Theorem 3 by extending it to calculate any odd number that may be included in a Collatz loop.

Theorem 4

Supposing a sequence of consecutive odd numbers $\{n_1, n_2, n_3, n_4 \dots, n_m\}$ that are included in a potential Collatz loop, where the numerical process to pass from one odd number among them to the following one is as described in (Equation 29). The unified mathematical formula to calculate any odd number n_{l+1} in this sequence (where $0 \leq l \leq m-1$), it is as shown in (Equation 30).

The used operator $ROT_s^m(\quad)$ in (Equation 30) is rotating the order of shown numbers in the sequence (Equation 29) to the left by "S" steps, whereas "m" is expressing the number of numbers in the sequence. This rotating operator is expressed as shown in (Equation 31).

$$3n_1 + 1 = 2^{k_1}n_2;\ 3n_2 + 1 = 2^{k_2}n_3;\ 3n_3 + 1 = 2^{k_3}n_4;\ \dots;\ 3n_{m-1} + 1 = 2^{k_{m-1}}n_m;\ 3n_m + 1 = 2^{k_m}n_1 \tag{29}$$

$$n_{l+1} = \frac{\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} ROT_l^m\left(k_{(j)}\right)}\right]}{2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m} \mid\ 0 \leq l \leq m-1 \ and\ k_0 = 0\ and\ k_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \tag{30}$$

$$ROT_l^m\left(k_{(j)}\right) = \begin{cases} k_{j+l},\ if(j + l \leq m) \\ k_{j+l-m},\ if(j + l > m) \\ k_0 = 0,\ if(j = 0) \end{cases} \tag{31}$$

Proof of Theorem 4

The proposed unified formula in (Equation 9) (Theorem 3) enables the calculation of the value of the first number n_1 from a sequence of odd numbers $\{n_1, n_2, n_3, \dots, n_m\}$ that can create a Collatz loop. This proposed formula can be extended to calculate the values of other numbers n_i in this sequence by relying on iterative rotations of this sequence in terms of order.

For example, we rotate the sequence of odd numbers $\{n_1, n_2, n_3, \dots, n_{m-1}, n_m\}$ to the left by one step in order to have the sequence $\{n_2, n_3, \dots, n_{m-1}, n_m, n_1\}$, where each number is expressed as shown in (Equation 32). :

$$3n_2 + 1 = 2^{k_2}n_3;\ 3n_3 + 1 = 2^{k_3}n_4;\ \dots;\ 3n_{m-1} + 1 = 2^{k_{m-1}}n_m;\ 3n_m + 1 = 2^{k_m}n_1;\ 3n_1 + 1 = 2^{k_1}n_2 \tag{32}$$

As a result, we can calculate the value of the odd number n_2 as shown in (Equation 33):

$$n_2 = \frac{2^{\sum_{j=0}^{j=m-1} k_{(j)}} + \sum_{i=1}^{i=m-2}\left[3^i * 2^{\sum_{j=2}^{j=m-i} k_{(j)}}\right] + 3^{m-1}}{2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m} \tag{33}$$

This time, we rotate the sequence $\{n_1, n_2, n_3, \dots, n_{m-1}, n_m\}$ by two steps to the left, which gives us the resulting sequence $\{n_3, \dots, n_{m-1}, n_m, n_1, n_2\}$ expressed in (Equation 34), where we can calculate the value of n_3 as shown in (Equation 35):

$$3n_3 + 1 = 2^{k_3}n_4;\ \dots;\ 3n_{m-1} + 1 = 2^{k_{m-1}}n_m;\ 3n_m + 1 = 2^{k_m}n_1;\ 3n_1 + 1 = 2^{k_1}n_2;\ 3n_2 + 1 = 2^{k_2}n_3 \tag{34}$$

$$n_3 = \frac{2^{\sum_{j=0}^{j=m-1} k_{(j)}} + 3 * 2^{k_1} * 2^{\sum_{j=3}^{j=m-1} k_{(j)}} + \sum_{i=2}^{i=m-2}\left[3^i * 2^{\sum_{j=3}^{j=m-i+1} k_{(j)}}\right] + 3^{m-1}}{2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m} \mid k_0 = 0\ and\ k_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \tag{35}$$

We deduce that we use the same unified formula to calculate any odd number n_i from the sequence $\{n_1, n_2, n_3, \dots, n_{m-1}, n_m\}$ that can create a Collatz loop. The only difference is made by rotating the sequence, iteratively, by "s" steps to the left and respecting the new order of the resulting sequence. Therefore, we create the operator $ROT_s^m(\quad)$ to express the iterative rotation of order as shown in (Equation 36):

$$ROT_s^m(k_i) = k_j\left[if\ (i == 0)\{j = 0;\}\ else\{if(i + s > m)\{j = i + s - m;\}\ else\{j = i + s;\}\}\right] \tag{36}$$

We can calculate the value of any number n_{l+1} from the sequence $\{n_1, n_2, n_3, \dots, n_{m-1}, n_m\}$, where the value of "$l$" respects the condition $(0 \leq l \leq m-1)$, by relying on the operator $ROT_s^m(\quad)$ shown in (Equation 36), while using the presented expression in (Equation 37):

$$n_{l+1} = \frac{\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} ROT_l^m(K_{(j)})}\right]}{2^{\sum_{l=0}^{l=m} K_{(l)}} - 3^m} \tag{37}$$

The next stage is relying on recurrence (induction) to prove the given statement and proposed formulas in Theorem 4.

We suppose that the used formula in (Equation 37) is correct to calculate a number n_{l+1}, by rotating the order of the sequence (Equation 29) by "l" steps to the left, in order to obtain the shown sequence in (Equation 38). Then, we will prove by recurrence that we can use the same logic to calculate n_{l+2} by rotating the resulting sequence in (Equation 38) by one other step to the left.

$$3ROT_l^m(n_1) + 1 = 2^{ROT_l^m(K_{(1)})} ROT_l^m(n_2);\ 3ROT_l^m(n_2) + 1 = 2^{ROT_l^m(K_{(2)})} ROT_l^m(n_3); \dots;\ 3ROT_l^m(n_{m-3}) + 1 = 2^{ROT_l^m(K_{(m-3)})} ROT_l^m(n_{m-2});\ 3ROT_l^m(n_{m-2}) + 1 = 2^{ROT_l^m(K_{(m-2)})} ROT_l^m(n_{m-1});\ 3ROT_l^m(n_{m-1}) + 1 = 2^{ROT_l^m(K_{(m-1)})} ROT_l^m(n_m);\ 3ROT_l^m(n_m) + 1 = 2^{ROT_l^m(K_{(m)})} ROT_l^m(n_1) \tag{38}$$

We extend the used logic in this proof by rotating the shown sequence in (Equation 38) by one step to the left. As a result, we obtain the shown sequence in (Equation 39).

$$3ROT_l^m(n_2) + 1 = 2^{ROT_l^m(K_{(2)})} ROT_l^m(n_3); \dots;\ 3ROT_l^m(n_{m-3}) + 1 = 2^{ROT_l^m(K_{(m-3)})} ROT_l^m(n_{m-2});\ 3ROT_l^m(n_{m-2}) + 1 = 2^{ROT_l^m(K_{(m-2)})} ROT_l^m(n_{m-1});\ 3ROT_l^m(n_{m-1}) + 1 = 2^{ROT_l^m(K_{(m-1)})} ROT_l^m(n_m);\ 3ROT_l^m(n_m) + 1 = 2^{ROT_l^m(K_{(m)})} ROT_l^m(n_1);\ 3ROT_l^m(n_1) + 1 = 2^{ROT_l^m(K_{(1)})} ROT_l^m(n_2) \tag{39}$$

As a result, we can calculate the value of the element $ROT_l^m(n_2)$ by using the shown formula in (Equation 40), which is based on extending the demonstrated formula in Theorem 3.

$$ROT_l^m(n_2) = \frac{2^{\sum_{j=0}^{j=m-1} ROT_l^m(K_{(j)})} + \sum_{i=1}^{i=m-2}\left[3^i * 2^{\sum_{j=2}^{j=m-i} ROT_l^m(K_{(j)})}\right] + 3^{m-1}}{2^{\sum_{l=0}^{l=m} K_{(l)}} - 3^m} \tag{40}$$

We deduce that we can re-express the shown formula in (Equation 40) to be presented as shown in (Equation 41).

$$ROT_{l+1}^m(n_1) = ROT_l^m(n_2) = \frac{\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} ROT_{l+1}^m(K_{(j)})}\right]}{2^{\sum_{l=0}^{l=m} K_{(l)}} - 3^m} \tag{41}$$

Therefore, the proposed statements and formulas in Theorem 4 are correct by recurrence (induction).

3.5. Fifth Unified Formula

This subsection presents a new theorem proposing a fifth unified formula scaling the second proposed formula in Theorem 2 by reversing its operations in terms of order, which allows for expressing an odd number in a function of its consecutive odd numbers according to the operations of the Collatz conjecture.

Theorem 5

Supposing a sequence of odd numbers $\{n_1, n_2, n_3, n_4 \dots, n_m\}$ that create a Collatz branch, where the numerical process to pass from one number of this sequence to the following one is as stated by the operations of the Collatz conjecture, which are re-expressed in Theorem 1. The unified mathematical formula to calculate any number n_s of this sequence in function of n_m where $(1 \leq s < m)$ is as shown in (Equation 42).

$$n_s = \frac{n_m * 2^{\sum_{l=0}^{l=m-1-s} K_{(-l+m-1)}} - \sum_{i=0}^{i=m-1-S}\left[3^i * 2^{\sum_{j=i}^{j=m-1-s} K_{(-j+m-2)}}\right]}{3^{m-s}} \mid K_{s-1} = 0 * K_{s-1} \text{ and } K_{(i \geq S)} \in \{\mathbb{N} - \{0\}\} \tag{42}$$

Proof of Theorem 5

In the case of having a sequence of two odd numbers $\{n_{m-1}, n_m\}$ that create a Collatz branch, where the numerical process to pass from one odd number of this sequence to the following one is as described in (Equation 1) (Theorem 1); we express the value of n_{m-1} as follows: $3n_{m-1} + 1 = 2^{k_{m-1}} n_m$. Therefore, we can express this value of n_{m-1} as shown in (Equation 43).

$$n_{(s=m-1)} = \frac{n_m * 2^{\sum_{l=0}^{l=0} k(-l+m-1)} - \sum_{i=0}^{i=0}\left[3^i * 2^{\sum_{j=i}^{j=m-1-s} k(-j+m-2)}\right]}{3^{m-s}} = \frac{\left(n_m * 2^{k(m-1)} - 1\right)}{3} \; | k_{m-2} = 0 * k_{m-2} \quad (43)$$

As a result, the proposed formula in (Equation 42) (Theorem 5) is correct for the case of having a sequence of two consecutive odd numbers $\{n_{m-1}, n_m\}$ that creates a Collatz branch.

In the case of having a sequence of consecutive three odd numbers $\{n_{m-2}, n_{m-1}, n_m\}$ that create a Collatz branch, where the numerical process to pass from one number of this sequence to the following one is as described in (Equation 1) (Theorem 1); we express the values of n_{m-1} and $n_{m-2,}$ as follows: $3n_{m-2} + 1 = 2^{k(m-2)} n_{m-1}$ *and* $3n_{m-1} + 1 = 2^{k(m-1)} n_m$.

After replacing the value of n_{m-1} by $\left(n_{m-1} = \frac{n_m * 2^{k(m-1)} - 1}{3}\right)$ in order to identify the expression of $\left(n_{m-2} = \frac{2^{k(m-2)} * n_{(m-1)} - 1}{3}\right)$ in terms of n_m, we obtain the following formula presenting the value of n_{m-2} in function of n_m: $n_{m-2} = \frac{n_m * 2^{k(m-2)+k(m-1)} - 2^{k(m-2)} - 3}{9}$. Therefore, we can express this value of n_{m-2} as shown in (Equation 44).

$$n_{(s=m-2)} = \frac{n_m * 2^{\sum_{l=0}^{l=1} k(-l+m-1)} - \sum_{i=0}^{i=1}\left[3^i * 2^{\sum_{j=i}^{j=m-1-s} k(-j+m-2)}\right]}{3^{m-s}} \; | k_{m-3} = 0 * k_{m-3}$$

$$\Rightarrow n_{m-2} = \frac{n_m * 2^{k(m-2)+k(m-1)} - 3^0 * 2^{k(m-3)+k(m-2)} - 3^1 * 2^{k(m-3)}}{9} \; | \; k_{m-3} = 0 \quad (44)$$

As a result, the proposed formula in (Equation 42) (Theorem 5) is correct for the case of having a sequence of three odd numbers $\{n_{m-2}, n_{m-1}, n_m\}$ that creates a Collatz branch.

In the case of having a sequence of four odd numbers $\{n_{m-3}, n_{m-2}, n_{m-1}, n_m\}$ that create a Collatz branch, where the numerical process to pass from one odd number of this sequence to the following one is as described in (Equation 1) (Theorem 1); we express the values of n_{m-3}, n_{m-2} and n_{m-1} as follows: $3n_{m-3} + 1 = 2^{k_{m-3}} n_{m-2}$ *and* $3n_{m-2} + 1 = 2^{k_{m-2}} n_{m-1}$ *and* $3n_{m-1} + 1 = 2^{k_{m-1}} n_m$.

After replacing the value of n_{m-1} by $\left(n_{m-1} = \frac{2^{k(m-1)} n_m - 1}{3}\right)$ in order to identify the expression of $\left(n_{m-2} = \frac{2^{k(m-2)} n_{(m-1)} - 1}{3}\right)$ in terms of n_m, we obtain the result $\left(n_{m-2} = \frac{n_m * 2^{k(m-2)+k(m-1)} - 2^{k(m-2)} - 3}{9}\right)$. Then, we replace the value of n_{m-2} by $\left(n_{m-2} = \frac{n_m * 2^{k(m-2)+k(m-1)} - 2^{k(m-2)} - 3}{9}\right)$ in order to identify the expression of $\left(n_{m-3} = \frac{2^{k(m-3)} n_{(m-2)} - 1}{3}\right)$ in terms of n_m, which allows us to determine the expression of n_{m-3} as follows: $n_{m-3} = \frac{n_m * 2^{k(m-3)+k(m-2)+k(m-1)} - 2^{k(m-3)+k(m-2)} - 3 * 2^{k(m-3)} - 9}{27}$. Therefore, we can express this value of n_{m-3} as shown in (Equation 45).

$$n_{(s=m-3)} = \frac{n_m * 2^{\sum_{l=0}^{l=m-1-s} k(-l+m-1)} - \sum_{i=0}^{i=m-1-s}\left[3^i * 2^{\sum_{j=i}^{j=m-1-s} k(-j+m-2)}\right]}{3^{m-s}} \; | \; k_{s-1} = 0 * k_{s-1}$$

$$\Rightarrow n_{m-3} = \frac{n_m * 2^{\sum_{l=0}^{l=2} k(-l+m-1)} - \sum_{i=0}^{i=2}\left[3^i * 2^{\sum_{j=i}^{j=2} k(-j+m-2)}\right]}{3^3} \; | \; k_{m-4} = 0 * k_{m-4}$$

$$\Rightarrow n_{m-3} = \frac{n_m * 2^{k(m-3)+k(m-2)+k(m-1)} - 3^0 * 2^{k(m-4)+k(m-3)+k(m-2)} - 3^1 * 2^{k(m-4)+k(m-3)} - 3^2 * 2^{k(m-4)}}{27} \; | \; k_{m-4} = 0$$

$$\Longrightarrow n_{m-3} = \frac{n_m * 2^{k_{(m-3)}+k_{(m-2)}+k_{(m-1)}} - 2^{k_{(m-3)}+k_{(m-2)}} - 3 * 2^{k_{(m-3)}} - 9}{27} \tag{45}$$

As a result, the proposed formula in (Equation 42) (Theorem 5) is correct for the case of having a sequence of four odd numbers $\{n_{m-3}, n_{m-2}, n_{m-1}, n_m\}$ that creates a Collatz branch.

The next step in this proof is using recurrence (induction). Therefore, we suppose that the proposed formula in (Equation 42) (Theorem 5) is correct for a sequence of $(p-1)$ odd numbers $\{n_{m-p+2}, n_{m-p+3}, n_{m-p+4}, \dots n_{m-3}, n_{m-2}, n_{m-1}, n_m\}$ that create a Collatz branch, where the numerical process to pass from one number of this sequence to the following one is as described in (Equation 1) (Theorem 1). Then, we add another odd number $n_{(m-p+1)}$ to this sequence, and we try to verify whether the proposed formula in (Equation 42) (Theorem 5) extends itself correctly by expressing the value of $n_{(m-p+1)}$ in terms of n_m .

We suppose having a sequence of (p) odd numbers $\{n_{m-p+1}, n_{m-p+2}, n_{m-p+3}, \dots n_{m-3}, n_{m-2}, n_{m-1}, n_m\}$ that create a Collatz branch, where the numerical process to pass from one number of this sequence to the following one is as described in (Equation 1) (Theorem 1). Then, we suppose that the proposed formula in (Equation 42) (Theorem 5) is correct for each element n_s from this sequence, where the value of "s" is in the following range $(m-p+2 \le s \le m-1)$. Therefore, we suppose that the calculated expression in (Equation 46) for $n_{(s=m-p+2)}$ by using the proposed formula in (Equation 42) (Theorem 5) is correct.

$$n_{(s=m-p+2)} = \frac{n_m * 2^{\sum_{l=0}^{l=m-1-s} k_{(-l+m-1)}} - \sum_{i=0}^{i=m-1-S}\left[3^i * 2^{\sum_{j=i}^{j=m-1-s} k_{(-j+m-2)}}\right]}{3^{m-s}} \quad | \; k_{s-1} = 0 * k_{s-1} \tag{46}$$

Finally, we replace $n_{(m-p+2)}$ in the equation $\left(n_{(m-p+1)} = \frac{n_{(m-p+2)} * 2^{k_{(m-p+1)}} - 1}{3}\right)$ by its shown expression in (Equation 46) in order to present the number $n_{(m-p+1)}$ in function of n_m, which gives us the shown result in (Equation 47).

$$n_{(s=m-p+1)} = \frac{2^{k_{(m-p+1)}} n_{(m-p+2)} - 1}{3}$$

$$\Longrightarrow n_{m-p+1} = \frac{2^{k_{m-p+1}} * 2^{\sum_{l=0}^{l=p-3} k_{(-l+m-1)}} n_m - 2^{k_{m-p+1}}\left[3^{p-3} + \sum_{i=0}^{i=p-4}\left(3^i * 2^{\sum_{j=i}^{j=p-4} k_{(-j+m-2)}}\right)\right] - 3^{p-2}}{3^{p-1}} \quad | \; k_{m-p} = 0 * k_{m-p}$$

$$\Longrightarrow n_{m-p+1} = \frac{2^{k_{m-p+1}} * 2^{\sum_{l=0}^{l=p-3} k_{(-l+m-1)}} n_m - 2^{k_{m-p+1}} * 3^{p-3} - \sum_{i=0}^{i=p-4}\left(3^i * 2^{\sum_{j=i}^{j=p-4} k_{(-j+m-2)}}\right) - 3^{p-2}}{3^{p-1}} \quad | \; k_{m-p} = 0 * k_{m-p}$$

$$\Longrightarrow n_{m-p+1} = \frac{2^{k_{m-p+1}} * 2^{\sum_{l=0}^{l=p-3} k_{(-l+m-1)}} n_m - \sum_{i=0}^{i=p-3}\left(3^i * 2^{\sum_{j=i}^{j=p-3} k_{(-j+m-2)}}\right) - 3^{p-2}}{3^{p-1}} \quad | \; k_{m-p} = 0 * k_{m-p}$$

$$\Longrightarrow n_{m-p+1} = \frac{2^{\sum_{l=0}^{l=p-2} k_{(-l+m-1)}} n_m - \sum_{i=0}^{i=p-3}\left[3^i * 2^{\sum_{j=i}^{j=p-3} k_{(-j+m-2)}}\right] - 3^{p-2}}{3^{p-1}} \quad | \; k_{m-p} = 0$$

$$\Longrightarrow n_{m-p+1} = \frac{2^{\sum_{l=0}^{l=p-2} k_{(-l+m-1)}} n_m - \sum_{i=0}^{i=p-2}\left[3^i * 2^{\sum_{j=i}^{j=p-2} k_{(-j+m-2)}}\right]}{3^{p-1}} \quad | \; k_{m-p} = 0$$

$$\Longrightarrow n_{(s=m-p+1)} = \frac{n_m * 2^{\sum_{l=0}^{l=m-1-s} k_{(-l+m-1)}} - \sum_{i=0}^{i=m-1-S}\left[3^i * 2^{\sum_{j=i}^{j=m-1-s} k_{(-j+m-2)}}\right]}{3^{m-s}} \quad | \; k_{s-1} = 0 \tag{47}$$

As a result, we deduce that Theorem 5 is true, because its proposed statements and proposed formula in (Equation 42) are correct by recurrence to calculate the value of each element n_i from a sequence containing (p) elements of odd numbers $\{n_{m-p+1}, n_{m-p+2}, n_{m-p+3}, \dots n_{m-3}, n_{m-2}, n_{m-1}, n_m\}$ that create a Collatz branch, where the numerical process to pass from one odd number of this sequence to the following one is as described in (Equation 1) (Theorem 1).

3.6. Sixth Unified Formula

This subsection presents a new theorem proposing a sixth unified formula re-expressing the statements of the Collatz conjecture by expressing only sequences of consecutive odd numbers, which is based on extending the first and the fifth proposed theorems in this paper.

Theorem 6

Supposing a sequence of odd numbers $\{n_1, n_2, n_3, n_4 \dots, n_m\}$ that create a Collatz branch, where the numerical process to pass from one odd number of this sequence to the following one is as stated by the operations of the Collatz conjecture, which are re-expressed in Theorem 1. If this Collatz branch is based on conducting the operations of the Collatz conjecture by starting from the number n_1 and converging into the number $(n_m = 1)$, then we can express n_1 according to the general formula shown in (Equation 48).

$$n_1 = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \quad | \; K_0 = 0 \; and \; K_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \tag{48}$$

Proof of Theorem 6

In order to prove this sixth theorem, we are going to extend Theorem 5 by scaling its presented formula in (Equation 42).

Based on Theorem five, which we already proved, when we have a sequence of odd numbers $\{n_1, n_2, n_3, n_4 \dots, n_m\}$ that creates a Collatz branch, where the numerical process to pass from one odd number of this sequence to the following one is as stated by the operations of the Collatz conjecture (which are re-expressed in Theorem 1); we can calculate the value of an odd number n_s of this sequence in terms of n_m as shown in (Equation 49):

$$n_s = \frac{n_m * 2^{\sum_{l=0}^{l=m-1-s} K_{(-l+m-1)}} - \sum_{i=0}^{i=m-1-s}\left[3^i * 2^{\sum_{j=i}^{j=m-1-s} K_{(-j+m-2)}}\right]}{3^{m-s}} \quad | \; K_{s-1} = 0 * K_{s-1} \tag{49}$$

Therefore, if the sequence of odd numbers $\{n_1, n_2, n_3, n_4 \dots, n_m\}$ is starting the conduction of Collatz operations from n_1 and then converging into an odd number $n_m = 1$; then we can express the number n_1 as presented in (Equation 50):

$$n_{(s=1)} = \frac{n_m * 2^{\sum_{l=0}^{l=m-1-s} K_{(-l+m-1)}} - \sum_{i=0}^{i=m-1-s}\left[3^i * 2^{\sum_{j=i}^{j=m-1-s} K_{(-j+m-2)}}\right]}{3^{m-s}} \quad | \; K_{s-1} = 0 * K_{s-1}$$

$$\Rightarrow n_1 = \frac{n_m * 2^{\sum_{l=0}^{l=m-2} K_{(-l+m-1)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=i}^{j=m-2} K_{(-j+m-2)}}\right]}{3^{m-1}} \quad | \; K_0 = 0$$

$$\Rightarrow n_1 = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=i}^{j=m-2} K_{(-j+m-2)}}\right]}{3^{m-1}} \quad | \; K_0 = 0$$

$$\Rightarrow n_1 = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \quad | \; K_0 = 0 \tag{50}$$

As a result, the proposed formula and the presented statements in Theorem 6 are correct.

3.7. Seventh Unified Formula

This subsection presents a new theorem proposing a seventh unified formula re-expressing the statements of the Collatz conjecture by expressing sequences of consecutive odd numbers and their neighbors, which is based on extending the first and the sixth proposed theorems in this paper.

Theorem 7

Supposing a group "L" containing natural numbers, where the minimum number in "L" is $\{n_1 = 1\}$ and the maximum number is $\{n_m = m\}$; as shown in (Equation 51). If every odd number n_s from this group, "L" is converging to a value $(P = 1)$ inferior to itself when repeating the operations of the Collatz conjecture, whereas having $(n_s \leq n_m)$; then we can express every natural number "n" from the group "L" where $(n \leq n_{m-1})$ as shown in (Equation 52).

$$L = \{n_1 = 1; n_2 = 2;\ n_3 = 3; n_4 = 4; \dots ;\ n_{m-1} = (m-1); n_m = m\} \tag{51}$$

$$n = 2^R \frac{2^{\sum_{l=0}^{l=t} K''_{(l)}} - \sum_{i=0}^{i=t-1}\left[3^i * 2^{\sum_{j=0}^{j=t-i-1} K''_{(j)}}\right]}{3^{t-1}} \mid R \in \mathbb{N}\ and\ K''_{(0)} = 0\ and\ K''_{(2)} \in \mathbb{Z}\ and\ K''_{(1)} \in \mathbb{Z}\ and\ K''_{(i \notin \{0,1,2\})} \in \{\mathbb{N} - \{0\}\}\ and \left(R + K''_1\right) \geq 0\ and\ \left(R + K''_1 + K''_2\right) > 0 \tag{52}$$

Proof of Theorem 7

In order to prove this theorem (Theorem 7), we are going to extend Theorem 6 by scaling its presented formula in (Equation 48).

We suppose having an odd number n_s from the group "L" where the minimum number in "L" is $\{n_1 = 1\}$ and the maximum number in "L" is $\{n_m = m\}$. We also suppose that every odd number in the group "L" is converging to the number "1" when repeating the operations of the Collatz conjecture.

Since we supposed that n_s is an odd number converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting them on n_s, we can express the generated branch of Collatz by these operations as shown in (Equation 53).

$$n_s = n'_1;\ 3n'_1 + 1 = 2^{k_1} n'_2;\ 3n'_2 + 1 = 2^{k_2} n'_3; \dots ;\ 3n'_{t-1} + 1 = 2^{k_{t-1}} n'_t = 2^{k_{t-1}} \tag{53}$$

By relying on Theorem 6 and the shown expressions in (Equation 53), we can present the number n_s as shown in (Equation 54).

$$n_s = n'_1 = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right]}{3^{t-1}} \mid k_0 = 0\ and\ k_i \in \{\mathbb{N} - \{0\}\} \tag{54}$$

Now, we calculate the value of the natural number "n" where $\left(n \in \{L - \{n_m\}\}\right)$ and $(n = n_s - x)$, whereas having $\left(x \in \{\mathbb{N} - \{0\}\}\ and\ x = \frac{(2^q + 3*2^r - 1)}{3}\ and\ q \in \mathbb{N}\ and\ r \in \mathbb{N}\right)$. By selecting the values of "q" and "r" as natural numbers, the value of "x" can be among the group $\left\{\frac{(2^0+3*2^0-1)}{3} = 1; \frac{(2^0+3*2^1-1)}{3} = 2; \frac{(2^2+3*2^1-1)}{3} = 3; \frac{(2^0+3*2^2-1)}{3} = 4; \frac{(2^2+3*2^2-1)}{3} = 5; \dots ; n_{s-1}\right\}$. As a result, we obtain the shown expression in (Equation 55).

$$n = n_s - x \Rightarrow n = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right]}{3^{t-1}} - x$$

$$\Rightarrow n = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right]}{3^{t-1}} - \frac{(2^q + 3*2^r - 1)}{3}$$

$$\Rightarrow n = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right] - (2^q + 3*2^r - 1)3^{t-2}}{3^{t-1}}$$

$$\Rightarrow n = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right] - 3^{t-2} - (2^q + 3*2^r - 1)3^{t-2}}{3^{t-1}}$$

$$\Rightarrow n = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}-\sum_{i=0}^{i=t-3}\left[3^i*2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right]-2^q*3^{t-2}-2^r*3^{t-1}}}{3^{t-1}} \quad (55)$$

The next step is adapting the shown expression in (Equation 55) by replacing "q" with $\left(k'_{(1)} = q\right)$ whereas conducting the following re-expressions $\{k_{(1)} = q + k'_{(2)};\ k_{(2\leq j\leq t-1)} = k'_{(j+1)}\ and\ k'_{(0)} = 0\}$, which gives us the shown result in (Equation 56). Since the value of $k_{(1)}$ is a natural number absolutely superior to zero, whereas we have$\left(k_{(1)} = q + k'_{(2)}\right)$, then we deduce $\left\{k'_{(2)} \in \mathbb{Z}\ and\ \left(q + k'_{(2)}\right) > 0\right\}$.

$$\Rightarrow n = \frac{2^{q+\sum_{l=1}^{l=t-1} k'_{(l+1)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{q+\sum_{j=1}^{j=t-i-2} k'_{(j+1)}}\right] - 2^q * 3^{t-2} - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n = \frac{2^{k'_{(1)}+\sum_{l=1}^{l=t-1} k'_{(l+1)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{k'_{(1)}+\sum_{j=1}^{j=t-i-2} k'_{(j+1)}}\right] - 2^{k'_{(1)}} * 3^{t-2} - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n = \frac{2^{k'_{(1)}+\sum_{l=2}^{l=t} k'_{(l)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{k'_{(1)}+\sum_{j=2}^{j=t-i-1} k'_{(j)}}\right] - 2^{k'_{(1)}} * 3^{t-2} - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n = \frac{2^{\sum_{l=1}^{l=t} k'_{(l)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{\sum_{j=1}^{j=t-i-1} k'_{(j)}}\right] - 2^{k'_{(1)}} * 3^{t-2} - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n = \frac{2^{\sum_{l=1}^{l=t} k'_{(l)}-\sum_{i=0}^{i=t-2}\left[3^i*2^{\sum_{j=1}^{j=t-i-1} k'_{(j)}}\right]-2^r*3^{t-1}}}{3^{t-1}} \quad (56)$$

The following step is adapting the shown expression in (Equation 56) by replacing $k'_{(1)}$ with $\left(k'_{(1)} = k''_{(1)} + r\right)$ whereas replacing $k'_{(j>1)}$ with $\left(k'_{(j>1)} = k''_{(j)}\right)$ and using $\left(k''_{(0)} = 0\right)$, which gives us the shown result in (Equation 57). The values $\left(k'_{(1)} = k''_{(1)} + r \geq 0\right)$, $\left(k'_{(1)} = q\right)$, $\left(q + k'_{(2)} > 0\right)$ and $\left(k'_{(j>1)} = k''_{(j)}\right)$ are inducing the conditions $\left(k''_{(1)} + r \geq 0\right)$ and $\left(k''_{(1)} + r + k''_{(2)}\right) > 0$ where the values of "r" and "q" are selected to be natural numbers.

$$\Rightarrow n = \frac{2^{r+\sum_{l=1}^{l=t} k''_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{r+\sum_{j=1}^{j=t-i-1} k''_{(j)}}\right] - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n = \frac{2^r * 2^{\sum_{l=1}^{l=t} k''_{(l)}} - 2^r \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=1}^{j=t-i-1} k''_{(j)}}\right] - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n = 2^r \frac{2^{\sum_{l=1}^{l=t} k''_{(l)}-\sum_{i=0}^{i=t-2}\left[3^i*2^{\sum_{j=1}^{j=t-i-1} k''_{(j)}}\right]-3^{t-1}}}{3^{t-1}} \quad (57)$$

Since we have $\left(k''_{(0)} = 0\right)$, we can re-express the shown value of "n" in (Equation 57) as presented in (Equation 58).

$$\Rightarrow n = 2^r \frac{2^{\sum_{l=0}^{l=t} k''_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-1} k''_{(j)}}\right] - 2^0 * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n = 2^r \frac{2^{\sum_{l=0}^{l=t} k''_{(l)}-\sum_{i=0}^{i=t-1}\left[3^i*2^{\sum_{j=0}^{j=t-i-1} k''_{(j)}}\right]}}{3^{t-1}} \quad (58)$$

As a result, we deduce that we can express the natural number "n" from the group $\{L - \{n_m\}\}$ according to the proposed formula in (Equation 58), whereas having $r \in \mathbb{N}\ and\ k''_{(0)} = 0\ and\ k''_{(2)} \in \mathbb{Z}\ and\ k''_{(1)} \in \mathbb{Z}\ and\ k''_{(i\notin\{0,1,2\})} \in \{\mathbb{N} - \{0\}\}\ and (r + k''_1) \geq 0\ and\ (r + k''_1 + k''_2) > 0$. Therefore, the given statements and the proposed formulas in Theorem 7 are correct.

4. New Theorems Providing Reciprocal Formulas for the Collatz Conjecture

This section presents two theorems proving the reciprocal logic of the presented statements and proposed formulas in Theorem 6 and Theorem 7, which we will use principally to prove further theorems and formulas in this paper about the correctness of the Collatz conjecture.

4.1. First Proposed Theorem for Reciprocity

This subsection presents the eighth theorem, which we rely on to prove the reciprocity of Theorem 6 in this paper.

This eighth theorem will allow us to prove in other sections that repeating the operations of the Collatz conjecture, whether starting them on any odd number or even number, will eventually converge to the number "1".

Theorem 8

In the group of natural numbers $\mathbb{N}$, if any odd number n_s is expressed as shown in (Equation 59), where all coefficients $k_{(i\geq1)}$ are natural numbers different from zero and $(k_0 = 0)$, then this odd number n_s is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_s.

$$n_s = \frac{2^{\sum_{l=0}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} k_{(j)}}\right]}{3^{m-1}} \mid k_0 = 0\ and\ k_{(i\geq1)} \in \{\mathbb{N} - \{0\}\} \quad (59)$$

Proof of Theorem 8

In this proof, we will rely on the presented statements and proposed formulas in Theorem 1 and Theorem 5.

Supposing an odd number $(n_s = n_1)$ expressed as shown in (Equation 59), where all coefficients $k_{(i\geq1)}$ are natural numbers different from zero, whereas $(k_0 = 0)$.

The first step is conducting Collatz operations on the odd number $(n_s = n_1)$ according to the presented formula in (Equation 1) (Theorem 1), which allows us to express n_s as shown in (Equation 60).

$$3n_s + 1 = 3n_1 + 1 = 2^{k_1} n_2 \mid k_1 \in \{\mathbb{N} - \{0\}\}\ \ and\ n_2\ is\ an\ odd\ number \quad (60)$$

To simplify expressing the calculations and formulas, we use the shifting operator $SHIFT_l^{m'}(k_{(j)})$ to shift the value of the coefficient $k_{(j)}$ to the left by l steps, whereas m' is the maximum value to reach.

$$SHIFT_l^{m'}(k_{(j)}) = \begin{cases} k_0 = 0, & if (j = 0) \\ k_{j+l}, & if (j > 0\ and\ 0 < j + l \leq m') \end{cases}$$

Based on the shown expressions in (Equation 60), we can present the odd number n_2 as shown in (Equation 61), where we replace n_s with its shown value in (Equation 59).

$$n_2 = \frac{(3n_s + 1)}{2^{k_1}} \Rightarrow n_2 = 3\frac{2^{\sum_{l=0}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} k_{(j)}}\right]}{2^{k_1} * 3^{m-1}} + \frac{1}{2^{k_1}}$$

$$\Rightarrow n_2 = \frac{2^{\sum_{l=0}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} k_{(j)}}\right] + 3^{m-2}}{2^{k_1} * 3^{m-2}}$$

$$\Rightarrow n_2 = \frac{2^{\sum_{l=0}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} k_{(j)}}\right] - 3^{m-2} + 3^{m-2}}{2^{k_1} * 3^{m-2}}$$

$$\Longrightarrow n_2 = \frac{2^{\sum_{l=0}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} k_{(j)}}\right]}{2^{k_1} * 3^{m-2}}$$

$$\Longrightarrow n_2 = \frac{2^{k_1} * 2^{\sum_{l=2}^{l=m-1} k_{(l)}} - 2^{k_1} * \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-3} SHIFT_1^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_1} * 3^{m-2}}$$

$$\Longrightarrow n_2 = 2^{k_1}\frac{2^{\sum_{l=2}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-3} SHIFT_1^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_1} * 3^{m-2}}$$

$$\Longrightarrow n_2 = \frac{2^{\sum_{l=2}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-3} SHIFT_1^{m-1}\left(k_{(j)}\right)}\right]}{3^{m-2}} \quad (61)$$

In this first step, the operator $SHIFT_l^{m'}\left(k_{(j)}\right)$ is allowing the shift in the sequence $\sum_{j=0}^{j=m-i-3} SHIFT_1^{m-1}\left(k_{(j)}\right)$ to the left by one step, in order to exclude the coefficient k_1 from the sequence.

The second step is conducting Collatz operations on the odd number n_2 according to the presented formula in (Equation 1) (Theorem 1), which allows us to express n_2 as shown in (Equation 62).

$$3n_2 + 1 = 2^{k_2} n_3 \mid k_2 \in \{\mathbb{N} - \{0\}\} \textit{ and } n_3 \textit{ is an odd number} \quad (62)$$

Based on the shown expressions in (Equation 62), we can present the odd number n_3 as shown in (Equation 63), where we replace n_2 with its shown value in (Equation 61).

$$n_3 = \frac{(3n_2 + 1)}{2^{k_2}} \Longrightarrow n_3 = 3\frac{2^{\sum_{l=2}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-3} SHIFT_1^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_2} * 3^{m-2}} + \frac{1}{2^{k_2}}$$

$$\Longrightarrow n_3 = \frac{2^{\sum_{l=2}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-3} SHIFT_1^{m-1}\left(k_{(j)}\right)}\right] + 3^{m-3}}{2^{k_2} * 3^{m-3}}$$

$$\Longrightarrow n_3 = \frac{2^{\sum_{l=2}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-4}\left[3^i * 2^{\sum_{j=0}^{j=m-i-3} SHIFT_1^{m-1}\left(k_{(j)}\right)}\right] - 3^{m-3} + 3^{m-3}}{2^{k_2} * 3^{m-3}}$$

$$\Longrightarrow n_3 = \frac{2^{\sum_{l=2}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-4}\left[3^i * 2^{\sum_{j=0}^{j=m-i-3} SHIFT_1^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_2} * 3^{m-3}}$$

$$\Longrightarrow n_3 = \frac{2^{k_2} * 2^{\sum_{l=3}^{l=m-1} k_{(l)}} - 2^{k_2} * \sum_{i=0}^{i=m-4}\left[3^i * 2^{\sum_{j=0}^{j=m-i-4} SHIFT_2^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_2} * 3^{m-3}}$$

$$\Longrightarrow n_3 = 2^{k_2}\frac{2^{\sum_{l=3}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-4}\left[3^i * 2^{\sum_{j=0}^{j=m-i-4} SHIFT_2^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_2} * 3^{m-3}}$$

$$\Longrightarrow n_3 = \frac{2^{\sum_{l=3}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-4}\left[3^i * 2^{\sum_{j=0}^{j=m-i-4} SHIFT_2^{m-1}\left(k_{(j)}\right)}\right]}{3^{m-3}} \quad (63)$$

In this second step, the operator $SHIFT_l^{m'}\left(k_{(j)}\right)$ is allowing the shift in the sequence $\sum_{j=0}^{j=m-i-4} SHIFT_2^{m-1}\left(k_{(j)}\right)$ to the left by two steps, in order to exclude the coefficients k_1 and k_2 from the sequence.

The third step is conducting Collatz operations on the odd number n_3 according to the presented formula in (Equation 1) (Theorem 1), which allows us to express n_3 as shown in (Equation 64).

$$3n_3 + 1 = 2^{k_3} n_4 \mid k_3 \in \{\mathbb{N} - \{0\}\} \text{ and } n_4 \text{ is an odd number} \tag{64}$$

Based on the shown expressions in (Equation 64), we can present the odd number n_4 as shown in (Equation 65), where we replace n_3 with its shown value in (Equation 63).

$$n_4 = \frac{(3n_3+1)}{2^{k_3}} \Rightarrow n_4 = 3\frac{2^{\sum_{l=3}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-4}\left[3^i * 2^{\sum_{j=0}^{j=m-i-4} SHIFT_2^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_3} * 3^{m-3}} + \frac{1}{2^{k_3}}$$

$$\Rightarrow n_4 = \frac{2^{\sum_{l=3}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-4}\left[3^i * 2^{\sum_{j=0}^{j=m-i-4} SHIFT_2^{m-1}\left(k_{(j)}\right)}\right] + 3^{m-4}}{2^{k_3} * 3^{m-4}}$$

$$\Rightarrow n_4 = \frac{2^{\sum_{l=3}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-5}\left[3^i * 2^{\sum_{j=0}^{j=m-i-4} SHIFT_2^{m-1}\left(k_{(j)}\right)}\right] - 3^{m-4} + 3^{m-4}}{2^{k_3} * 3^{m-4}}$$

$$\Rightarrow n_4 = \frac{2^{\sum_{l=3}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-5}\left[3^i * 2^{\sum_{j=0}^{j=m-i-4} SHIFT_2^{m-i-1}\left(k_{(j)}\right)}\right]}{2^{k_3} * 3^{m-4}}$$

$$\Rightarrow n_4 = \frac{2^{k_3} * 2^{\sum_{l=4}^{l=m-1} k_{(l)}} - 2^{k_3} * \sum_{i=0}^{i=m-5}\left[3^i * 2^{\sum_{j=0}^{j=m-i-5} SHIFT_3^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_3} * 3^{m-4}}$$

$$\Rightarrow n_4 = 2^{k_3}\frac{2^{\sum_{l=4}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-5}\left[3^i * 2^{\sum_{j=0}^{j=m-i-5} SHIFT_3^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_3} * 3^{m-4}}$$

$$\Rightarrow n_4 = \frac{2^{\sum_{l=4}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-5}\left[3^i * 2^{\sum_{j=0}^{j=m-i-5} SHIFT_3^{m-1}\left(k_{(j)}\right)}\right]}{3^{m-4}} \tag{65}$$

In this third step, the operator $SHIFT_l^{m'}\left(k_{(j)}\right)$ is allowing the shift in the sequence $\sum_{j=0}^{j=m-i-5} SHIFT_3^{m-1}\left(k_{(j)}\right)$ to the left by three steps, in order to exclude the coefficients k_1, k_2 and k_3 from the sequence.

Now, we will rely on recurrence (induction) by supposing that n_t is expressed as shown in (Equation 66), where ($t \in [\![1, m-2]\!]$). Then, we calculate the value of $\left(n_{t+1} = \frac{(3n_t+1)}{2^{k_t}}\right)$, which is presented in (Equation 67).

$$n_t = \frac{2^{\sum_{l=t}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-t-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-t-1} SHIFT_{t-1}^{m-1}\left(k_{(j)}\right)}\right]}{3^{m-t}} \tag{66}$$

$$n_{t+1} = \frac{(3n_t+1)}{2^{k_t}} \Rightarrow n_{t+1} = 3\frac{2^{\sum_{l=t}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-t-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-t-1} SHIFT_{t-1}^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_t} * 3^{m-t}} + \frac{1}{2^{k_t}}$$

$$\Rightarrow n_{t+1} = \frac{2^{\sum_{l=t}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-t-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-t-1} SHIFT_{t-1}^{m-1}\left(\mathrm{k_{(j)}}\right)}\right] + 3^{m-t-1}}{2^{k_t} * 3^{m-t-1}}$$

$$\Rightarrow n_{t+1} = \frac{2^{\sum_{l=t}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-t-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-t-1} \mathrm{SHIFT}_{t-1}^{m-1}\left(\mathrm{k_{(j)}}\right)}\right] - 3^{m-t-1} + 3^{m-t-1}}{2^{k_t} * 3^{m-t-1}}$$

$$\Longrightarrow n_{t+1} = \frac{2^{\sum_{l=t}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-t-2}\left[3^{i} * 2^{\sum_{j=0}^{j=m-i-t-1} \mathrm{SHIFT}_{t-1}^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_t} * 3^{m-t-1}}$$

$$\Longrightarrow n_{t+1} = \frac{2^{k_t} * 2^{\sum_{l=t+1}^{l=m-1} k_{(l)}} - 2^{k_t} * \sum_{i=0}^{i=m-t-2}\left[3^{i} * 2^{\sum_{j=0}^{j=m-i-t-2} \mathrm{SHIFT}_{t}^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_t} * 3^{m-t-1}}$$

$$\Longrightarrow n_{t+1} = 2^{k_t}\frac{2^{\sum_{l=t+1}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-t-2}\left[3^{i} * 2^{\sum_{j=0}^{j=m-i-t-2} SHIFT_{t}^{m-1}\left(k_{(j)}\right)}\right]}{2^{k_t} * 3^{m-t-1}}$$

$$\Longrightarrow n_{t+1} = \frac{2^{\sum_{l=t+1}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-t-2}\left[3^{i} * 2^{\sum_{j=0}^{j=m-i-t-2} SHIFT_{t}^{m-1}\left(k_{(j)}\right)}\right]}{3^{m-t-1}} \tag{67}$$

According to the used method of recurrence (induction), we deduce that whatever the value of the natural number v where ($v \in [\![1, m-1]\!]$), the value of the odd number n_v can be calculated as shown in (Equation 68), whereas having ($n_s = n_1$). The operator $SHIFT_{l}^{m'}\left(k_{(j)}\right)$ is allowing the shift in the sequence $2^{\sum_{j=0}^{j=m-i-v-1} SHIFT_{v-1}^{m-1}\left(k_{(j)}\right)}$ to the left by $(v-1)$ steps, in order to exclude all the coefficients $\{k_h\}$ ($h \in [\![1, v-1]\!]$) from the sequence

$$n_v = \frac{2^{\sum_{l=v}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-v-1}\left[3^{i} * 2^{\sum_{j=0}^{j=m-i-v-1} SHIFT_{v-1}^{m-1}\left(k_{(j)}\right)}\right]}{3^{m-v}} \;|k_0 = 0 \text{ and } k_{(j\geq 1)} \in \{\mathbb{N} - \{0\}\} \tag{68}$$

$$SHIFT_{l}^{m'}\left(k_{(j)}\right) = \begin{cases} k_0 = 0, & if\,(j = 0) \\ k_{j+l}, & if\,(j > 0 \text{ and } 0 < j + l \leq m') \end{cases}$$

Therefore, we can use the shown expression in (Equation 68) to calculate the value of the odd number n_{m-1} as shown in (Equation 69), which we converge to by repeating the operations of the Collatz conjecture, whereas starting these operations on the odd number $n_s = n_1$.

$$n_{m-1} = \frac{2^{\sum_{l=m-1}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-(m-1)-1}\left[3^{i} * 2^{\sum_{j=0}^{j=m-i-(m-1)-1} \mathrm{SHIFT}_{m-2}^{m-1}\left(k_{(j)}\right)}\right]}{3^{m-(m-1)}}$$

$$\Longrightarrow n_{m-1} = \frac{2^{k_{(m-1)}} - \sum_{i=0}^{i=0}\left[3^{i} * 2^{\sum_{j=0}^{j=-i} \mathrm{SHIFT}_{m-2}^{m-1}\left(k_{(j)}\right)}\right]}{3^{1}}$$

$$\Longrightarrow n_{m-1} = \frac{2^{k_{(m-1)}} - 3^{0} * 2^{\sum_{j=0}^{j=0} \mathrm{SHIFT}_{m-2}^{m-1}\left(k_{(j)}\right)}}{3^{1}} = \frac{2^{k_{(m-1)}} - 3^{0} * 2^{k_{(0)}}}{3^{1}}$$

$$\Longrightarrow n_{m-1} = \frac{2^{k_{(m-1)}} - 1}{3} \tag{69}$$

The next step is calculating the value of the odd number n_m by conducting the operations of the Collatz conjecture on the odd numbers n_{m-1} according to (Equation 1) (Theorem 1), which allows us to express the value of n_m as shown in (Equation 70).

$$n_m = \frac{(3n_{m-1} + 1)}{2^{k_{(m-1)}}} \Longrightarrow n_m = 3\,\frac{2^{k_{(m-1)}} - 1}{2^{k_{(m-1)}} * 3} + \frac{1}{2^{k_{(m-1)}}}$$

$$\Longrightarrow n_m = \frac{2^{k_{(m-1)}} - 1 + 1}{2^{k_{(m-1)}}}$$

$$\Rightarrow n_m = \frac{2^{k(m-1)}}{2^{k(m-1)}}$$

$$\Rightarrow n_m = 1 \quad (70)$$

Therefore, we deduce that by repeating the operations of the Collatz conjecture, starting these operations on an odd number $(n_s = n_1)$; we eventually converge to an odd number $(n_m = 1)$, where the generated Collatz branch "L" by conducting these operations is as shown in (Equation 71).

$$L = \left\{n_s = n_1, n_2 = \frac{(3n_1+1)}{2^{k_1}}, n_3 = \frac{(3n_2+1)}{2^{k_2}}, n_4 = \frac{(3n_3+1)}{2^{k_3}}, \dots, n_{m-1}, n_m = \frac{(3n_{m-1}+1)}{2^{k_{m-1}}} = 1\right\} \quad (71)$$

As a result, we deduce that the presented statements and proposed formula in Theorem 8 are correct.

4.2. Second Proposed Theorem for Reciprocity

This subsection presents the ninth theorem, which we rely on to prove the reciprocity of Theorem 7 in this paper.

This ninth theorem is based on scaling the proposed formulas and presented statements in Theorem 8, which we will use in other sections to prove, algebraically, that repeating the operations of the Collatz conjecture, starting these operations on any odd number or even number, will eventually converge to the number "1".

Theorem 9

In the group of natural numbers $\mathbb{N}$, if any odd number n_s is expressed as shown in (Equation 72), where all coefficients $K''_{(i>2)}$ are natural numbers different from zero, whereas $(K''_2 \in \mathbb{Z})$, $(K''_1 \in \mathbb{Z})$, $(K''_0 = 0)$, $(R + K''_1) \geq 0$ and $(R + K''_1 + K''_2) > 0$; then the odd number n'_s shown in (Equation 73) $(n'_s = n_s + y)$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s.

$$n_s = 2^R \frac{2^{\sum_{l=0}^{l=t} K''_{(l)}} - \sum_{i=0}^{i=t-1}\left[3^i * 2^{\sum_{j=0}^{j=t-i-1} K''_{(j)}}\right]}{3^{t-1}} \mid K''_0 = 0 \; and \; K''_{i\notin\{0,1,2\}} \in \{\mathbb{N} - \{0\}\} \; and \; K''_2 \in \mathbb{Z} \; and \; K''_1 \in \mathbb{Z} \; and \; R \in \mathbb{N} \; and \; (R + K''_1) \geq 0 \; and \; (R + K''_1 + K''_2) > 0 \quad (72)$$

$$n'_s = n_s + y \mid y \in \{\mathbb{N} - \{0\}\} \; and \; n'_s \; MOD[2] = 1 \quad (73)$$

Proof of Theorem 9

In this proof, we suppose having an odd number n_s expressed as shown in (Equation 74), where all coefficients $k''_{(i>2)}$ are natural numbers different from zero, whereas having $(k''_2 \in \mathbb{Z})$, $(k''_1 \in \mathbb{Z})$ and $(k''_0 = 0)$.

$$n_s = 2^r \frac{2^{\sum_{l=0}^{l=t} k''_{(l)}} - \sum_{i=0}^{i=t-1}\left[3^i * 2^{\sum_{j=0}^{j=t-i-1} k''_{(j)}}\right]}{3^{t-1}} \mid k''_0 = 0 \; and \; k''_{i\notin\{0,1,2\}} \in \{\mathbb{N} - \{0\}\} \; and \; k''_2 \in \mathbb{Z} \; and \; k''_1 \in \mathbb{Z} \; and \; r \in \mathbb{N} \; and \; (r + k''_1) \geq 0 \; and \; (r + k''_1 + k''_2) > 0 \quad (74)$$

$$\Rightarrow n_s = 2^r \frac{2^{\sum_{l=0}^{l=t} k''_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-1} k''_{(j)}}\right] - 2^0 * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n_s = \frac{2^r * 2^{\sum_{l=1}^{l=t} k''_{(l)}} - 2^r \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=1}^{j=t-i-1} k''_{(j)}}\right] - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n_s = \frac{2^{r+\sum_{l=1}^{l=t} k''_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{r+\sum_{j=1}^{j=t-i-1} k''_{(j)}}\right] - 2^r * 3^{t-1}}{3^{t-1}} \quad (75)$$

The following step is adapting the shown expression in (Equation 75) by replacing $k''_{(1)}$ with $(k''_{(1)} = k'_{(1)} - r)$ whereas replacing $k''_{(j)}$ with $(k''_{(j>1)} = k'_{(j>1)})$ and using $(k'_{(0)} = 0)$, which gives us the shown result in (Equation 76).

$$\Rightarrow n_s = \frac{2^{\sum_{l=1}^{l=t} k'_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=1}^{j=t-i-1} k'_{(j)}}\right] - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n_s = \frac{2^{\sum_{l=1}^{l=t} k'_{(l)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{\sum_{j=1}^{j=t-i-1} k'_{(j)}}\right] - 2^{k'_{(1)}} * 3^{t-2} - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n_s = \frac{2^{k'_{(1)}+\sum_{l=2}^{l=t} k'_{(l)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{k'_{(1)}+\sum_{j=2}^{j=t-i-1} k'_{(j)}}\right] - 2^{k'_{(1)}} * 3^{t-2} - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n_s = \frac{2^{k'_{(1)}+\sum_{l=1}^{l=t-1} k'_{(l+1)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{k'_{(1)}+\sum_{j=1}^{j=t-i-2} k'_{(j+1)}}\right] - 2^{k'_{(1)}} * 3^{t-2} - 2^r * 3^{t-1}}{3^{t-1}} \quad (76)$$

The next step is adapting the shown expression in (Equation 76) by replacing $k'_{(1)}$ with $\left(q = k'_{(1)}\right)$ whereas conducting the following re-expressions $\{k_{(1)} = q + k'_{(2)};\ k_{(2\le j\le t-1)} = k'_{(j+1)}\ and\ k_{(0)} = 0\}$, which gives us the shown result in (Equation 77). The values $\left(k'_{(1)} = k''_{(1)} + r \ge 0\right)$, $\left(k'_{(1)} = q\right)$ and $\left(k_{(1)} = q + k'_{(2)}\right)$, are relying on the conditions $\left(k''_{(1)} + r \ge 0\right)$ and $\left(k_{(1)} = k''_{(2)} + r + k''_{(1)} > 0\right)$, where the values of "$r$" and "$q$" are to be selected as natural numbers, whereas the value of $k_{(1)}$ is in the group $\{\mathbb{N} - \{0\}\}$.

$$\Rightarrow n_s = \frac{2^{q+\sum_{l=1}^{l=t-1} k'_{(l+1)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{q+\sum_{j=1}^{j=t-i-2} k'_{(j+1)}}\right] - 2^q * 3^{t-2} - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n_s = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right] - 2^q * 3^{t-2} - 2^r * 3^{t-1}}{3^{t-1}}$$

$$\Rightarrow n_s = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-3}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right] - 3^{t-2} - (2^q + 3 * 2^r - 1)3^{t-2}}{3^{t-1}}$$

$$\Rightarrow n_s = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right] - (2^q + 3 * 2^r - 1)3^{t-2}}{3^{t-1}}$$

$$\Rightarrow n_s = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right]}{3^{t-1}} - \frac{(2^q + 3 * 2^r - 1)}{3} \mid k_0 = 0\ and\ k_{(i\ge1)} \in \{\mathbb{N} - \{0\}\} \quad (77)$$

Now, we re-express $\frac{(2^q+3*2^r-1)}{3}$ as $\left(\frac{(2^q+3*2^r-1)}{3} = y\right)$ where $\left(y \in \{\mathbb{N} - \{0\}\}\ and\ y = \frac{(2^q+3*2^r-1)}{3}\ and\ q \in \mathbb{N}\ and\ r \in \mathbb{N}\right)$. By selecting the values of "q" and "r" as natural numbers, the value of "y" can be among the group $\left\{\frac{(2^0+3*2^0-1)}{3} = 1;\ \frac{(2^0+3*2^1-1)}{3} = 2;\ \frac{(2^2+3*2^1-1)}{3} = 3; \frac{(2^0+3*2^2-1)}{3} = 4; \frac{(2^2+3*2^2-1)}{3} = 5; \ldots; n_s - 1\right\}$. As a result, we obtain the shown expression in (Equation 78).

$$\Rightarrow n_s = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right]}{3^{t-1}} - y \mid k_0 = 0\ and\ k_{(i\ge1)} \in \{\mathbb{N} - \{0\}\}$$

$$\Rightarrow n_s + y = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right]}{3^{t-1}} \mid k_0 = 0\ and\ k_{(i\ge1)} \in \{\mathbb{N} - \{0\}\} \quad (78)$$

Based on (Equation 78), we deduce that the odd number n'_s $\{n'_s = n_s + y\}$ is expressed as shown in (Equation 79), where n_s is an odd number and $\left(y = \frac{(2^q+3*2^r-1)}{3}\right)$ is an even number.

$$\Rightarrow n'_s = \frac{2^{\sum_{l=0}^{l=t-1} k_{(l)}} - \sum_{i=0}^{i=t-2}\left[3^i * 2^{\sum_{j=0}^{j=t-i-2} k_{(j)}}\right]}{3^{t-1}} \mid k_0 = 0 \ and \ k_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \quad (79)$$

Based on the shown formula of the odd number n'_s in (Equation 79), we deduce that n'_s is expressed according to the proposed formula and presented statements in Theorem 8, which means that we eventually converge to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on an odd number n'_s.

As a result, we deduce that the presented statements and proposed formulas in Theorem 9 are correct.

5. New Theorems Proving the Collatz Conjecture by Treating Possible Loops

This section presents nine new theorems along with detailed poofs, which rely on the proposed formulas in the previous theorems in this paper, in order to demonstrate that there is no possible Collatz loop where the operations of the Collatz conjecture start from an odd number n_1 and loop back to the same number, except the loop where $(n_1 = 1)$.

5.1. Proposed Theorem on Collatz Loops Consisted of One Element

This subsection presents the tenth theorem in this paper, which treats possible Collatz loops consisting of one odd number according to the proposed formulas in Theorem 3 and Theorem 4 in this paper.

Theorem 10

In the group of Natural numbers $\mathbb{N}$, there is only one subgroup containing one odd number $\{n_1\}$ that can create a Collatz loop where $3n_1 + 1 = 2^{k_1}n_1$ and $k_1 \in \{\mathbb{N} - \{0\}\}$, and this unique subgroup is $\{n_1 = 1\}$ as shown in (Equation 80).

$$\forall\, G \in \mathbb{N}\, ; \left(if\ \exists\{n_1\} \in G \ \mid 3n_1 + 1 = 2^{k_1}n_1 \ and \ k_1 \in \{\mathbb{N} - \{0\}\}\right) \Rightarrow (\{n_1\} = \{1\}) \quad (80)$$

Proof of Theorem 10

The proposed unified formula in (Equation 9) (Theorem 3) for Collatz loops is based on dividing the value of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]$ on the value of the denominator $\left[2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right]$. Therefore, this proposed formula allows us to calculate values of real numbers.

However, in order to have an integer number as a result of this formula (Equation 9 in Theorem 3), the value of the denominator should be less than or equal to the numerator, and also should be a divisor of an integer value.

In the case of having the smallest Collatz loop, which consists of only one odd number $\{n_1\}$, the value of the number n_1 is calculated as $\left(n_1 = \frac{1}{2^{k_1}-3}\right)$ where $\left(k_1 \in \{\mathbb{N} - \{0\}\}\right)$. Therefore, in order to have an integer value as a result of this expression, we need to start by having the condition. $(2^{k_1} - 3) \leq 1$, which can also be expressed as follows: $2^{k_1} \leq 4$. Consequently, the value of k_1 should be limited as follows: $1 \leq k_1 \leq 2$

In addition, the value of the denominator $(2^{k_1} - 3)$ should be dividing the value of the numerator (number 1) by a positive integer value, which means that there is only one value that can be given to k_1 which is $(k_1 = 2)$. Therefore, we can calculate the exact value of n_1 as follows: $n_1 = \frac{1}{2^{k_1}-3} = 1$.

As a result, Theorem 10 and its shown expression in (Equation 80) are correct, because there is only one subgroup consisting of one odd number $\{n_1\}$ that can create a Collatz loop, and this subgroup is where $\{n_1 = 1\}$

5.2. Proposed Theorem on Collatz Loops Consisted of Two Elements

This subsection presents the eleventh theorem, which treats possible Collatz loops consisting of two odd numbers according to the proposed formulas in Theorem 3 and Theorem 4 in this paper.

Theorem 11

In the group of Natural numbers ℕ, there is no subgroup of two different odd numbers $\{n_1, n_2\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)$ and $(3n_2 + 1 = 2^{k_2}n_1)$ whereas having $(k_i \in \{\mathbb{N} - \{0\}\})$ as shown in (Equation 81), because the maximum value for n_1 will be $\left(\max(n_1) = \frac{11}{7}\right)$.

$$\forall\, G \in \mathbb{N}\ ;\ \nexists\{n_1, n_2\} \in G \mid n_1 \neq n_2\ and\ 3n_1 + 1 = 2^{k_1}n_2\ and\ 3n_2 + 1 = 2^{k_2}n_1\ and\ k_i \in \{\mathbb{N} - \{0\}\} \quad (81)$$

Proof of Theorem 11

The proposed unified formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4) for Collatz loops are based on dividing the values of specific numerators by specific denominators when calculating values of odd numbers involved in possible Collatz loops.

Since both formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4) are based on the same structure of terms, while rotating the order to calculate the values of consecutive odd numbers according to Theorem 4, we will focus on analyzing the division of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]$ on the value of the denominator $\left[2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right]$.

Therefore, these proposed formulas in (Equation 9) and (Equation 29) allow for calculating the values of real numbers.

However, in order to have an integer number as a result of these formulas (Equation 9 and Equation 29), the value of the denominator should be inferior (or equal) to the numerator and also should be divided by an integer value.

In the case of having the second smallest Collatz loop, which consists of two different odd numbers $\{n_1, n_2\}$, the value of the number n_1 is calculated as $\left(n_1 = \frac{(2^{k_1}+3)}{2^{k_1+k_2}-9}\right)$ whereas having $(k_i \in \{\mathbb{N} - \{0\}\})$. Therefore, in order to have an integer value as a result of this expression of n_1, we need to start by having the condition $(2^{k_1+k_2} - 9) \leq (2^{k_1} + 3)$, which can also be expressed as follows: $2^{k_1}(2^{k_2} - 1) \leq 12$. Consequently, the values of k_1 and k_2 should be limited as follows: $1 \leq k_1 \leq 3$ and $1 \leq k_2 \leq (4 - k_1)$.

Therefore, it is impossible to have a Collatz loop consisting of two different odd numbers $\{n_1, n_2\}$, where n_1 is outside the range $[\![1;\ 2[\![$, because the maximum real value for n_1 is $\frac{11}{7}$ where $(k_1 = 3)$ and $(k_2 = 1)$.

The maximum value of n_1 is calculated by identifying the minimum positive value of the denominator $\left[2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right]$, then identifying the maximum positive value of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]$.

As a result, the expression (Equation 81) and the presented statements by the eleventh theorem in this paper are correct, because there is no subgroup of two different odd numbers $\{n_1, n_2\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)$ and $(3n_2 + 1 = 2^{k_2}n_1)$ and $(k_i \in \{\mathbb{N} - \{0\}\})$.

5.3. Proposed Theorem on Collatz Loops Consisted of Three Elements

This subsection presents the twelfth theorem, which treats possible Collatz loops consisting of three odd numbers according to the proposed formulas in Theorem 3 and Theorem 4 in this paper.

Theorem 12

In the group of natural numbers ℕ, there is no subgroup of three different odd numbers $\{n_1, n_2, n_3\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)$ and $(3n_2 + 1 = 2^{k_2}n_3)$ and $(3n_3 + 1 = 2^{k_3}n_1)$ and $(k_i \in \{\mathbb{N} - \{0\}\})$ as shown in (Equation 82), because the maximum value for n_1 will be $\left(\max(n_1) = \frac{49}{5}\right)$.

$$\forall\, G \in \mathbb{N}\ ;\ \nexists\{n_1, n_2, n_3\} \in G\ |n_i \neq n_{j(j\neq i)}\ and\ 3n_1 + 1 = 2^{k_1}n_2\ and\ 3n_2 + 1 = 2^{k_2}n_3\ and\ 3n_3 + 1 = 2^{k_3}n_1\ and\ k_i \in \{\mathbb{N} - \{0\}\} \quad (82)$$

Proof of Theorem 12

As mentioned in the proof of Theorem 11, both formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4) are based on the same structure of terms, while rotating the order to calculate the values of consecutive odd numbers according to Theorem 4. Therefore, we will focus on analyzing the division of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]$ on denominator $\left[2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right]$, which generate values of real numbers.

As it was mentioned in the proof of Theorem 11, in order to have an integer number as a result of the shown formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4), the value of the denominator should be inferior (or equal) to the numerator, and also should be a divisor of an integer value.

In the case of having the third smallest loop of Collatz, which consists of three different odd numbers $\{n_1, n_2, n_3\}$, the value of the first number n_1 the length of this sequence is calculated as $\left(n_1 = \frac{(2^{k_2+k_1}+3*2^{k_1}+9)}{2^{k_1+k_2+k_3}-27}\right)$ whereas having $\left(k_i \in \{\mathbb{N} - \{0\}\}\right)$. Therefore, in order to have an integer value as a result of this expression of n_1, we need to start by having the condition $(2^{k_1+k_2+k_3} - 27) \leq (2^{k_2+k_1} + 3 * 2^{k_1} + 9)$, which can also be expressed as follows: $2^{k_1}(2^{k_3+k_2} - 2^{k_2} - 3) \leq 36$. Consequently, the values of k_1, k_2 and k_3 should be limited as follows: $1 \leq k_1 \leq 5$ and $1 \leq k_2 \leq (6 - k_1)$ and $1 \leq k_3 \leq (7 - k_1 - k_2)$.

Therefore, it is impossible to have a Collatz loop consisting of three different odd numbers $\{n_1, n_2, n_3\}$, where n_1 is outside the range $[\![1;\ 9]\!]$, because the maximum value for n_1 is $\frac{49}{5}$ where $(k_1 = 3)$, $(k_2 = 1)$ and $(k_3 = 1)$.

The maximum value of n_1 is calculated by identifying the minimum positive value of the denominator $\left[2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right]$, then identifying the maximum positive value of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]$.

As a result, the expression (Equation 82) and the presented statements by the twelfth theorem in this paper are correct, because there is no subgroup of three different odd numbers $\{n_1, n_2, n_3\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)\ and\ (3n_2 + 1 = 2^{k_2}n_3)\ and\ (3n_3 + 1 = 2^{k_3}n_1)\ and\ \left(k_i \in \{\mathbb{N} - \{0\}\}\right)$.

5.4. Proposed Theorem on Collatz Loops Consisted of Four Elements

This subsection presents the thirteenth theorem, which treats possible Collatz loops consisting of four odd numbers according to the proposed formulas in Theorem 3 and Theorem 4 in this paper.

Theorem 13

In the group of Natural numbers $\mathbb{N}$, there is no subgroup of four different odd numbers $\{n_1, n_2, n_3, n_4\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)\ and\ (3n_2 + 1 = 2^{k_2}n_3)\ and\ (3n_3 + 1 = 2^{k_3}n_4)\ and\ (3n_4 + 1 = 2^{k_4}n_1)\ and\ \left(k_i \in \{\mathbb{N} - \{0\}\}\right)$ as shown in (Equation 83), because the maximum value for n_1 will be $\left(\max(n_1) = \frac{331}{47}\right)$.

$$\forall\, G \in \mathbb{N}\ ;\ \nexists\{n_1, n_2, n_3, n_4\} \in G \mid n_i \neq n_{j(j\neq i)}\ and\ 3n_1 + 1 = 2^{k_1}n_2\ and\ 3n_2 + 1 = 2^{k_2}n_3\ and\ 3n_3 + 1 = 2^{k_3}n_4\ and\ 3n_4 + 1 = 2^{k_4}n_1\ and\ k_i \in \{\mathbb{N} - \{0\}\} \quad (83)$$

Proof of Theorem 13

As mentioned in the proof of Theorem 11, both formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4) are based on the same structure of terms, while rotating the order to calculate the values of consecutive odd numbers according to Theorem 4. Therefore, we will focus on analyzing the division of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} K_{(j)}}\right]$ on the value of the denominator $\left[2^{\sum_{l=0}^{l=m} K_{(l)}} - 3^m\right]$, which generate values of real numbers.

As it was mentioned in the proof of Theorem 11, in order to have an integer number as a result of the shown formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4), the value of denominator should be less than (or equal) to the numerator, and also should be dividing it into an integer value.

In the case of having a loop of Collatz consisting of four different odd numbers $\{n_1, n_2, n_3, n_4\}$, the value of the first number n_1, the length of this sequence is calculated as $\left(n_1 = \frac{(2^{k_3+k_2+k_1}+3*2^{k_2+k_1}+9*2^{k_1}+27)}{2^{k_1+k_2+k_3+k_4}-81}\right)$ whereas having $\left(k_i \in \{\mathbb{N} - \{0\}\}\right)$. Therefore, in order to have an integer value as a result of this expression of n_1, we need to start by having the condition $(2^{k_1+k_2+k_3+k_4} - 81) \leq (2^{k_3+k_2+k_1} + 3 * 2^{k_2+k_1} + 9 * 2^{k_1} + 27)$, which can also be expressed as follows: $2^{k_1}(2^{k_4+k_3+k_2} - 2^{k_3+k_2} - 3 * 2^{k_2} - 9) \leq 108$. Consequently, the values of k_1, k_2, k_3 and k_4 should be limited as follows: $1 \leq k_1 \leq 6$ and $1 \leq k_2 \leq (7 - k_1)$ and $1 \leq k_3 \leq (8 - k_1 - k_2)$ and $1 \leq k_4 \leq (9 - k_1 - k_2 - k_3)$.

Therefore, it is impossible to have a Collatz loop consisting of four different odd numbers $\{n_1, n_2, n_3, n_4\}$, where n_1 is outside the range $[\![1;\ 7]\!]$, because the maximum value for n_1 is $\frac{331}{47}$ where $(k_1 = 4)$, $(k_2 = 1)$, $(k_3 = 1)$ and $(k_4 = 1)$.

The maximum value of n_1 is calculated by identifying the minimum positive value of the denominator $\left[2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right]$, then identifying the maximum positive value of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]$.

As a result, the expression (Equation 83) and the presented statements by the thirteenth theorem in this paper are correct, because there is no subgroup of four different odd numbers $\{n_1, n_2, n_3, n_4\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)\ and\ (3n_2 + 1 = 2^{k_2}n_3)\ and\ (3n_3 + 1 = 2^{k_3}n_4)\ and\ (3n_4 + 1 = 2^{k_4}n_1)\ and\ \left(k_i \in \{\mathbb{N} - \{0\}\}\right)$.

5.5. Proposed Theorem on Collatz Loops Consisted of Five Elements

This subsection presents the fourteenth theorem, which treats possible Collatz loops consisting of five odd numbers according to the proposed formulas in Theorem 3 and Theorem 4 in this paper.

Theorem 14

In the group of Natural numbers $\mathbb{N}$, there is no subgroup of five different odd numbers $\{n_1, n_2, n_3, n_4, n_5\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)\ and\ (3n_2 + 1 = 2^{k_2}n_3)\ and\ (3n_3 + 1 = 2^{k_3}n_4)\ and\ (3n_4 + 1 = 2^{k_4}n_5)\ and\ (3n_5 + 1 = 2^{k_5}n_1)\ and\ \left(k_i \in \{\mathbb{N} - \{0\}\}\right)$ as shown in (Equation 84), because the maximum value for n_1 will be $\left(\max(n_1) = \frac{1121}{13}\right)$.

$$\forall\ G \in \mathbb{N}\ ;\ \nexists\{n_1, n_2, n_3, n_4, n_5\} \in G \mid n_i \neq n_{j(j\neq i)}\ and\ 3n_1 + 1 = 2^{k_1}n_2\ and\ 3n_2 + 1 = 2^{k_2}n_3\ and\ 3n_3 + 1 = 2^{k_3}n_4\ and\ 3n_4 + 1 = 2^{k_4}n_5\ and\ 3n_5 + 1 = 2^{k_5}n_1\ and\ k_i \in \{\mathbb{N} - \{0\}\} \quad (84)$$

Proof of Theorem 14

As mentioned in the proof of Theorem 11, both formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4) are based on the same structure of terms while rotating the structure to calculate the values of consecutive odd numbers according to Theorem 4.

Therefore, we will focus on analyzing the division of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} K_{(j)}}\right]$ on the value of the denominator $\left[2^{\sum_{l=0}^{l=m} K_{(l)}} - 3^m\right]$, which generate values of real numbers.

As it was mentioned in the proof of Theorem 11, in order to have an integer number as a result of the shown formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4), the value of denominator should be less than (or equal) to the numerator, and also should be dividing it into an integer value.

In the case of having a loop of Collatz consisting of five different odd numbers $\{n_1, n_2, n_3, n_4, n_5\}$, the value of the first number n_1 the value of this sequence is calculated as shown in (Equation 85), where $k_i \in \{\mathbb{N} - \{0\}\}$:

$$n_1 = \frac{(2^{k_4+k_3+k_2+k_1}+3*2^{k_3+k_2+k_1}+9*2^{k_2+k_1}+27*2^{k_1}+81)}{2^{k_1+k_2+k_3+k_4+k_5}-243} \quad (85)$$

Therefore, in order to have an integer value as a result of this expression of n_1 (Equation 85), we need to start by having the condition $(2^{k_1+k_2+k_3+k_4+k_5} - 243) \leq (2^{k_4+k_3+k_2+k_1} + 3 * 2^{k_3+k_2+k_1} + 9 * 2^{k_2+k_1} + 27 * 2^{k_1} + 81)$, which can also be expressed as follows: $2^{k_1}(2^{k_5+k_4+k_3+k_2} - 2^{k_4+k_3+k_2} - 3 * 2^{k_3+k_2} - 9 * 2^{k_2} - 27) \leq 324$. Consequently, the values of k_1, k_2, k_3, k_4 and k_5 should be limited as follows: $1 \leq k_1 \leq 8$ and $1 \leq k_2 \leq (9 - k_1)$ and $1 \leq k_3 \leq (10 - k_1 - k_2)$ and $1 \leq k_4 \leq (11 - k_1 - k_2 - k_3)$ and $1 \leq k_5 \leq (12 - k_1 - k_2 - k_3 - k_4)$.

Therefore, it is impossible to have a Collatz loop consisting of five different odd numbers $\{n_1, n_2, n_3, n_4, n_5\}$, where n_1 is outside the range $[\![1;\ 86]\!]$, because the maximum value for n_1 is $\frac{1121}{13}$ where $k_1 = 4$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$ and $k_5 = 1$.

The maximum value of n_1 is calculated by identifying the minimum positive value of the denominator $\left[2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right]$, then identifying the maximum positive value of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]$.

As a result, the expression (Equation 84) and the presented statements by the fourteenth theorem in this paper are correct, because there is no subgroup of five different odd numbers $\{n_1, n_2, n_3, n_4, n_5\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)\ and\ (3n_2 + 1 = 2^{k_2}n_3)\ and\ (3n_3 + 1 = 2^{k_3}n_4)\ and\ (3n_4 + 1 = 2^{k_4}n_5)\ and\ (3n_5 + 1 = 2^{k_5}n_1)\ and\ \left(k_i \in \{\mathbb{N} - \{0\}\}\right)$.

5.6. Proposed Theorem on Collatz Loops Consisted of Six Elements

This subsection presents the fifteenth theorem, which treats possible Collatz loops consisting of six odd numbers according to the proposed formulas in Theorem 3 and Theorem 4 in this paper.

Theorem 15

In the group of Natural numbers $\mathbb{N}$, there is no subgroup of six different odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)\ and\ (3n_2 + 1 = 2^{k_2}n_3)\ and\ (3n_3 + 1 = 2^{k_3}n_4)\ and\ (3n_4 + 1 = 2^{k_4}n_5)\ and\ (3n_5 + 1 = 2^{k_5}n_6)\ and\ (3n_6 + 1 = 2^{k_6}n_1)\ and\ \left(k_i \in \{\mathbb{N} - \{0\}\}\right)$ as shown in (Equation 86), because the maximum value for n_1 will be $\left(\max(n_1) = \frac{6995}{295}\right)$.

$$\forall\ G \in \mathbb{N}\ ;\ \nexists\{n_1, n_2, n_3, n_4, n_5, n_6\} \in G \mid n_i \neq n_{j(j\neq i)}\ and\ 3n_1 + 1 = 2^{k_1}n_2\ and\ 3n_2 + 1 = 2^{k_2}n_3\ and\ 3n_3 + 1 = 2^{k_3}n_4\ and\ 3n_4 + 1 = 2^{k_4}n_5\ and\ 3n_5 + 1 = 2^{k_5}n_6\ and\ 3n_6 + 1 = 2^{k_6}n_1\ and\ k_i \in \{\mathbb{N} - \{0\}\} \quad (86)$$

Proof of Theorem 15

As mentioned in the proof of Theorem 11, both formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4) are based on the same structure of terms, while rotating the order to calculate the value of consecutive odd numbers according to Theorem 4. Therefore, we will focus on analyzing the division of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]$ on the value of the denominator $\left[2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right]$, which generate values of real numbers.

As it was mentioned in the proof of Theorem 11, in order to have an integer number as a result of the shown formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4), the value of denominator should be less than (or equal) to the numerator, and also should be dividing it into an integer value.

In the case of having a loop of Collatz consisting of six different odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6\}$, the value of the first number n_1 of this sequence is calculated as shown in (Equation 87), whereas having $\left(k_i \in \{\mathbb{N} - \{0\}\}\right)$.

$$n_1 = \frac{(2^{k_5+k_4+k_3+k_2+k_1} + 3 * 2^{k_4+k_3+k_2+k_1} + 9 * 2^{k_3+k_2+k_1} + 27 * 2^{k_2+k_1} + 81 * 2^{k_1} + 243)}{2^{k_1+k_2+k_3+k_4+k_5+k_6} - 729} \quad (87)$$

Therefore, in order to have an integer value as a result of this expression of n_1 (Equation 87), we need to start by having the condition $(2^{k_1+k_2+k_3+k_4+k_5+k_6} - 729) \leq (2^{k_5+k_4+k_3+k_2+k_1} + 3 * 2^{k_4+k_3+k_2+k_1} + 9 * 2^{k_3+k_2+k_1} + 27 * 2^{k_2+k_1} + 81 * 2^{k_1} + 243)$, which can also be expressed as follows: $2^{k_1}(2^{k_6+k_5+k_4+k_3+k_2} - 2^{k_5+k_4+k_3+k_2} - 3 * 2^{k_4+k_3+k_2} - 9 * 2^{k_3+k_2} - 27 * 2^{k_2} - 81) \leq 972$.

Consequently, the values of k_1, k_2, k_3, k_4, k_5 and k_6 should be limited as follows: $1 \le k_1 \le 9$ and $1 \le k_2 \le (10 - k_1)$ and $1 \le k_3 \le (11 - k_1 - k_2)$ and $1 \le k_4 \le (12 - k_1 - k_2 - k_3)$ and $1 \le k_5 \le (13 - k_1 - k_2 - k_3 - k_4)$ and $1 \le k_6 \le (14 - k_1 - k_2 - k_3 - k_4 - k_5)$.

Therefore, it is impossible to have a Collatz loop consisting of six different odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6\}$, where $n_1$is outside the range ⟦1; 23⟧, because the maximum value for n_1 is $\frac{6995}{295}$ where $(k_1 = 5)$, $(k_2 = 1)$, $(k_3 = 1)$, $(k_4 = 1)$, $(k_5 = 1)$ and $(k_6 = 1)$.

As a result, the expression (Equation 86) and the presented statements by the fifteenth theorem in this paper are correct, because there is no subgroup of six different odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)$ *and* $(3n_2 + 1 = 2^{k_2}n_3)$ *and* $(3n_3 + 1 = 2^{k_3}n_4)$ *and* $(3n_4 + 1 = 2^{k_4}n_5)$ *and* $(3n_5 + 1 = 2^{k_5}n_6)$ *and* $(3n_6 + 1 = 2^{k_6}n_1)$ *and* $\left(k_i \in \{\mathbb{N} - \{0\}\}\right)$.

5.7. Proposed Theorem on Collatz Loops Consisted of Seven Elements

This subsection presents the sixteenth theorem, which treats possible Collatz loops consisting of seven odd numbers according to the proposed formulas in Theorem 3 and Theorem 4 in this paper.

Theorem 16

In the group of Natural numbers $\mathbb{N}$, there is no subgroup of seven different odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)$ *and* $(3n_2 + 1 = 2^{k_2}n_3)$ *and* $(3n_3 + 1 = 2^{k_3}n_4)$ *and* $(3n_4 + 1 = 2^{k_4}n_5)$ *and* $(3n_5 + 1 = 2^{k_5}n_6)$ *and* $(3n_6 + 1 = 2^{k_6}n_7)$ *and* $(3n_7 + 1 = 2^{k_7}n_1)$ *and* $\left(k_i \in \{\mathbb{N} - \{0\}\}\right)$ as shown in (Equation 88), because the maximum value for n_1 will be $\left(\max(n_1) = \frac{43289}{1909}\right)$.

$$\forall\, G \in \mathbb{N}\ ;\ \nexists\{n_1, n_2, n_3, n_4, n_5, n_6, n_7\} \in G \mid n_i \neq n_{j(j \neq i)}\ and\ 3n_1 + 1 = 2^{k_1}n_2\ and\ 3n_2 + 1 = 2^{k_2}n_3\ and\ 3n_3 + 1 = 2^{k_3}n_4\ and\ 3n_4 + 1 = 2^{k_4}n_5\ and\ 3n_5 + 1 = 2^{k_5}n_6\ and\ 3n_6 + 1 = 2^{k_6}n_7\ and\ 3n_7 + 1 = 2^{k_7}n_1\ and\ k_i \in \{\mathbb{N} - \{0\}\} \quad \textbf{(88)}$$

Proof of Theorem 16

As mentioned in the proof of Theorem 11, both formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4) are based on the same structure of terms, while rotating the order to calculate the values of consecutive odd numbers according to Theorem 4. Therefore, we will focus on analyzing the division of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]$ on the value of the denominator $\left[2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right]$, which generate values of real numbers.

As it was mentioned in the proof of Theorem 11, in order to have an integer number as a result of shown formulas in (Equation. 9) and (Equation. 29) (Theorem 3 and Theorem 4), the value of denominator should be inferior (or equal) to the numerator, and also should be dividing it into an integer value.

In the case of having a loop of Collatz consisting of seven different odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7\}$, the value of the first number n_1 of this sequence is calculated as shown in (Equation 89), whereas having $\left(k_i \in \{\mathbb{N} - \{0\}\}\right)$.

$$n_1 = \frac{\left(2^{k_6+k_5+k_4+k_3+k_2+k_1} + 3*2^{k_5+k_4+k_3+k_2+k_1} + 9*2^{k_4+k_3+k_2+k_1} + 27*2^{k_3+k_2+k_1} + 81*2^{k_2+k_1} + 243*2^{k_1} + 729\right)}{2^{k_1+k_2+k_3+k_4+k_5+k_6+k_7} - 2187} \quad (89)$$

Therefore, in order to have an integer value as a result of this expression of n_1 (Equation 89), we need to start by having the condition $(2^{k_1+k_2+k_3+k_4+k_5+k_6+k_7} - 2187) \le (2^{k_6+k_5+k_4+k_3+k_2+k_1} + 3 * 2^{k_5+k_4+k_3+k_2+k_1} + 9 * 2^{k_4+k_3+k_2+k_1} + 27 * 2^{k_3+k_2+k_1} + 81 * 2^{k_2+k_1} + 243 * 2^{k_1} + 729)$, which can also be expressed as follows: $2^{k_1}(2^{k_7+k_6+k_5+k_4+k_3+k_2} - 2^{k_6+k_5+k_4+k_3+k_2} - 3 * 2^{k_5+k_4+k_3+k_2} - 9 * 2^{k_4+k_3+k_2} - 27 * 2^{k_3+k_2} - 81 * 2^{k_2} - 243) \le 2916$. Consequently, the values of $k_1, k_2, k_3, k_4, k_5, k_6$ and k_7 should be limited as follows: $1 \le k_1 \le 11$ and $1 \le k_2 \le (12 - k_1)$ and $1 \le k_3 \le (13 - k_1 - k_2)$ and $1 \le k_4 \le (14 - k_1 - k_2 - k_3)$ and $1 \le k_5 \le (15 - k_1 - k_2 - k_3 - k_4)$ and $1 \le k_6 \le (16 - k_1 - k_2 - k_3 - k_4 - k_5)$ and $1 \le k_7 \le (17 - k_1 - k_2 - k_3 - k_4 - k_5 - k_6)$.

Therefore, it is impossible to have a Collatz loop consisting of seven different odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7\}$, where n_1 is outside the range $⟦1;\ 23⟧$, because the maximum value for n_1 is $\frac{43289}{1909}$ where $(k_1 = 6)$, $(k_2 = 1)$, $(k_3 = 1)$, $(k_4 = 1)$, $(k_5 = 1)$, $(k_6 = 1)$ and $(k_7 = 1)$.

As a result, the expression (Equation 88) and the presented statements by the sixteenth theorem in this paper are correct, because there is no subgroup of seven different odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)\ and\ (3n_2 + 1 = 2^{k_2}n_3)\ and\ (3n_3 + 1 = 2^{k_3}n_4)\ and\ (3n_4 + 1 = 2^{k_4}n_5)\ and\ (3n_5 + 1 = 2^{k_5}n_6)\ and\ (3n_6 + 1 = 2^{k_6}n_7)\ and\ (3n_7 + 1 = 2^{k_7}n_1)\ and\ \left(k_i \in \{\mathbb{N} - \{0\}\}\right)$.

5.8. Proposed Theorem on Limited Collatz Loops

This subsection presents the seventeenth theorem, which treats possible Collatz loops consisting of "L" odd numbers ($L \in ⟦2;\ 64⟧$) according to the proposed formulas in Theorem 3 and Theorem 4 in this paper.

By having "L" odd numbers in a possible Collatz loop, the range of these odd numbers can have a theoretical value that may exceed $\left(\frac{3}{2}\right)^L$.

Theorem 17

In the group of Natural numbers $\mathbb{N}$, there is no subgroup of "L" ($L \in ⟦2;\ 64⟧$) different odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7, \dots, n_{L-1}, n_L\}$ that can create a Collatz loop where $(3n_1 + 1 = 2^{k_1}n_2)\ and\ (3n_2 + 1 = 2^{k_2}n_3)\ and\ (3n_3 + 1 = 2^{k_3}n_4)\ and\ (3n_4 + 1 = 2^{k_4}n_5)\ and\ (3n_5 + 1 = 2^{k_5}n_6)\ and\ (3n_6 + 1 = 2^{k_6}n_7)\ and\ (3n_7 + 1 = 2^{k_7}n_8)\ and\ \dots and\ (3n_{L-1} + 1 = 2^{k_{L-1}}n_L)\ and\ (3n_L + 1 = 2^{k_L}n_1)\ \ and\ \left(k_i \in \{\mathbb{N} - \{0\}\}\right)$, as shown in (Equation 90).

$$(\forall\ G \in \mathbb{N}\ )\ and\ (\forall\ L \in ⟦2;\ 64⟧\ );\ \nexists\{n_1, n_2, n_3, n_4, n_5, n_6, n_7, \dots, n_{L-1}, n_L\} \in G\ |\ \forall\ i \in ⟦1, L-1⟧;\ n_i \neq n_{j(j\neq i)}\ and\ 3n_i + 1 = 2^{k_i}n_{i+1}\ and\ 3n_L + 1 = 2^{k_L}n_1\ and\ k_i \in \{\mathbb{N} - \{0\}\} \quad (90)$$

Proof of Theorem 17

As mentioned in the proof of Theorem 11, both formulas in (Equation 9) and (Equation 29) (Theorem 3 and Theorem 4) are based on the same structure of terms, while rotating the order to calculate the values of consecutive odd numbers. Therefore, we will focus on analyzing the division of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k(j)}\right]$ on the value of the denominator $\left[2^{\sum_{l=0}^{l=m} k(l)} - 3^m\right]$, which generate values of real numbers.

As it was mentioned in the proof of Theorem 11, in order to have an integer number as a result of shown formulas in (Equation. 9) and (Equation. 29) (Theorem 3 and Theorem 4), the value of denominator should be inferior (or equal) to the value of numerator, and also should be dividing it into an integer value.

As long as we forward calculations toward identifying a finite (or infinite) Collatz loop consisting of "m" different odd numbers. $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7, \dots, n_{m-1}, n_m\}$, we will always need to have a value for the denominator $\left[2^{\sum_{l=0}^{l=m} k(l)} - 3^m\right]$ less than the value of the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k(j)}\right]$.

As a result, we can identify the shown condition in (Equation 91).

$$\left[2^{\sum_{l=0}^{l=m} k(l)} - 3^m\right] \leq \sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k(j)}\right]$$

$$\Rightarrow 2^{\sum_{l=0}^{l=m} k(l)} - \sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k(j)}\right] \leq 3^m$$

$$\Rightarrow 2^{\sum_{l=0}^{l=m} k(l)} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k(j)}\right] \leq 3^m + 3^{m-1}$$

$$\Rightarrow 2^{k_1}\left[2^{\sum_{l=2}^{l=m}k_{(l)}} - 3^{m-2} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=2}^{j=m-i-1}k_{(j)}}\right]\right] \leq 3^m\left[\frac{4}{3}\right] \quad (91)$$

Considering that every coefficient $k_{(i\neq 0)}$ is a natural number different from zero $\left(k_i \in \{\mathbb{N} - \{0\}\}\right)$ and $(k_0 = 0)$; the minimum value for k_i is one when $(1 \leq i \leq m)$, whereas the expected Collatz loop will consist of "m" elements. Therefore, the minimum value of $\left[\sum_{i=2}^{i=m} k_{(i)}\right]$ is $(m-1)$.

The maximum value of the coefficient k_1 is also identified by having the minimum positive value for the denominator $\left[2^{\sum_{l=0}^{l=m}k_{(l)}} - 3^m\right]$ where $\left(3^m \leq 2^{k_1} * 2^{\sum_{l=2}^{l=m}k_{(l)}}\right)$, which gives the shown result in (Equation 92).

$$\Rightarrow 2 \leq 2^{k_1} \leq ceil\left(\frac{3^m}{2^{m-1}}\right)\ and\ 2^{m-1} \leq 2^{\sum_{l=2}^{l=m}k_{(l)}} \leq ceil\left(\frac{3^m}{2^{k_1}}\right)$$

$$\Rightarrow 1 \leq k_1 \leq ceil\left(m * \frac{\ln(3)}{\ln(2)} - (m-1)\right)\ and\ (m-1) \leq \sum_{l=2}^{l=m} k_{(l)} \leq ceil\left(m * \frac{\ln(3)}{\ln(2)} - (k_1)\right)$$

$$\Rightarrow k_1 \in \left[\!\left[1; ceil\left(m * \frac{ln(3)}{ln(2)} - m + 1\right)\right]\!\right]\ and\ \sum_{l=2}^{l=m} k_{(l)} \in \left[\!\left[(m-1); ceil\left(m * \frac{ln(3)}{ln(2)} - k_1\right)\right]\!\right] \quad (92)$$

Since we reached the result $\left(k_1 \in \left[\!\left[1; ceil\left(m * \frac{\ln(3)}{\ln(2)} - m + 1\right)\right]\!\right]\right)$ and $\left(\sum_{l=2}^{l=m} k_{(l)} \in \left[\!\left[(m-1); ceil\left(m * \frac{\ln(3)}{\ln(2)} - k_1\right)\right]\!\right]\right)$, we deduce the minimum and maximum values of k_1 shown in (Equation 93), whereas we deduce the minimum and maximum values of $\sum_{l=1}^{l=m-1} k_{(l+1)}$ shown in (Equation 94).

$$MIN(k_1) = 1;\ MAX(k_1) = ceil\left(m * \frac{ln(3)}{ln(2)} - m + 1\right) \quad (93)$$

$$MIN\left(\sum_{l=1}^{l=m-1} k_{(l+1)}\right) = (m-1);\ MAX\left(\sum_{l=1}^{l=m-1} k_{(l+1)}\right) = ceil\left(m * \frac{ln(3)}{ln(2)}\right) \quad (94)$$

We deduce that when the value of "m" goes to infinity, the value of $k_1$is limited by having a maximum value that cannot exceed 60% of (m), because of the shown result in (Equation 95).

$$MAX(k_1) = ceil\left(m * \frac{ln(3)}{ln(2)} - m + 1\right) \approx ceil(m * 0{,}5849625 + 1) \quad (95)$$

We also deduce that the value of $\sum_{l=1}^{l=m-1} k_{(l+1)}$ is limited by having a maximum value that cannot exceed 159% of (m) when the value of "m" goes to infinity, because of the shown result in (Equation 96).

$$MAX\left(\sum_{l=1}^{l=m-1} k_{(l+1)}\right) = \left[ceil\left(m\frac{ln(3)}{ln(2)}\right)\right] \approx ceil(m * 1{,}5849625) \quad (96)$$

Now, we express the formula to calculate the maximum value of n_1 which is obtained when k_1 is taking its MAX value shown in (Equation 95). Then, we use Python (Figure 4) to provide computational results highlighting the evolution of the maximum value of n_1.

The maximum value of n_1 It is mainly defined by identifying the maximum value for the numerator $\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1}k_{(j)}}\right]$, after determining the first positive value for the denominator $\left(2^{\sum_{l=0}^{l=m}k_{(l)}} - 3^m\right)$.

.

Codes in Python to calculate the maximum and minimum values of n1 in possible Collatz loops

Code part 1

```
import numpy
import matplotlib.pyplot as plt
from mpmath import *

mp.dps = 10; mp.pretty = True
def n1_function(m):
    m=int(m)
    max_k1=m*(numpy.log(3)-
numpy.log(2))/numpy.log(2)+1
    max_k1=numpy.ceil(max_k1)
    numerator=3**(m-1)
    for i in range(1,m):
        numerator+=(pow(3,(m-1-
i)))*(pow(2,max_k1+i-1))
    denominator=(pow(2,max_k1+m-1))-(pow(3,m))
    max_n1=numerator/denominator
    min_n1=(pow(3,m-1)-pow(2,m-1))/denominator
    return [max_n1,min_n1]
m=input("Enter the value of m: ")
m=int(m)
x_array=[]
y_max_array=[]
y_min_array=[]
for j in range(2,m+1):
    x_array.append(j)
    y=n1_function(j)
    print("y is: ",y)
    y_max_array.append(y[0])
    y_min_array.append(y[1])
print("array y max is: ",y_max_array)
print("array y min is: ",y_min_array)
fig, ax1 = plt.subplots(figsize=(8, 8))
```

Code part 2

```
ax2 = ax1.twinx()
ax1.set_ylim(0,max(y_max_array))
ax2.set_ylim(0,max(y_max_array))
ax1.plot(x_array, y_max_array, marker='.',
linestyle='dotted', linewidth=3,
markersize=10)
ax2.plot(x_array, y_min_array, marker='x',
linestyle='dotted', color="#FF0000",
linewidth=3, markersize=10)

font2 =
{'family':'serif','color':'blue','size':16}
font3 =
{'family':'serif','color':'red','size':16}
plt.rcParams.update({'font.size': 16})

ax1.set_xlabel("The amount of M elements in a
potential Collatz loop", fontdict=font2)
ax1.set_ylabel("The maximum value of n1",
fontdict=font2)
ax1.tick_params(axis="y",
labelcolor="#0000FF", size=14)
plt.xticks(fontsize=14)
ax2.set_ylabel("The minimum value of n1",
fontdict=font3)
ax2.tick_params(axis="y",
labelcolor="#FF0000", size=14)
plt.yticks(fontsize=14)
fig.suptitle("(Graph D) \n The maximum and
minimum values of n1 in function of M",
color="black", family="serif", size=16)
plt.show()
```

Fig. 4 Programmed codes in python to generate graphical illustrations of the maximum and minimum values of n_1.

When the coefficient k_1 takes its maximum value, the value of $\sum_{l=1}^{l=m-1} k_{(l+1)}$ will be at its minimum, which is $(m-1)$.

As a result, the maximum value of n_1 (Equation 97) is calculated by giving each coefficient k_i a value equal to "1" when $(2 \le i \le m)$, whereas k_1 is taking the maximum value of $MAX(k_1)$.

$$MAX(n_1) = \frac{\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m} \text{ where } k_1 = ceil\left(m * \frac{ln(3)}{ln(2)} - m + 1\right) \text{ and } (\forall\, i \in [\![2, m]\!] \mid k_i = 1) \quad (97)$$

The next step is expressing the formula to calculate the minimum value of n_1 (Equation 98), which is obtained when k_m is taking its MAX value.

$$MIN(n_1) = \frac{\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m} \text{ where } k_m = ceil\left(m * \frac{ln(3)}{ln(2)} - m + 1\right) \text{ and } (\forall\, i \in [\![1, m-1]\!] \mid k_i = 1) \quad (98)$$

We can re-express the shown formula of $MAX(n_1)$ in (Equation 97) to be as presented in (Equation 99):

$$MAX(n_1) = \frac{3^{m-1} + \sum_{i=0}^{i=m-2}\left[3^i * 2^{(m-2-i+K_1)}\right]}{2^{(m-1+K_1)} - 3^m} \text{ where } K_1 = ceil\left(m * \frac{ln(3)}{ln(2)} - m + 1\right) \quad (99)$$

We can re-express the shown formula of $MIN(n_1)$ in (Equation 98) to be as presented in (Equation 100):

$$MIN(n_1) = \frac{3^{m-1} + \sum_{i=0}^{i=m-2}\left[3^i * 2^{(m-1-i)}\right]}{2^{(m-1+K_m)} - 3^m} = \frac{\sum_{i=0}^{i=m-1}\left[3^i * 2^{(m-1-i)}\right]}{2^{(m-1+K_m)} - 3^m} \text{ where } K_m = ceil\left(m * \frac{ln(3)}{ln(2)} - m + 1\right) \quad (100)$$

We used the Python programming language to program the shown code in Figure 4, in order to compute the values of $MAX(n_1)$ and $MIN(n_1)$ when "m" goes from 2 to 64, whereas "m" is the number of elements in prospect Collatz loops.

When we vary the value of "m" in the range $⟦2;\ 64⟧$, the maximum value $MAX(n_1)$ among all values illustrated in Figure 5, is the value (851995363565), which is obtained when $m = 63$, whereas all the numbers in the range ($⟦1;\ 18446744073709551616⟧ = ⟦1; 2^{64}⟧$) are already verified by computation that they converge toward the value "1" when we keep repeating the operations of the Collatz conjecture on them [10, 32]. Therefore, there is no Collatz loop consisting of "m" elements where "m" is in the range $⟦2;\ 64⟧$. As a result, the expression (Equation 90) and the presented statements in Theorem 17 are correct, which means there is no Collatz loop of "m" elements that can contradict the Collatz conjecture, where "m" is in the range $⟦2;\ 64⟧$.

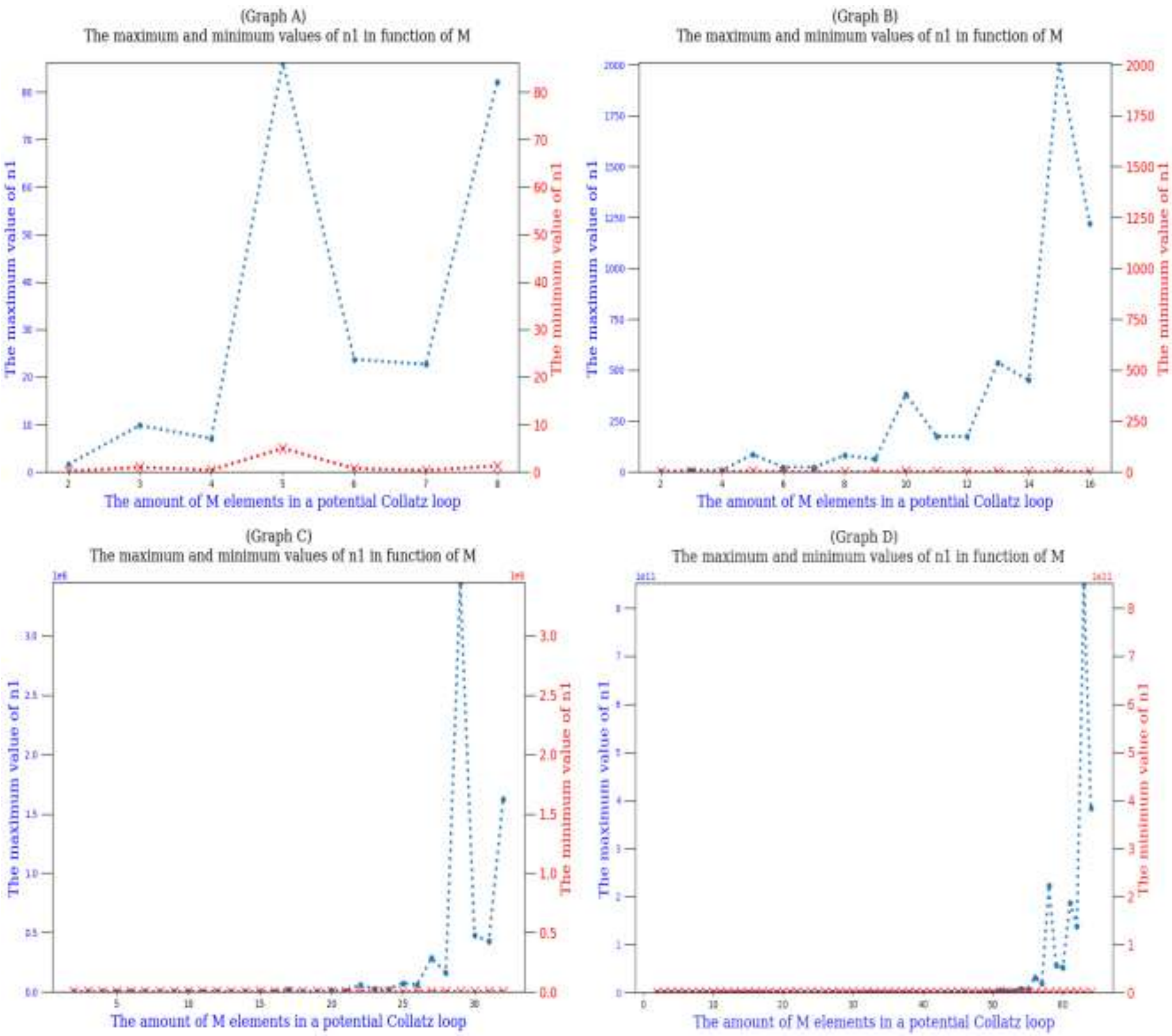

Fig. 5 Computed graphs showing the evolution of maximum and minimum values of n_1 in function of the amount M of odd numbers in potential Collatz loops.

5.9. Proposed Theorem on General Collatz Loops in the Group N

This subsection presents the eighteenth theorem in this paper, which treats possible Collatz loops consisting of L odd numbers ($L \in \{\mathbb{N}\}$) according to the proposed formulas in Theorems 3 and 4 in this paper.

Theorem 18
In the group of Natural Numbers $\{\mathbb{N}\}$, there is no subgroup of L $(L \in \{\mathbb{N}-\{1\}\})$ odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7, \dots, n_{L-1}, n_L\}$ that can create a Collatz loop where $3n_1 + 1 = 2^{k_1}n_2$ *and* $3n_2 + 1 = 2^{k_2}n_3$ *and* $3n_3 + 1 = 2^{k_3}n_4$ *and* $3n_4 + 1 = 2^{k_4}n_5$ *and* $3n_5 + 1 = 2^{k_5}n_6$ *and* $3n_6 + 1 = 2^{k_6}n_7$ *and* $3n_7 + 1 = 2^{k_7}n_8$ *and* ... *and* $3n_{L-1} + 1 = 2^{k_{L-1}}n_L$ *and* $3n_L + 1 = 2^{k_L}n_1$.

$$(\forall\, G \in \mathbb{N})\ and\ (\forall\, L \in \{\mathbb{N}-\{0,1\}\});\ \nexists\{n_1, n_2, n_3, n_4, n_5, n_6, n_7, \dots, n_{L-1}, n_L\} \in G \mid\ \forall\, i \in [\![1, L-1]\!]\ 3n_i + 1 = 2^{k_i}n_{i+1}\ and\ 3n_L + 1 = 2^{k_L}n_1 \qquad (101)$$

Proof of Theorem 18
In the proof of Theorem 17, we demonstrated that in the group of Natural Numbers $\{\mathbb{N}\}$, there is no subgroup of L $(L \in [\![2, 64]\!])$ odd numbers $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7, \dots, n_{L-1}, n_L\}$ that can create a Collatz loop. For that, we relied on calculating the maximum value and the minimum value of n_1 as shown in (Equation 99) and (Equation 100).

In Theorem 3 and Theorem 4, we proved that if an odd number n_1 is included in a possible Collatz loop $\{n_1, n_2, n_3, n_4, n_5, n_6, n_7, \dots, n_{m-1}, n_m\}$ expressed as shown in (Equation 102), then the value of this odd number n_1 can be calculated by using (Equation 103). Therefore, we can re-express the odd number n_1 to be as presented in (Equation 104).

$$3n_1 + 1 = 2^{k_1}n_2;\ 3n_2 + 1 = 2^{k_2}n_3;\ 3n_3 + 1 = 2^{k_3}n_4; \dots;\ 3n_{m-1} + 1 = 2^{k_{m-1}}n_m;\ 3n_m + 1 = 2^{k_m}n_1 \qquad (102)$$

$$n_1 = \frac{\sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right]}{2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m} \mid k_0 = 0\ and\ k_{(i\geq 1)} \in \{\mathbb{N}-\{0\}\} \qquad (103)$$

$$\Rightarrow n_1\left(2^{\sum_{l=0}^{l=m} k_{(l)}} - 3^m\right) = \sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} k_{(j)}}\right] \mid k_0 = 0\ and\ k_{(i\geq 1)} \in \{\mathbb{N}-\{0\}\}$$

$$\Rightarrow \boldsymbol{n_1} = \frac{\boldsymbol{n_1} * 2^{\sum_{l=0}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} k_{(j)}}\right]}{3^{m-1}} \mid k_0 = 0\ and\ k_{(i\geq 1)} \in \{\mathbb{N}-\{0\}\} \qquad (104)$$

Proving that there is no Collatz loop consisting of "m" elements, where $(m > 2)$, is based on demonstrating that the value of n_1 on the right-hand side of the shown equation in (Equation 104), it is always equal to "1", and it is different from the value of n_1 shown on the left-hand side of the same equation.

In Theorem 20, we prove that if an odd number n_s is eventually converging toward "1" when conducting the operations of the Collatz conjecture on it, then the odd number $(2n_s + 1)$ is also converging toward "1" when we keep repeating the Collatz operation, whereas starting these operations on $(2n_s + 1)$.

In Theorem 21, we prove that if an odd number n_s is eventually converging toward "1" when conducting the operations of the Collatz conjecture on it, then the odd number $(2n_s + 3)$ is also converging toward "1" when we keep repeating the Collatz operation, whereas starting these operations on $(2n_s + 3)$.

In Theorem 20 and Theorem 21, the odd numbers $\{n_s; 2n_s + 1; 2n_s + 3\}$ are proven to converge toward the number "1" when repeating Collatz operations by proving that each one among them is expressed according to the shown formula in (Equation 105).

$$n = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0\ and\ K_{(i\geq 1)} \in \{\mathbb{N}-\{0\}\} \qquad (105)$$

In Theorem 25, all odd numbers are proven to be expressed according to the shown formula in (Equation 105) by scaling Theorem 20 and Theorem 21. Therefore, based on Theorem 8, every odd number "n" in the group of natural numbers $\mathbb{N}$ converges to the number "1" by repeating the conduction of Collatz operations, whereas starting these operations on "n", because these odd numbers are expressed as shown in (Equation 105). As a result, the expression (Equation 101) and the presented statements in Theorem 18 are correct, because there is no Collatz loop consisting of "m" elements (odd numbers) where $m \in \{\mathbb{N} - \{0\}\}$ except the Collatz loop, where $(m = 1)$.

6. New Theorems Proving the Collatz Conjecture by Treating Convergence

This section provides foundation theorems proving that when starting Collatz operations on any odd number, calculations do not diverge and they actually converge to values inferior to the starting number. Then, this section presents more general theorems built on the foundations, which allow us to prove that when repeating Collatz operations, starting them on any natural number different from zero, calculations eventually lead to the number "1".

6.1. Proposed Theorem for One-Step Convergence of Collatz Operations

This subsection presents a theorem identifying when a one-step operation of the Collatz conjecture on an odd number n_1 will actually converge to a value n_2 inferior than n_1. The presented results in this subsection are based on the statements and formulas of Theorem 1 in this paper, which re-express the operations of the Collatz conjecture.

Theorem 19

Supposing two odd numbers n_1 and n_2 connected according to the operations of the Collatz conjecture, which we can express as follows: $n_2 = \frac{3n_1+1}{2^{k_1}}$. If we have k_1 superior or equal "2" *(*$k_1 \geq 2$*)* and we have n_1 absolutely superior to "1" *(*$n_1 > 1$); then we will have $(n_2 < n_1)$ as shown in (Equation 106).

$$\forall \{n_1, n_2\} \in \{\mathbb{N} - \{0\}\}\ where\ \{n_1, n_2\}\ are\ odd\ numbers\ and\ n_2 = \frac{3n_1 + 1}{2^{k_1}}\ and\ n_1 > 1; If\ \ k_1 \geq 2\ is\ true \Rightarrow n_2 < n_1 \tag{106}$$

Proof of Theorem 19

When having a group of two consecutive odd numbers $\{n_1, n_2\}$ connected according to the operations of the Collatz conjecture, we can express these two odd numbers by using the first proposed formula (Equation 1) in Theorem 1 of this paper as follows: $n_2 = \frac{3n_1+1}{2^{k_1}}$which we demonstrated in the proof of Theorem 1.

By having $n_2 = \frac{3n_1+1}{2^{k_1}}$ and supposing the condition $n_1 > 1$ is true; we have the following logic that leads to the shown result in (Equation 107):

$$If\ \ k_1 \geq 2\ is\ true\ \Rightarrow n_2 = \frac{3n_1 + 1}{2^{k_1}} \leq \frac{3n_1 + 1}{2^2}$$

$$\Rightarrow n_2 = \frac{3n_1 + 1}{2^{k_1}} \leq \frac{3n_1 + 1}{4}$$

$$\Rightarrow 4n_2 = 4\frac{3n_1 + 1}{2^{k_1}} \leq 3n_1 + 1$$

$$\Rightarrow 4n_2 - 4n_1 \leq -n_1 + 1$$

$$\Rightarrow n_2 - n_1 < 0\ (since\ we\ supposed\ n_1 > 1)$$

$$\Rightarrow n_2 < n_1\ (since\ we\ supposed\ n_1 > 1) \tag{107}$$

As a result, (Equation 106) and the proposed statements in Theorem 19 are correct.

6.2. First Proposed Theorem for Multi-Step Convergence of Collatz Operations

This subsection presents a theorem proving that if we have a group of odd numbers L_1 ($L_1 \in \mathbb{N}$), where every odd number (n_s) in this group is converging to inferior values than itself when repeating the operations of the Collatz conjecture, while starting these operations on n_s, then, we will have other odd numbers expressed as $(2n_s + 1)$ also converging to inferior values than themselves when repeating the operations of the Collatz conjecture, whereas starting on them.

Theorem 20

Supposing a group L_1 containing odd numbers, where the minimum odd number in L_1 is $\{n_1 = 1\}$ and the maximum odd number is $\{n_h = 2(h-1)+1\}$; as shown in (Equation 108). If every odd number n_s from this group (L_1) is converging to a value ($P_1 = 1$) inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then every odd number ($n_{s_1} = 2n_s + 1$) is also converging to an inferior value ($P_1 = 1$) (where $P_1 \leq n_h$) when repeating the operations of the Collatz conjecture, whereas starting on n_{s_1}.

$$L_1 = \{n_1 = 1; n_2 = 3;\ n_3 = 5;\ n_4 = 7; \dots; n_{h-1} = \ 2(h-2)+1;\ n_h = 2(h-1)+1\} \qquad (108)$$

Proof of Theorem 20

To prove Theorem 20, we start by considering n_s from the group L_1 where n_s is converging to the number "1" when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s. Therefore, we will adopt the group L' expressing the odd numbers generated when starting Collatz operations on the odd numbers n_s, which is to be expressed as shown in (Equation 109).

$$L' = \left\{n'_1 = n_s\,; n'_2 = \frac{3n'_1+1}{2^{K_1}};\ n'_3 = \frac{3n'_2+1}{2^{K_2}};\ n'_4 = \frac{3n'_3+1}{2^{K_3}}; \dots; n'_{m-1};\ n'_m = \frac{3n'_{m-1}+1}{2^{K_{m-1}}} = P = 1\right\} \qquad (109)$$

The next step is relying on Theorem 6 to express n_s by using the presented formula in (Equation 110), which we already demonstrated in the proof of Theorem six.

$$n_s \ = n'_1 = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \ \mid K_0 = 0 \ and \ K_{i\geq 1} \in \{\mathbb{N} - \{0\}\} \qquad (110)$$

Now, we consider the odd number $(n_{s_1} = 2n_s + 1)$, which we need to prove that it is expressed according to the proposed formula (Equation 59) in Theorem 8. Therefore, we will prove that $(n'_{s_1} = n_{s_1} - 4)$ is expressed according to the proposed formula (Equation 72) in Theorem 9; because Theorem 9 is forwarding Theorem 8.

We calculate the value of $(n'_{s_1} = n_{s_1} - 4 = 2n_s - 3)$ whereas replacing n_s with its shown expression in (Equation 110), which allows us to obtain (Equation 111).

$$n'_{s_1} = 2n_s - 3 \ \Rightarrow n'_{s_1} = 2\left(\frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}}\right) - 3$$

$$\Rightarrow n'_{s_1} = \left(\frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}}\right) - 3$$

$$\Rightarrow n'_{s_1} = \frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 3^m}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 9 * 3^{m-2}}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 2 * 3^{m-2} - 9 * 3^{m-2}}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 11 * 3^{m-2}}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2*2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2*\sum_{i=0}^{i=m-3}\left[3^i*2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 8*3^{m-2} - 3*3^{m-2}}{3^{m-1}} \quad (111)$$

The next step is adapting the shown expression in (Equation 111) by replacing the number "8" with $\left(2^{K'_{(1)}} = 2^3\right)$ whereas conducting the following re-expressions $\{K'_{(1)} = 3;\ K_{(j=1)} = K'_{(j+1)} + 2;\ \ K_{(2\le j\le m-1)} = K'_{(j+1)}\ and\ K'_{(0)} = 0\}$, which gives us the shown result in (Equation 112). The value $\left(K_{(j=1)} = K'_{(j+1)} + 2\right)$ is leading to the result $(K'_{(2)} \in [\![-1, +\infty[\![\,)$, because $\left(K_{(1)} \in \{\mathbb{N} - \{0\}\}\right)$.

$$\Rightarrow n'_{s_1} = \frac{2^{K'_{(1)}-2} * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2^{K'_{(1)}-2} * \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 2^{K'_{(1)}} * 3^{m-2} - 3^{m-1}}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2^{K'_1-2} * 2^{2+\sum_{l=1}^{l=m-1} K'_{(l+1)}} - 2^{K'_1-2} * \sum_{i=0}^{i=m-3}\left[3^i * 2^{2+\sum_{j=1}^{j=m-i-2} K'_{(j+1)}}\right] - 2^{K'_1} * 3^{m-2} - 3^{m-1}}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2^{K'_{(1)}+\sum_{l=1}^{l=m-1} K'_{(l+1)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{K'_{(1)}+\sum_{j=1}^{j=m-i-2} K'_{(j+1)}}\right] - 2^{K'_{(1)}} * 3^{m-2} - 3^{m-1}}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2^{K'_{(1)}+\sum_{l=2}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{K'_{(1)}+\sum_{j=2}^{j=m-i-1} K'_{(j)}}\right] - 2^{K'_{(1)}} * 3^{m-2} - 3^{m-1}}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2^{\sum_{l=1}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=1}^{j=m-i-1} K'_{(j)}}\right] - 2^{K'_{(1)}} * 3^{m-2} - 3^{m-1}}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2^{\sum_{l=1}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i*2^{\sum_{j=1}^{j=m-i-1} K'_{(j)}}\right] - 3^{m-1}}{3^{m-1}} \quad (112)$$

Since we have $\left(K'_{(0)} = 0\right)$, we can re-express the shown value of n'_{s_1} in (Equation 112) to be presented as shown in (Equation 113):

$$\Rightarrow n'_{s_1} = \frac{2^{\sum_{l=0}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} K'_{(j)}}\right] - 3^{m-1}}{3^{m-1}}$$

$$\Rightarrow n'_{s_1} = \frac{2^{\sum_{l=0}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-1}\left[3^i*2^{\sum_{j=0}^{j=m-i-1} K'_{(j)}}\right]}{3^{m-1}} \quad (113)$$

Based on the shown expression in (Equation 113), we can present the value of n'_{s_1} as shown in (Equation. 114).

$$\Rightarrow n'_{s_1} = 2^R \frac{2^{\sum_{l=0}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-1}\left[3^i*2^{\sum_{j=0}^{j=m-i-1} K'_{(j)}}\right]}{3^{m-1}} \mid R = 0\ and\ K'_{(0)} = 0\ and\ K'_1 = 3\ and\ K'_{(2)} \in [\![-1, +\infty[\![\ and\ K'_{(i\notin\{0,2\})} \in \{\mathbb{N} - \{0\}\}\ and\ (R + K'_1) \ge 0\ and\ (R + K'_1 + K'_2) > 0 \quad (114)$$

We deduce that $\left(n'_{s_1} = n_{s_1} - 4\right)$ is expressed while respecting the presented statements and proposed formula (Equation 72) in Theorem 9. Therefore, based on the statements of Theorem 9, the odd number $\left(n_{s_1} = 2n_s + 1\right)$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_{s_1}, because it is expressed according to the proposed formula (Equation 59) in Theorem 8. As a result, the given statements in Theorem 20 are correct.

6.3. Second Proposed Theorem for Multi-Step Convergence of Collatz Operations

This subsection presents a theorem proving that if we have a group of odd numbers L_2 ($L_2 \in \mathbb{N}$), where every odd number (n_s) in this group is converging to inferior values than itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then, we will have other odd numbers expressed as $(2n_s + 3)$ also converging to inferior values than themselves when repeating the operations of the Collatz conjecture, whereas starting on them.

Theorem 21

Supposing a group L_2 containing odd numbers, where the minimum odd number in L_2 is $\{n_1 = 1\}$ and the maximum odd number is $\{n_h = 2(h-1)+1\}$; as shown in (Equation. 115). If every odd number n_s from this group (L_2) is eventually converging to a value ($P_2 = 1$) inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then every odd number ($n_{s_2} = 2n_s + 3$) is also converging to an inferior value ($P_2 = 1$) (where $P_2 \leq n_m$) when repeating the operations of the Collatz conjecture, whereas starting them on n_{s_2}.

$$L_2 = \{n_1 = 1; n_2 = 3;\ n_3 = 5; n_4 = \ 7; \dots;\ n_{h-1} = 2(h-2)+1; n_h = \ 2(h-1)+1\} \tag{115}$$

Proof of Theorem 21

To prove Theorem 21, we start by considering n_s from the group L_2 where n_s is converging to the number "1" when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s. Therefore, we will adopt the group L' expressing the odd numbers generated when starting Collatz operations on the odd numbers n_s, which is expressed as shown in (Equation. 116):

$$L' = \left\{n'_1 = n_s\,; n'_2 = \ \frac{3n'_1+1}{2^{k_1}}\ ;\ n'_3 = \ \frac{3n'_2+1}{2^{k_2}};\ n'_4 = \ \frac{3n'_3+1}{2^{k_3}} \dots; n'_m = \ P_2 = 1\right\} \tag{116}$$

The next step is relying on Theorem 6 to express n_s by using the presented formula in (Equation 117), which we already demonstrated in the proof of Theorem six.

$$n_s \ = n'_1 = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \ \mid K_0 = 0\ and\ K_{i\geq 1} \in \{\mathbb{N} - \{0\}\} \tag{117}$$

Now, we consider the odd number $\left(n_{s_2} = 2n_s + 3\right)$, which we need to prove that it is expressed according to the proposed formula (Equation 59) in Theorem 8. Therefore, we will prove that $\left(n'_{s_2} = n_{s_2} - 4\right)$ is expressed according to the proposed formula (Equation 72) in Theorem 9; because Theorem 9 is forwarding Theorem 8.

We calculate the value of $\left(n'_{s_2} = n_{s_2} - 4 = 2n_s - 1\right)$ while replacing n_s with its shown expression in (Equation 117), which allows us to obtain (Equation 118).

$$n'_{s_2} = 2n_s - 1 \ \Rightarrow n'_{s_2} = 2\left(\frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}}\right) - 1$$

$$\Rightarrow n'_{s_2} = \left(\frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}}\right) - 1$$

$$\Rightarrow n'_{s_2} = \frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 3^{m-1}}{3^{m-1}}$$

$$\Rightarrow n'_{s_2} = \frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 3 * 3^{m-2}}{3^{m-1}}$$

$$\Longrightarrow n'_{s_2} = \frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 2 * 3^{m-2} - 3 * 3^{m-2}}{3^{m-1}}$$

$$\Longrightarrow n'_{s_2} = \frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 5 * 3^{m-2}}{3^{m-1}}$$

$$\Longrightarrow n'_{s_2} = \frac{2 * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2 * \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 2 * 3^{m-2} - 3 * 3^{m-2}}{3^{m-1}} \quad (118)$$

The next step is adapting the shown expression in (Equation 118) by replacing the number "2" with $2^{K'_{(1)}} = 2^1$ whereas conducting the following re-expressions $\{K'_{(1)} = 1;\ K_{(1 \leq j \leq m-1)} = K'_{(j+1)}\ and\ K'_{(0)} = 0\}$, which gives us the shown result in (Equation 119). The value $\left(K_{(j=1)} = K'_{(j+1)}\right)$ is leading to the result $\left(K'_{(2)} \in \{\mathbb{N} - \{0\}\}\right)$, because $\left(K_{(1)} \in \{\mathbb{N} - \{0\}\}\right)$.

$$\Longrightarrow n'_{s_2} = \frac{2^{K'_{(1)}} * 2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 2^{K'_{(1)}} * \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] - 2^{K'_{(1)}} * 3^{m-2} - 3^{m-1}}{3^{m-1}}$$

$$\Longrightarrow n'_{s_2} = \frac{2^{K'_{(1)} + \sum_{l=1}^{l=m-1} K'_{(l+1)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{K'_{(1)} + \sum_{j=1}^{j=m-i-2} K'_{(j+1)}}\right] - 2^{K'_{(1)}} * 3^{m-2} - 3^{m-1}}{3^{m-1}}$$

$$\Longrightarrow n'_{s_2} = \frac{2^{K'_{(1)} + \sum_{l=2}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{K'_{(1)} + \sum_{j=2}^{j=m-i-1} K'_{(j)}}\right] - 2^{K'_{(1)}} * 3^{m-2} - 3^{m-1}}{3^{m-1}}$$

$$\Longrightarrow n'_{s_2} = \frac{2^{\sum_{l=1}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-3}\left[3^i * 2^{\sum_{j=1}^{j=m-i-1} K'_{(j)}}\right] - 2^{K'_{(1)}} * 3^{m-2} - 3^{m-1}}{3^{m-1}}$$

$$\Longrightarrow n'_{s_2} = \frac{2^{\sum_{l=1}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=1}^{j=m-i-1} K'_{(j)}}\right] - 3^{m-1}}{3^{m-1}} \quad (119)$$

Since we have $\left(K'_{(0)} = 0\right)$, we can re-express the shown value of n'_{s_2} in (Equation 119), in order to be presented as shown in (Equation 120):

$$\Longrightarrow n'_{s_2} = \frac{2^{\sum_{l=0}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} K'_{(j)}}\right] - 3^{m-1}}{3^{m-1}}$$

$$\Longrightarrow n'_{s_2} = \frac{2^{\sum_{l=0}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} K'_{(j)}}\right]}{3^{m-1}} \quad (120)$$

Based on the expression (Equation 120), we can represent the value of n'_{s_2} to be as shown in (Equation 121):

$$\Longrightarrow n'_{s_2} = 2^R \frac{2^{\sum_{l=0}^{l=m} K'_{(l)}} - \sum_{i=0}^{i=m-1}\left[3^i * 2^{\sum_{j=0}^{j=m-i-1} K'_{(j)}}\right]}{3^{m-1}} \,|R = 0\ and\ K'_{(0)} = 0\ and\ K'_{(i \geq 1)} \in \{\mathbb{N} - \{0\}\}\ and\ (R + K'_1) > 0\ and\ (R + K'_1 + K'_2) > 0 \quad (121)$$

We deduce that $\left(n'_{s_2} = n_{s_2} - 4\right)$ is expressed while respecting the presented statements and proposed formula (Equation 72) in Theorem 9. Therefore, based on the statements of Theorem 9, the odd number $\left(n_{s_2} = 2n_s + 3\right)$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_{s_2}, because it is expressed according to the proposed formula (Equation 59) in Theorem 8. As a result, the given statements in Theorem 21 are correct.

6.4. Third Proposed Theorem for Multi-Step Convergence of Collatz Operations

This subsection presents a theorem proving that if we have a group of odd numbers L_3 ($L_3 \in \mathbb{N}$), where every odd number (n_s) in this group is converging to inferior values than itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then, we will have even numbers expressed as $2^k n_s$ ($k \in [\![1, +\infty[\![$) also converging to inferior values than themselves when repeating the operations of the Collatz conjecture, whereas starting on them.

Theorem 22

Supposing a group L_3 containing odd numbers, where the minimum odd number in L_3 is $\{n_1 = 1\}$ and the maximum odd number is $\{n_h = 2(h-1)+1\}$; as shown in (Equation. 122). If every odd number n_s from this group (L_3) is converging to a value ($P_3 = 1$) inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then every even number $2^k n_s$ ($k \in [\![1, +\infty[\![$) is also converging to an inferior value ($P_3 = 1$) (where $P_3 \leq n_h$) when repeating the operations of the Collatz conjecture, whereas starting on $2^k n_s$.

$$L_3 = \{n_1 = 1; n_2 = 3;\ n_3 = 5; n_4 = \ 7; \ldots;\ n_{h-1} = 2(h-2)+1; n_h = \ 2(h-1)+1\} \tag{122}$$

Proof of Theorem 22

Supposing that every odd number n_s from the group L_3 is converging to the number ($P_3 = 1$) when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s.

In the case of having a number $\left(n_{s_3} = 2^k n_s\right)$ $where$ ($k \in [\![1, +\infty[\![$), it will be even. Therefore, we will keep dividing this number. n_{s_3} on "2" until reaching the next odd number, which is n_s. Consequently, repeating the operations of the Collatz conjecture, starting on $\left(n_{s_3} = 2^k n_s\right)$ will eventually converge to the number "1".

As a result, the given statements and formulas in Theorem 22 are correct, because every even number $\left(n_{s_3} = 2^k n_s\right)$ will be converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_{s_3}.

6.5. Fourth Proposed Theorem for Multi-Step Convergence of Collatz Operations

This subsection presents a theorem proving that if we have a group of odd numbers L_4 ($L_4 \in N$), where every odd number (n_s) in this group is converging to inferior values than itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then, we will have other even numbers and odd numbers expressed as $\{2^k n_s, 2n_s + 1, 2n_s + 3\}$ also converging to inferior values than themselves when repeating the operations of the Collatz conjecture, whereas starting on them.

Considering the odd number $(2n_s + 5)$; we can re-express it as $(2n_s + 5 = 2n_{s+1} + 1)$ where $\{n_s = 2(s-1) - 1\}$ and $\{n_{s+1} = 2(s) - 1\}$ are two consecutive odd numbers. Therefore, the odd number $(2n_s + 5)$ is already covered by the developed theorems. The same logic can be applied to the odd number $(2n_s + 4a + 1 \mid a \in [\![1, +\infty[\![)$ which we can express as $(2n_s + 4a + 1 = 2n_{s+a} + 1 \mid a \in [\![1, +\infty[\![)$.

Theorem 23

Supposing having a group L_4 containing odd numbers, where the minimum odd number in L_4 is $\{n_1 = 1\}$ and the maximum odd number is $\{n_m = 2(m-1)+1\}$; as shown in (Equation. 123). If every odd number n_s from this group (L_4) is converging to a value ($P_4 = 1$) inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then every natural number n'_s $\left(\{n'_s = 2^k n_s \ or \ n'_s = 2n_s + 1 \ or \ n'_s = 2n_s + 3\}(k \in \mathbb{N})\right)$ is also converging to an inferior value ($P_4 = 1$) (where $P_4 \leq n_m$) when repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s.

$$L_4 = \{n_1 = 1; n_2 = 3;\ n_3 = 5; n_4 = \ 7; \ldots;\ n_{m-1} = 2(m-2)+1; n_m = \ 2(m-1)+1\} \tag{123}$$

Proof of Theorem 23

Supposing having the group L_4 containing odd numbers where the minimum is $\{n_1 = 1\}$ and the maximum is $\{n_m = 2(m-1)+1\}$, whereas supposing every odd number n_s from this group (L_4) is converging to a value $(P_4 = 1)$ inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s.

According to the subsection of Theorem 20, we proved that if an odd number n_s is converging to the number "1" when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then the odd number expressed as $(n'_s = 2n_s + 1)$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting on n'_s.

According to the subsection of Theorem 21, we proved that if an odd number n_s is converging to the number "1" when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then the odd number expressed as $(n'_s = 2n_s + 3)$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting on n'_s.

According to the subsection of Theorem 22, we proved that if an odd number n_s is converging to the number "1" when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then the even number expressed as $(n'_s = 2^k n_s)$ $\left(where\ k \in \{\mathbb{N} - \{0\}\}\right)$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting on n'_s.

Therefore, we deduce that the given statements by Theorem 23 are correct.

6.6. Fifth Proposed Theorem for Multi-Step Convergence of Collatz Operations

This subsection presents a theorem proving that if we have a group of natural numbers L_5 $\left(L_5 \in \{N - \{0\}\}\right)$, where every odd number (n_s) in this group is converging to inferior values than itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then, we will have a larger group L_6 $(L_6 \in \{N - \{0\}\})$ twice the size of L_5 where every natural number $(n'_s \in L_6)$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting on n'_s .

Theorem 24

Supposing a group L_5 containing natural numbers, where the minimum number in L_5 is $\{n_1 = 1\}$ and the maximum number is $\{n_m = m\}$; as shown in (Equation. 124). If every odd number n_s from this group (L_5) is converging to a value $(P_5 = 1)$ inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then every natural number n'_s from the group $(L_6 = [\![1, 2n_m]\!])$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting on n'_s.

$$L_5 = \{n_1 = 1; n_2 = 2;\ n_3 = 3; n_4 =\ 4; \ldots;\ n_{m-1} = (m-1); n_m =\ m\} \tag{124}$$

Proof of Theorem 24

Supposing that the group L_5 is containing natural numbers where the minimum is $\{n_1 = 1\}$ and the maximum is $\{n_m = m\}$, whereas supposing every odd number n_s from this group (L_5) is converging to a value $(P_5 = 1)$ inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s.

According to the subsection of Theorem 20, we proved that if the odd number n_s is converging to the number "1" when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then the odd number expressed as $(n'_s = 2n_s + 1)$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting on n'_s.

Therefore, based on using Theorem 20, we deduce that every odd number $(n'_s = 2n_s + 1)$ in the interval $[\![1, 2n_m]\!]$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s.

According to the subsection of Theorem 21, we proved that if the odd number n_s is converging to the number "1" when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then the odd number expressed as

$(n'_s = 2n_s + 3)$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting on n'_s.

Therefore, based on using Theorem 21, we deduce that every odd number expressed as $(n'_s = 2n_s + 3)$ in the interval $[\![1, 2n_m]\!]$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s.

Considering the odd number $(2n_s + 5)$; we can re-express it as $(2n_s + 5 = 2n_{s+1} + 1)$ where $\{n_s = 2(s' - 1) - 1\}$ and $\{n_{s+1} = 2(s') - 1\}$ are two consecutive odd numbers. Therefore, the odd number $(2n_s + 5)$ is already covered by the developed theorems. The same logic can be applied to the odd number $(2n_s + 4a + 1 \mid a \in [\![1, +\infty[\![)$ which we can express as $(2n_s + 4a + 1 = 2n_{s+a} + 1 \mid a \in [\![1, +\infty[\![)$.

According to the subsection of Theorem 22, we proved that if an odd number n_s is converging to the number "1" when repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then every even number expressed as $(n'_s = 2^k n_s)$ $\left(\text{where } k \in \{N - \{0\}\}\right)$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting on n'_s.

Therefore, based on using Theorem 22, we deduce that every even number n'_s in the interval $[\![1, 2n_m]\!]$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s.

As a result, we deduce that the given statements by Theorem 24 are correct.

7. New Theorems Providing Algebraic Proofs on the Correctness of the Collatz Conjecture

This section presents new theorems proving the correctness of the Collatz conjecture while proposing new unified formulas re-expressing this conjecture algebraically on natural numbers.

7.1. First Proposed Theorem Proving Collatz Conjecture on Odd Numbers

This subsection presents a theorem proving that all odd numbers are converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on them, and they are all expressed according to a unified formula composed of distributed terms based on the group $\{2^k, 3^j\}$.

Theorem 25

Every odd number $\left(n_{s_1} \in \mathbb{N}\right)$ where we have $\left(n_{s_1} > 0\right)$, this number is converging to "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_{s_1}. This convergence is allowing us to express any odd number n_{s_1} $(n_{s_1} > 0)$ according to a unified formula composed of distributed terms as shown in (Equation 125).

$$n_{s_1} = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0 \; and \; K_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \tag{125}$$

Proof of Theorem 25

Supposing having a group L_1 containing natural numbers, where the minimum number in L_1 is $\{n_1 = 1\}$ and the maximum number is $\{n_m = m\}$.

Supposing every odd number n_{s_1} from this group (L_1) is converging to a value $(P = 1)$ inferior to itself when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_{s_1}; then, according to Theorem 24 in this paper, every odd number n'_{s_1} from the group $(L_2 = [\![1, 2^1 n_m]\!])$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n'_{s_1}.

Now, we consider the group of natural numbers. $(L_2 = [\![1, 2^1 n_m]\!])$ where the minimum odd number is $\{n'_1 = 1\}$ and the maximum odd number is $\{2n_m - 1\}$

Since every odd number n'_{s_1} from this group (L_2) is converging to a value $(P' = 1)$ inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n'_{s_1}; then, according to Theorem 24 in this paper, every odd number n''_{s_1} from the group ($L_3 = [\![1, 2^2 n_m]\!]$) is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_{s_1}.

The next step is relying on recurrence (induction) by considering the group of natural numbers $\left(L_j = [\![1, 2^{j-1} n_m]\!]\right)$ where the minimum odd number is $\{n'_1 = 1\}$ and the maximum odd number is $\{2^{j-1} n_m - 1\}$.

Since every odd number n'_{s_1} from the group (L_j) is converging to a value ($P' = 1$) inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n'_{s_1}; then, according to Theorem 24 in this paper, every odd number n''_{s_1} from the group $\left(L_{j+1} = [\![1, 2^j n_m]\!]\right)$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_{s_1}.

Therefore, by recurrence, we deduce that every odd number in the group $\left(L_{j+1} = [\![1, 2^j n_m]\!]\right)$ is converging to the number "1" when we keep repeating Collatz operations, which allow us to extend this deduction to the group L_{j+1} where the value of the natural number "j" is going to infinity.

Consequently, we deduce that every odd number n''_{s_1} from the group $\{\mathbb{N} - \{0\}\}$ $\left(\{\mathbb{N} - \{0\}\} = \lim_{j \to +\infty} L_{j+1} = \lim_{j \to +\infty} [\![1, 2^j n_m]\!]\right)$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_{s_1}.

Based on Theorem 6 in this paper, since every odd number n''_{s_1} from the group $\{\mathbb{N} - \{0\}\}$ $\left(\{\mathbb{N} - \{0\}\} = \lim_{j \to +\infty} L_{j+1} = \lim_{j \to +\infty} [\![1, 2^j n_m]\!]\right)$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_{s_1}; we deduce that we can express n''_{s_1} as shown in (Equation. 126):

$$n''_{s_1} = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0 \; and \; K_{(i \geq 1)} \in \{\mathbb{N} - \{0\}\} \quad (126)$$

As a result, we deduce that the given statements and the proposed formula (Equation 125) in Theorem 25 are correct, which means that the given statements in the Collatz conjecture are correct for odd numbers, because every odd number n''_{s_1} from the group $\{\mathbb{N} - \{0\}\}$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_{s_1} .

7.2. Second Proposed Theorem Proving Collatz Conjecture on Even Numbers

This subsection presents a theorem proving that all even numbers, except zero, converge to the number "1" when we keep repeating the operations of the Collatz conjecture, and they are all expressed according to a unified formula composed of distributed terms based on the group $\{2^k, 3^j\}$.

Theorem 26

Every even number $\left(n_{s_2} \in \mathbb{N}\right)$ where we have $\left(n_{s_2} > 0\right)$, this number is converging to "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_{s_2}. This convergence is allowing us to express any even number n_{s_2} ($n_{s_2} > 0$) according to a unified formula composed of distributed terms as shown in (Equation 127).

$$n_{s_2} = 2^k \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0 \; and \; K_{(i \geq 1)} \in \{\mathbb{N} - \{0\}\} \; and \; k \in \{\mathbb{N} - \{0\}\} \quad (127)$$

Proof of Theorem 26

Supposing having a group L_1 containing natural numbers, where the minimum number in L_1 is $\{n_1 = 1\}$ and the maximum number is $\{n_m = m\}$.

Supposing every odd number n_s from this group (L_1) is converging to a value $(P = 1)$ inferior to itself when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then, according to Theorem 24 in this paper, every odd number and even number n'_{s_2} from the group $(L_2 = [\![1, 2^1 n_m]\!])$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n'_{s_2}.

Now, we consider the group of natural numbers $(L_2 = [\![1, 2^1 n_m]\!])$ where the minimum odd number is $\{n'_1 = 1\}$ and the maximum odd number is $\{2n_m - 1\}$

Since every odd number n'_s from this group (L_2) is converging to a value $(P' = 1)$ inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s; then, according to Theorem 24 in this paper, every odd number and even number n''_{s_2} from the group $(L_3 = [\![1, 2^2 n_m]\!])$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_{s_2}.

The next step is relying on recurrence (induction) by considering the group of natural numbers $\left(L_j = [\![1, 2^{j-1} n_m]\!]\right)$ where the minimum odd number is $\{n'_1 = 1\}$ and the maximum odd number is $\{2^{j-1} n_m - 1\}$.

Since every odd number n'_s from the group (L_j) is converging to a value $(P' = 1)$ inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s ; then, according to Theorem 24 in this paper, every odd number and even number n''_{s_2} from the group $\left(L_{j+1} = [\![1, 2^j n_m]\!]\right)$ is also converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_{s_2}.

Therefore, by recurrence, we deduce that every even number from the group $\left(L_{j+1} = [\![1, 2^j n_m]\!]\right)$ is converging to the number "1" when we keep repeating Collatz operations, which allow us to extend this deduction to the group L_{j+1} where the value of the natural number *"j"* is going to infinity.

Consequently, we deduce that every even number n''_{s_2} from the group $\{\mathbb{N} - \{0\}\}$ $\left(\{\mathbb{N} - \{0\}\} = \lim_{j\to+\infty} L_{j+1} = \lim_{j\to+\infty} [\![1, 2^j n_m]\!]\right)$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_{s_2}.

Based on Theorem 6 and Theorem 25 in this paper, since every odd number n''_s from the group $\{\mathbb{N} - \{0\}\}$ $\left(\{\mathbb{N} - \{0\}\} = \lim_{j\to+\infty} L_{j+1} = \lim_{j\to+\infty} [\![1, 2^j n_m]\!]\right)$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_s; we deduce that we can express n''_s as shown in (Equation 128):

$$n''_s = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}}{3^{m-1}} \mid k_0 = 0 \text{ and } k_{(i\geq 1)} \in \mathbb{N} \tag{128}$$

Therefore, we can express every even number n''_{s_2} in the group $\{\mathbb{N} - \{0\}\}$ $\left(\{\mathbb{N} - \{0\}\} = \lim_{j\to+\infty} L_{j+1} = \lim_{j\to+\infty} [\![1, 2^j n_m]\!]\right)$ as shown in (Equation 129), since we can express every odd number in the same group as shown in (Equation 128).

$$n''_{s_2} = 2^k \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}}{3^{m-1}} \mid k_0 = 0 \text{ and } k_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \text{ and } k \in \{\mathbb{N} - \{0\}\} \tag{129}$$

As a result, we deduce that the given statements and the proposed formula (Equation 127) in Theorem 26 are correct, which means that the given statements in the Collatz conjecture are correct for even numbers, because every even number n''_{s_2} from the group $\{\mathbb{N} - \{0\}\}$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_{s_2}.

7.3. Third Proposed Theorem Proving Collatz Conjecture on Natural Numbers

This section presents a theorem proving that all natural numbers, except zero, converge to the number “1” when we keep repeating the operations of the Collatz conjecture, and they are all expressed according to a unified formula composed of distributed terms based on the group $\{2^k, 3^j\}$.

Theorem 27

Every natural number $(n_s \in \mathbb{N})$ where we have $(n_s > 0)$, this number is converging to “1” when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_s. This convergence is allowing us to express any natural number n_s $(n_s > 0)$ according to a unified formula composed of distributed terms as shown in (Equation 130).

$$n_s = 2^{k'} \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0 \text{ and } K_{(i\geq1)} \in \{\mathbb{N} - \{0\}\} \text{ and } k' \in \mathbb{N} \quad (130)$$

Proof of Theorem 27

Supposing having a group L_1 containing natural numbers, where the minimum number in L_1 is $\{n_1 = 1\}$ and the maximum number is $\{n_m = m\}$.

Supposing every odd number n_s from this group (L_1) is converging to a value $(P = 1)$ inferior to itself when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n_s; then, according to Theorem 21 in this paper, every natural number n'_s from the group $(L_2 = [\![1, 2^1 n_m]\!])$ is also converging to the number “1” when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s.

Now, we consider the group of natural numbers $(L_2 = [\![1, 2^1 n_m]\!])$ where the minimum odd number is $\{n'_1 = 1\}$ and the maximum odd number is $\{2n_m - 1\}$.

Since every odd number n'_s from this group (L_2) is converging to a value $(P' = 1)$ inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s; then, according to Theorem 21 in this paper, every natural number n''_s from the group $(L_3 = [\![1, 2^2 n_m]\!])$ is also converging to the number “1” when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_s.

The next step is relying on recurrence (induction) by considering the group of natural numbers $\left(L_j = [\![1, 2^{j-1} n_m]\!]\right)$ where the minimum odd number is $\{n'_1 = 1\}$ and the maximum odd number is $\{2^{j-1} n_m - 1\}$.

Since every odd number n'_s from the group (L_j) is converging to a value $(P' = 1)$ inferior to itself when repeating the operations of the Collatz conjecture, whereas starting these operations on n'_s; then, according to Theorem 21 in this paper, every natural number n''_s from the group $\left(L_{j+1} = [\![1, 2^j n_m]\!]\right)$ is also converging to the number “1” when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_s.

Therefore, by recurrence, we deduce that every natural number in the group $\left(L_{j+1} = [\![1, 2^j n_m]\!]\right)$ is converging to the number “1” when we keep repeating Collatz operations, which allow us to extend this deduction to the group L_{j+1} where the value of the natural number “j” is going to infinity.

Consequently, we deduce that every natural number n''_s from the group $\{\mathbb{N} - \{0\}\}$ $\left(\{\mathbb{N} - \{0\}\} = \lim_{j\to+\infty} L_{j+1} = \lim_{j\to+\infty} [\![1, 2^j n_m]\!]\right)$ is converging to the number “1” when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on n''_s.

Based on the proposed formulas in Theorem 25 and Theorem 26 in this paper, we can express every natural number n''_s where $\left(n''_s > 0\right)$ as shown in (Equation. 131).

$$n''_s = 2^{k'} \frac{2^{\sum_{l=0}^{l=m-1} k_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} k_{(j)}}\right]}{3^{m-1}} \mid k_0 = 0 \text{ and } k_{(i\geq1)} \in \{\mathbb{N} - \{0\}\} \text{ and } k' \in \mathbb{N} \quad (131)$$

As a result, we deduce that the given statements and the proposed formula in Theorem 27 are correct, which means that the given statements in the Collatz conjecture are correct on natural numbers, because every natural number $n_s^{"}$ from the group $\{\mathbb{N} - \{0\}\}$ is converging to the number "1" when we keep repeating the operations of the Collatz conjecture, whereas starting these operations on $n_s^{"}$.

8. New Results Providing Insights on Prime Numbers basing on the Collatz Conjecture

This section uses the presented unified formulas in this paper for the Collatz conjecture, in order to analyze characteristics of prime numbers, including their distribution.

8.1. Using the Collatz conjecture on prime numbers

Prime numbers have been intriguing for mathematicians for centuries, with the vision of finding unified formulas expressing them was always an axis of focus [19].

There have been many patterns in algebra leading toward identifying some prime numbers like Mersenne primes [20], whereas the majority of prime numbers were not solidly linked into one conclusive pattern reinforced by algebraic proofs.

Some published papers focused on counting the number of prime numbers and their distribution instead of identifying the values of primes themselves [21-22], since there has been no visible unified thread allowing the algebraic calculation of all prime numbers with perfect precision. However, there are some other analytic methods and probabilistic methods to approximate prime numbers and their distribution.

The importance of prime numbers surpasses theoretical concepts by using them in cryptography [23], where encrypted data becomes more secure when using prime numbers for the generation of encryption keys, especially by using massive prime values or massive pseudoprime numbers in asymmetric cryptography [24].

The Collatz conjecture allows us to interconnect all natural numbers according to a tree (Figure 1), which inspired us toward developing unified formulas in this paper in order to re-express the operations of the Collatz conjecture. As a result, we were able to develop a unified formula interconnecting all odd numbers by re-expressing the operations of the Collatz conjecture as presented in Theorem 25, Theorem 8, and Theorem 6.

In the tree presenting the Collatz conjecture in Figure 1, natural numbers are interconnected according to branches. Therefore, Theorem 25 in this paper presents a unified formula (Equation 132) interconnecting all odd numbers, whereas prime numbers are expected to be distributed according to different branches as shown in Figure 1.

$$n = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^{i} * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0 \; and \; K_{(i \geq 1)} \in \{\mathbb{N} - \{0\}\} \tag{132}$$

In order to study the characteristics of prime numbers over the shown branches illustrating the Collatz conjecture according to a tree in (Figure 1), we will rely on analyzing the included variables and sub-expressions in the unified formula (Equation 132), which we demonstrated in the proofs of Theorem 6 and Theorem 8.

8.2. Using the Collatz Conjecture to Identify Patterns of Prime Numbers

This subsection specifies six expressions deduced from the unified formula (Equation 132), which will allow analyzing potential patterns among prime numbers. These expressions are as presented from (Equation 133) up to (Equation 138).

$$P_1 = \sum_{l=0}^{l=m-1} K_{(l)} \tag{133}$$

$$P_2 = m - 1 \tag{134}$$

$$P_3 = \left[2^{\sum_{l=0}^{l=m-1} K_{(l)}}\right] \tag{135}$$

$$P_4 = [3^{m-1}] \tag{136}$$

$$P_5 = \left[2^{\sum_{l=0}^{l=m-1} K_{(l)}} - n * 3^{m-1}\right] \tag{137}$$

$$P_6 = \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] \quad (138)$$

The first promising expression deduced from the shown formula in (Equation 132) is as presented in (Equation 140), which generates a specific pattern expressed by (Equation 141).

$$r_n = \left[\sum_{l=0}^{l=m-1} K_{(l)}\right] MOD\ [n-1] \quad (139)$$

Codes in Python to calculate the values of $PATTERN_n$ in function of odd numbers while using Collatz operations

Code part 1

```
import matplotlib.pyplot as plt
from mpmath import *
import threading
m=0
odd_numbers_array=[]
collatz_itterations_amount=[]
ki_arrays=[]
ki_array=[]
save_array=[]
shaping_array=[]
pattern_array=[]
def ki_counting_loop(R):
    s=3*R+1
    ki_counter=0
    while(s%2==0):
        s=s/2
        ki_counter+=1
    return [s,ki_counter]
def collatz_operations(i):
    odd_numbers_array.append(i)
    convergency_reached=False
    itterations_counter=0
    ki_array=[]
    R=i
    while(convergency_reached==False):
        A=ki_counting_loop(R)
        itterations_counter+=1
        ki_array.append(A[1])
        R=A[0]
        if R == 1:
            convergency_reached=True
collatz_itterations_amount.append(itterations_counter)
    ki_arrays.append(ki_array)
    d=i-1-(sum(ki_array)%(i-1))
    s=0
    k=0
    e=0
    for l in range(itterations_counter):
        e=pow(3,(itterations_counter-1-l));
s+=e*pow(2,k);
        s=s%i
        k+=ki_array[l]
    #p=((2**(d))*(s))%i
    p=(pow(2,d)*s)%i
    pattern_array.append(p)
m=input("Enter the value of an odd number m: ")
m=int(m)
threads = []
```

Code part 2

```
for i in range(3,m+1,2):
    #print("the number i is: ", i)
    t =
threading.Thread(target=collatz_operations,
args=(i,), kwargs={})
    threads.append(t)
for t in threads:
    t.start()
for t in threads:
    t.join()
with open('odd-numbers-list.txt', 'w+') as f:
    for items in odd_numbers_array:
        f.write('%s\n' %items)
    print("File written successfully for odd
numbers")
f.close()
with open('collatz-iterations-list.txt', 'w+')
as f:
    for items in collatz_itterations_amount:
        f.write('%s\n' %items)
    print("File written successfully for
collatz iterations")
f.close()
with open('collatz-ki-list.txt', 'w+') as f:
    for items in ki_arrays:
        f.write('%s\n' %items)
    print("File written successfully for ki
lists")
f.close()
with open('collatz-pattern-list.txt', 'w+') as
f:
    for items in pattern_array:
        f.write('%s\n' %items)
    print("File written successfully for
pattern lists")
f.close()
plt.plot(odd_numbers_array, pattern_array,
marker='.', linestyle='dotted')
font1 =
{'family':'serif','color':'blue','size':18}
font2 =
{'family':'serif','color':'darkred','size':15}
plt.title(" (Graph 5) Evolution of pattern
values \n in function of odd numbers",
fontdict=font1)
plt.xlabel("Values of Odd numbers",
fontdict=font2)
plt.ylabel("values of (PATTERN)",
fontdict=font2)
plt.show()
```

Fig. 6 Programmed codes in python to calculate values of PATTERN_n and graphically illustrate their evolutions.

$$R_n = 2^{[(n-1)-r_n]} * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] \quad (140)$$

$$PATTERN_n = [R_n]\ MOD\ [n] \quad (141)$$

We use the Python programming language, as shown in Figure 6, to create the codes that allow computing the results of the shown expressions in (Equation 139), (Equation 140), and (Equation 141), which allows calculating the values of the deduced pattern in (Equation 141). As a result, we obtain the graphical evolutions of the pattern as shown in Figure 7.

The graphical evolutions in Figure 7 show a specific pattern of static values mainly repeated among prime numbers, where the value of $([R_n]MOD[n])$ is equal to "1" when "n" is a prime number.

There are a few odd numbers {s} which are not prime numbers, but they still give values of pattern equal to "1" $([R_s]MOD[s] = 1)$. However, these odd numbers are rare in the obtained results from the shown codes in Figure 6 and the illustrated pattern in Figure 7. In addition, these rare odd numbers, which are not primes, do not appear in the pattern until reaching the range $[\![2^8, +\infty[\![$, like the value (341) where $([R_{341}]MOD[341] = 1)$, and these non-prime numbers are presenting a negligible quantity in the horizontal line of pattern (where $PATTERN_n = 1$) by comparison to the quantity of prime numbers in the same line.

The shown expression of R_n in (Equation 141) is proposed in this paper as an alternative to using $(2^{n-1}MOD[n])$, whereas the expression $(2^{n-1}MOD[n])$ was proposed by Pierre de Fermat [25-26], which was also forwarded by scaling up its logic toward verifying massive Mersenne primes [27-28], especially by using the Lucas-Lehmer primality test.

The proposed pattern for R_n This paper is composed of lower exponential values and is also composed of distributed terms, in comparison to the term. $(2^{n-1}MOD[n])$.

The composition of the expression $(R_nMOD[n])$ makes it take lower values at the computational level than $(2^{n-1}MOD[n])$ when "n" goes to infinity, and also allows calculating the values of distributed terms $\left(\sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]\right)$ by using a reduction pattern and parallel computation [29], which can minimize the computation time exponentially.

The advantage of the proposed pattern $(R_nMOD[n])$ in this paper, we are allowed to identify potential prime numbers, especially infinite primes, because we can compute the calculation of each distributed term $\left(\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] MOD[n]\right)$ according to a separate thread in parallel with other terms, while having the possibility of using reduction patterns, we can replace the value of $\left(\left[2^{\sum_{l=0}^{l=m-1} K_{(l)}} - 3^{(m-1)} * n\right]\ MOD\ [n]\right)$ with the sum of terms $\left(\left[\sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}} MOD\ [n]\right]\right]\ MOD\ [n]\right)$. As a result, the proposed pattern $(R_nMOD[n])$ can exponentially reduce the computation time of verifying infinity prime numbers by comparison to using $(2^{n-1}MOD[n])$.

Furthermore, the pattern $(R_nMOD[n])$ can be expressed over less RAM (Random Access Memory) space than $(2^{n-1}MOD[n])$, because we can calculate each term $\left(\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] MOD[n]\right)$ from the proposed pattern according to a converging reduction pattern while using parallel threads, whereas each one among these terms $\left(\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] MOD[n]\right)$ can have a lower exponential value than $(2^{n-1}MOD[n])$, which can massively optimize the use of RAM space while reducing the computation time.

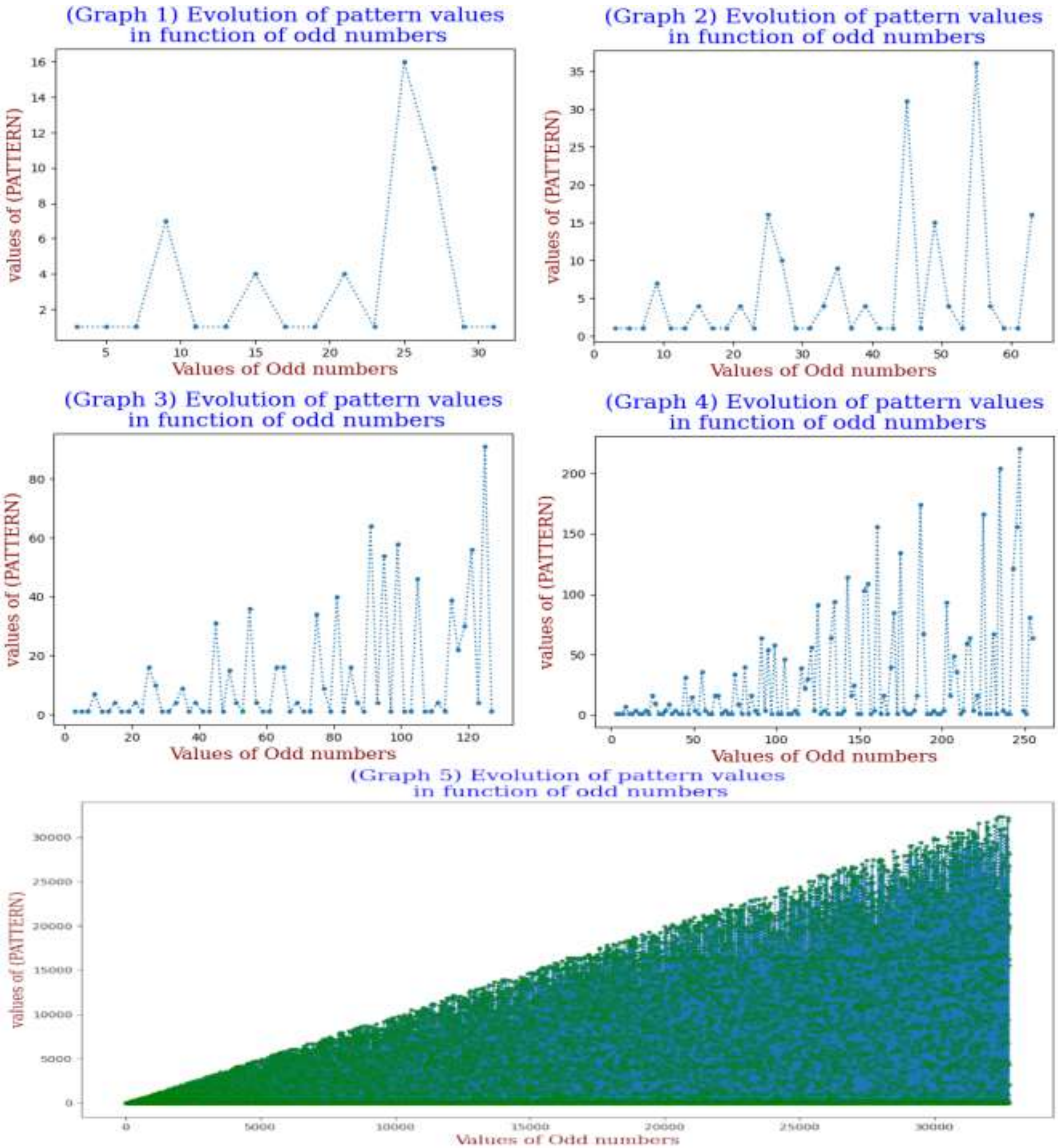

Fig. 7 Computed graphs showing the evolution of the values of PATTERN_n in function of odd numbers.

Therefore, we are proposing four new conjectures based on the operations of the Collatz conjecture and the proposed unified formulas in this paper to verify whether an odd number may be a prime or not, while preparing the proofs of these four new conjectures to be published in a future article.

When verifying a possible prime number by using the proposed conjectures in this paper, the exponential value of (3^i) can be less than the exponential value of (2^{n-1}) because the value of (i) is in the range $[\![0, m]\!]$, whereas m is the number of the coefficients $\{K_{(i)}\}$. In addition, the exponential value of $\left(2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right)$ can also be less than the exponential value of (2^{n-1}), whereas we will have the possibility of computing the distributed terms $\left(\sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}} MOD[n]\right]\right)$ by using a reduction pattern and parallel computation to identify the values of the proposed patterns.

The advantage of these four new conjectures in this paper is relying on the Collatz conjecture to verify finite and infinite prime numbers while consuming less computing time by allowing more parallel computation of distributed terms, whereas using reduction patterns. In addition, these four new conjectures are designed to allow the reduction of the use of RAM (Random Access Memory) space, where the value of the new proposed patterns can occupy portions of memory space by comparison to the pattern $(2^{n-1} MOD[n])$.

Conjecture 1

If (n) is an odd number, then we can conduct the operations of the Collatz conjecture, which are re-expressed in Theorem 25, in order to present this odd number as shown in (Equation 142).

After expressing the odd number (n) as shown in (Equation 142), whereas expressing the values of r_n and R_n as consecutively shown in (Equation 143) and (Equation 144); if $PATTERN_n$ shown in (Equation 145) is equal to "1", then the odd number (n) can be a prime number as shown in (Equation 146).

To enhance the computation of the shown pattern expression $(PATTERN_n)$ in (Equation 145) at the levels of exploiting memory space and processing time, we can re-express this pattern as shown in (Equation 147).

$$n = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0 \; and \; K_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \quad (142)$$

$$r_n = \left[\sum_{l=0}^{l=m-1} K_{(l)}\right] MOD\ [n-1] \quad (143)$$

$$R_n = 2^{[(n-1)-r_n]} * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] \quad (144)$$

$$PATTERN_n = [R_n]\ MOD\ [n] \quad (145)$$

$$if\ (PATTERN_n = 1)\ is\ true \Rightarrow n\,can\ be\ a\ prime\ number \quad (146)$$

$$PATTERN_n \equiv \left[\left(2^{[(n-1)-r_n]} MOD\ [n]\right) * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}} MOD\ [n]\right]\right] MOD\ [n] \quad (147)$$

Conjecture 2

If (p) is an odd number, then we can conduct the operations of the Collatz conjecture, which are re-expressed in Theorem 25, in order to present this odd number as shown in (Equation 148).

After expressing the odd number (p) as shown in (Equation 148), whereas expressing the values of r_p, r'_p, s'_p and R'_p as consecutively shown from (Equation 149) up to (Equation 152); if $PATTERN'_p$ shown in (Equation 153) is equal to "1", then the odd number (p) can be a prime number as shown in (Equation 154).

To enhance the computation of the shown pattern expression $(PATTERN'_p)$ in (Equation 153) at the levels of exploiting memory space and processing time, we can re-express this pattern as shown in (Equation 155).

$$p = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0 \; and \; K_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \quad (148)$$

$$r_p = \left[\sum_{l=0}^{l=m-1} K_{(l)}\right] MOD\ [p-1] \quad (149)$$

$$r'_p = [(p-1) - r_p]\ MOD\ [r_p] \quad (150)$$

$$s'_p = math.floor\left[\frac{((p-1)-r_p)}{r_p}\right] + 1 \quad (151)$$

$$R'_p = 2^{r'_p} * \left[\sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]\right]^{s'_p} \quad (152)$$

$$PATTERN'_p = [R'_p]\ MOD\ [p] \quad (153)$$

$$if\ (PATTERN'_p = 1)\ is\ true\ \Rightarrow \text{p} can\ be\ a\ prime\ number \quad (154)$$

$$PATTERN'_p \equiv \left[\left(2^{r'_p}\ MOD\ [p]\right) * \left[\left(\sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\ MOD\ [p]\right]\right)^{s'_p} MOD\ [p]\right]\right] MOD\ [p] \quad (155)$$

Conjecture 3

If (q) is an odd number, then we can conduct the operations of the Collatz conjecture, which are re-expressed in Theorem 25, in order to present this odd number as shown in (Equation 156).

After expressing the odd number (q) as shown in (Equation 156), whereas expressing the values of r'_q and R'_q as consecutively shown in (Equation 157) and (Equation 158); if $PATTERN'_q$ shown in (Equation 159) is equal to “1” or if it is equal $(q-1)$, then the odd number (q) can be a prime number as shown in (Equation 160).

To optimize the computation of the shown pattern expression $(PATTERN'_q)$ in (Equation 159) at the levels of exploiting memory space and processing time, we can re-express this pattern as shown in (Equation 161).

$$q = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0\ and\ K_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \quad (156)$$

$$r'_q = \left[\sum_{l=0}^{l=m-1} K_{(l)}\right] MOD\ \left[\frac{q-1}{2}\right] \quad (157)$$

$$R'_q = 2^{\left[\frac{(q-1)}{2} - r'_q\right]} * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right] \quad (158)$$

$$PATTERN'_q = [R'_q]\ MOD\ [q] \quad (159)$$

$$if\ (PATTERN'_q = 1\ or\ PATTERN'_q = q-1)\ is\ true\ \Rightarrow "q"\ can\ be\ a\ prime\ number \quad (160)$$

$$PATTERN'_q \equiv \left[\left(2^{\left[\frac{(q-1)}{2} - r'_q\right]}\ MOD\ [q]\right) * \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\ MOD\ [q]\right]\right] MOD\ [q] \quad (161)$$

Conjecture 4

If (v) is an odd number, then we can conduct the operations of the Collatz conjecture, which are re-expressed in Theorem 25, in order to present this odd number as shown in (Equation 162).

After expressing the odd number (v) as shown in (Equation 162), whereas expressing the values of r_v, r'_v, s'_v and R'_v as consecutively shown from (Equation 163) up to (Equation 166); if $PATTERN'_v$ shown in (Equation 167) is equal to “1” or if it is equal $(v-1)$, then the odd number (v) can be a prime number as shown in (Equation 168).

To optimize the computation of the shown pattern expression $(PATTERN'_v)$ in (Equation 167) at the levels of exploiting memory space and processing time, we can re-express this pattern as shown in (Equation 169).

$$v = \frac{2^{\sum_{l=0}^{l=m-1} K_{(l)}} - \sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]}{3^{m-1}} \mid K_0 = 0 \; and \; K_{(i\geq 1)} \in \{\mathbb{N} - \{0\}\} \quad (162)$$

$$r_v = \left[\sum_{l=0}^{l=m-1} K_{(l)}\right] MOD \left[\frac{v-1}{2}\right] \quad (163)$$

$$r'_v = \left[\frac{(v-1)}{2} - r_v\right] MOD \, [r_v] \quad (164)$$

$$s'_v = math.floor\left[\frac{\left(\frac{(v-1)}{2} - r_v\right)}{r_v}\right] + 1 \quad (165)$$

$$R'_v = 2^{r'_v} * \left[\sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]\right]^{s'_v} \quad (166)$$

$$PATTERN'_v = [R'_v] \, MOD \, [v] \quad (167)$$

$$if \; (PATTERN'_v = 1 \; or \; PATTERN'_v = v - 1) \; is \; true \implies "v" \; can \; be \; a \; prime \; number \quad (168)$$

$$PATTERN'_v \equiv \left[\left(2^{r'_v} \, MOD \, [v]\right) * \left(\left[\sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}} \, MOD \, [v]\right]\right]^{s'_v} MOD \, [v]\right)\right] MOD \, [v] \quad (169)$$

8.3. Using the Collatz conjecture to analyze the distribution of prime numbers

This subsection identifies specific distribution patterns of prime numbers over the group of natural numbers $\mathbb{N}$ while relying on the proposed formulas re-expressing the Collatz conjecture in this paper.

In the previous subsection, we proposed four conjectures identifying patterns of prime numbers, where the goal is to deduce whether an odd number can be a prime or not while relying on the proposed formulas re-expressing the Collatz conjecture in this paper.

The four patterns $\{PATTERN_n \, ; \, PATTERN'_q \, ; \, PATTERN'_p \, ; \, PATTERN'_v\}$ The four proposed conjectures presented in the previous subsection extend the pattern $\{2^{n-1} \, MOD \, [n]\}$ to verify whether an odd number "n" can be a prime or not. In addition, these proposed patterns are enhancing the used mathematical expressions by relying on proposed formulas re-expressing the Collatz conjecture in Theorem 25, Theorem 8, and Theorem 6 in this paper.

The four proposed patterns $\{PATTERN_n \, ; \, PATTERN'_q \, ; \, PATTERN'_p \, ; \, PATTERN'_v\}$ are composed of mathematical expressions with lower exponential values than $\{2^{n-1} \, MOD \, [n]\}$, and also composed of distributed terms $\left\{\sum_{i=0}^{i=m-2}\left[3^i * 2^{\sum_{j=0}^{j=m-i-2} K_{(j)}}\right]\right\}$, which opens the way to minimize the use of RAM (Random Access Memory) space for exponential values, especially when the odd number "n" is scaling up to infinity. In addition, these distributed terms allow the use of reduction patterns while deploying more parallel computation by relying on multithreading, parallel streams, and distributed computation on CPUs (Central Processing Units) and cores, which can minimize processing time.

In this subsection, we are going to use Conjecture 1 to identify possible prime numbers while counting them according to specific ranges expressed as $[\![0, MAX_L]\!]$ where $L \in \mathbb{N}$.

The proposed formulas in Theorem 25, Theorem 8, and Theorem 6 in this paper re-express the Collatz conjecture while being composed of a distributed architecture of terms based on the group $\{2^i, 3^k\}$. Therefore, we are going to count and analyze the distribution of prime numbers according to exponential ranges expressed as $[\![0, 2^L]\!]$ where $L \in \mathbb{N}$. We are selecting the smallest natural number $\{2\}$ from the group $\{2^i, 3^k\}$ as an essence to divide ranges as $[\![0, 2^L]\!]$ in order to analyze the distribution of prime numbers.

Usually, we use the sieve algorithm to identify prime numbers in computation because of its perfect theoretical precision. However, this algorithm of sieve relies on saving arrays or lists representing the majority of natural numbers in memory while identifying prime numbers among them, which can consume a lot of RAM (Random Access Memory) space that can exceed Gigabytes or even exceed dozens of Gigabytes when scaling up to infinity numbers. Therefore, it will be useful to rely on specific patterns to identify the primality of odd numbers without saving them in memory space before and after their verification.

In Conjecture 1, we identified a pattern allowing us to verify whether an odd number can be a prime number or not, where $PATTERN_n$ shown in (Equation 136) is equal to “1” when majorly "n" is a prime number, which we deduced from the shown results in Figure 7.

All values of $PATTERN_n$ in Figure 7, the edges are distributed according to a triangular shape, where the maximum edge can be expressed by a linear equation, whereas the minimum edge is identified by a static value ($PATTERN_n = 1$) obtained majorly when "n" is a prime number.

Codes to count possible prime numbers according to Conjecture 1 in the range $[\![0, 2^n]\!]$ (Section 1)

Code Part 1

```java
public class Collatz_Primes_Pi_C1  {
    public static BigInteger[]
primes_counter_in_range;
    public static BigInteger[] primes_counter;
    public static BigInteger counter;
    public static int range_o;
    public static int l, pl, lm, dlm, m;
    private static CategoryDataset createDataset(int
l, BigInteger[] result, BigInteger[] sieve) {
        DefaultCategoryDataset dataset = new
DefaultCategoryDataset();
        String series1 = "Approximation_PI";
        String series2 = "Sieve_PI";
        String series3 = "Approximation_PI_Error";
        for(int ii=0; ii<l+1; ii++) {
            dataset.addValue(result[ii], series1,
ii);
            dataset.addValue(sieve[ii], series2,
ii);
            dataset.addValue(result[ii].subtract(sie
ve[ii]), series3, ii);
        }
        return dataset;
    }

    public static void ranges_initializing() {
        primes_counter_in_range=new
BigInteger[dlm+1];
        primes_counter=new BigInteger[l+1];
    }

    public static void values_initializing() {
        for(int i=0; i<l+1; i++) {
            primes_counter[i]=BigInteger.ZERO;
        }
        for(int i=0; i<dlm+1; i++) {
            primes_counter_in_range[i]=BigInteger.ZE
RO;
        }
        counter=BigInteger.ZERO;
    }
/* testing whether an odd number can be a prime
number */
    public static void prime_testing(int n, int r) {
        int ki=0;
        int m_count=1;
        int sum_ki=0;
        int n2=n;
        ArrayList<Integer> ki_array = new
ArrayList<>();
        ki_array.add(0);
```

Code Part 2

```java
        if(n2>1) {
            while (n2 >1) {
                ki=0;
                n2=3*n2+1;
                while(n2%2 == 0) {
                    n2=n2/2;
                    ki+=1;
                }
                m_count+=1;
                ki_array.add(ki);
                sum_ki+=ki;
            }
            BigInteger c, nr, nr_2, nr_3, nr_2_3;
            nr=BigInteger.ZERO;
            nr_3=BigInteger.ZERO;
            int k=0;
            BigInteger nn=BigInteger.valueOf(n);
            for(int j=0; j<m_count-1; j++) {
/*Using reduction pattern on the group (ki) */
                    k+=ki_array.get(j);
                    nr_3=BigInteger.valueOf(3).po
w(m_count-2-j);
                    nr_2=BigInteger.TWO.pow(k);
                    nr_2_3=nr_2.multiply(nr_3);
                    nr=nr.add(nr_2_3);
                    nr=nr.mod(nn);
                }
            int kk =sum_ki%(n-1);
            nr=nr.multiply(BigInteger.TWO.pow(n-
1-kk));
            nr=nr.mod(nn);
            if(nr.equals(BigInteger.ONE)) {
                primes_counter_in_range[r]=primes
_counter_in_range[r].add(BigInteger.ONE);

            }
        }
    }
/* testing a range of odd numbers to identify
primes among them */
    public static void range_primes_testing(int
min, int max, int r) {
        int rr=(int)(max-min)/2+1;
        //Using Multithreading
        Thread[] thread=new Thread[rr];
        int min2=min;
        if(min%2==0) {
            min2=min+1;
        }
```

Fig. 8 Section 1 of developed codes in Java programming language for comparison between using Conjecture 1 and Sieve algorithm to count the amount of prime numbers in the range $[\![0, 2^n]\!]$

The shown expression of $PATTERN_n$ in (Equation 136) is based on R_n which is presented in (Equation 135), whereas R_n consists of the subexpression $\{2^{[(n-1)-r_n]}\}$. In addition, all odd numbers are expressed as shown in (Equation 133), which consists of the subexpression $\left(2^{\sum_{l=0}^{l=m-1} K_{(l)}}\right)$. Therefore, we use the Java programming language to create the shown codes in {Figures 8, 9, and 10}, in order to count prime numbers in each range $[\![0, 2^{(n)}]\!]$, which generates the illustrated evolution in blue color shown in Figure 11.

Codes to count possible prime numbers according to Conjecture 1 in the range $[\![0, 2^n]\!]$ (Section 2)

Code Part 3

```
        for(int jj=min2; jj<max+1; jj+=2) {
            int jj2=jj;
            int jt=(int)(jj-min2)/2;
/* Creating a thread to test each odd number in
parallel with others */
            thread[jt]=new Thread(() ->
prime_testing(jj2, r));
            thread[jt].start();
        }
        for(int jj=min2; jj<max+1; jj+=2) {
            int jt2=(int)(jj-min2)/2;
            try {
                thread[jt2].join();
                } catch (InterruptedException e) {
                   e.printStackTrace();
                }
        }
        if(r==0) {
            primes_counter_in_range[r]=primes_counte
r_in_range[r].add(BigInteger.ONE);
        }
    }
/* Identifying the minimum and maximum values of a
specific range to count prime numbers in it */
    public static void range_function(int s) {
        int ss=s;
        int c=(int)Math.pow(2,m);
        int s2=s*c;
        int s_min= s2;
        int s_max= s_min+c;
        System.out.println("range id is: "+s+" ,
s_min is "+s_min+" , s_max is "+s_max);
        range_primes_testing(s_min, s_max, ss);
    }
    public static void main(String[] args) {
        l=20;
        pl=(int) Math.pow(2, l);
        m=2;
        range_o=(int) Math.pow(2, m);
        lm=l-m;
        dlm=(int)Math.pow(2,lm);
        ranges_initializing() ;
        values_initializing();
        System.out.println("Amount of expected
ranges: "+dlm+" , size of each range: "+range_o);
        ArrayList<Integer> numbers = new
ArrayList<>(dlm+1);
        for(int i = 0; i < dlm+1; i++){
            numbers.add(i);
        }
```

Code Part 4

```
        System.out.println("*******Launching
primes counting in each range*******");
/* Using parallel stream to distribute processes
on cores of CPUs */
        numbers.parallelStream().forEach(e ->
range_function(e));

        counter=primes_counter_in_range[0];
        primes_counter[m]=counter;
        for(int s=0; s<dlm+1; s++) {
            double t=Math.log(s+1);
            int t2;
            double tt=t/Math.log(2);
            if(tt%1==0.0) {
                primes_counter[(int)tt+m]=counter
;
            }
            if(s<dlm) {
                counter=counter.add(primes_counte
r_in_range[s+1]);
            }
        }
        int[] sieve_counter= {0, 1, 2, 4, 6, 11,
18, 31, 54, 97, 172, 309, 564, 1028, 1900, 3512,
6542, 12251, 23000, 43390, 82025, 155611, 295947,
564163, 1077871, 2063689,  3957809, 7603553,
14630843, 28192750, 54400028, 105097565,
203280221};
        BigInteger[] sieve_counter_biginteger=new
BigInteger[sieve_counter.length];
        for(int ii=0; ii<sieve_counter.length;
ii++) {
            sieve_counter_biginteger[ii]=BigInteg
er.valueOf(sieve_counter[ii]);
            if(ii<m) {
                primes_counter[ii]=BigInteger.val
ueOf(sieve_counter[ii]);
            }

        }
        System.out.println("********Results of
primes counting********");
        for(int n=0; n<l+1; n++) {
            System.out.println("the value of n
is: "+n+" and its counter is:
"+primes_counter[n]);
        }
```

Fig. 9 Section 2 of developed codes in Java programming language for comparison between using Conjecture 1 and Sieve algorithm to count the amount of prime numbers in the range $[\![0, 2^n]\!]$

Codes to count possible prime numbers according to Conjecture 1 in the range $[0, 2^n]$ (Section 3)

Code Part 5

```java
/* Identifying a dataset of arrays to be displayed
according to Graphs */
        DefaultCategoryDataset dataset = new
DefaultCategoryDataset();
        dataset=  (DefaultCategoryDataset)
createDataset(1,primes_counter,
sieve_counter_biginteger);
        JFreeChart chart =
ChartFactory.createLineChart(
                "Values of Sieve PI and
Approximation PI by Conjecture 1 in the range \n [0,
Math.pow(2,n)] (Graph 4)",
                "value of n",
                "APP_PI and Sieve_PI",
                dataset);
       final Font oldTitleFont =
chart.getTitle().getFont();
        final Font titleFont = new Font("Times New
Roman", oldTitleFont.getStyle(), 25);
        chart.getTitle().setFont(titleFont);
        chart.getLegend().setItemFont(new
Font("SansSerif", Font.BOLD, 24));
        CategoryPlot p = chart.getCategoryPlot();
        ValueAxis axis = p.getRangeAxis();
        CategoryAxis axisy = p.getDomainAxis();

        Font newFont = new Font("Times New Roman",
Font.BOLD, 24);
        axis.setTickLabelFont(newFont);
        axisy.setTickLabelFont(newFont);
        p.getDomainAxis().setLabelFont(newFont);
        p.getRangeAxis().setLabelFont(newFont);
```

Code Part 6

```java
        LineAndShapeRenderer renderer =
(LineAndShapeRenderer)
chart.getCategoryPlot().getRenderer();
        renderer.setAutoPopulateSeriesStroke(fals
e);
        renderer.setDefaultStroke(new
BasicStroke(5.0f));
        renderer.setSeriesStroke(1, new
BasicStroke(3.0f));
        renderer.setSeriesStroke(2, new
BasicStroke(3.0f));
        renderer.setSeriesPaint(1, new
Color(0x00, 0xdd, 0x00));
        renderer.setSeriesPaint(2, new
Color(0xff, 0x00, 0x00));
        renderer.setSeriesPaint(0, new
Color(0x00, 0x00, 0xff));

        ChartPanel chartPanel = new
ChartPanel(chart);

        JFrame frame = new JFrame();
        frame.setSize(1024, 1024);
        frame.setContentPane(chartPanel);
        frame.setLocationRelativeTo(null);
        frame.setDefaultCloseOperation(JFrame.EXI
T_ON_CLOSE);
        frame.setVisible(true);
    }
}
```

Fig. 10 Section 3 of developed codes in Java programming language for comparison between using Conjecture 1 and Sieve algorithm to count the amount of prime numbers in the range $[\![0, 2^n]\!]$

The results shown in blue in Figure 11 present the calculated amount of prime numbers $\left(\pi_{c1}\left(2^{(n)}\right)\right)$ expressed in (Equation 171), which shows an exponential evolution of the number of prime numbers as a function of the natural number "n".

The presented function $\left(\pi_{c1}\left(2^{(n)}\right)\right)$, in (Equation 171), the expression is used $\Gamma_{PATTERN_i}$ shown in (Equation 170) to identify the primality of odd numbers while relying on Conjecture 1.

$$\Gamma_{PATTERN_i} = \begin{cases} 1, & if(PATTERN_i = 1) \\ 0, & if(PATTERN_i \neq 1) \end{cases} \tag{170}$$

$$\pi_{c1}\left(2^{(n)}\right) = 1 + \sum_{i=0}^{2^{(n-1)}} \Gamma_{PATTERN_{(2i+1)}} \mid n \in \{\mathbb{N} - \{0\}\} \tag{171}$$

The green color in Figure 11 presents the actual number of prime numbers in each range $[\![0, 2^{(n)}]\!]$ by using the sieve algorithm to calculate $(\pi_s(2^{(n)})$, whereas the red color presents the error difference $\left[\pi_{c1}\left(2^{(n)}\right) - \pi_s\left(2^{(n)}\right)\right]$.

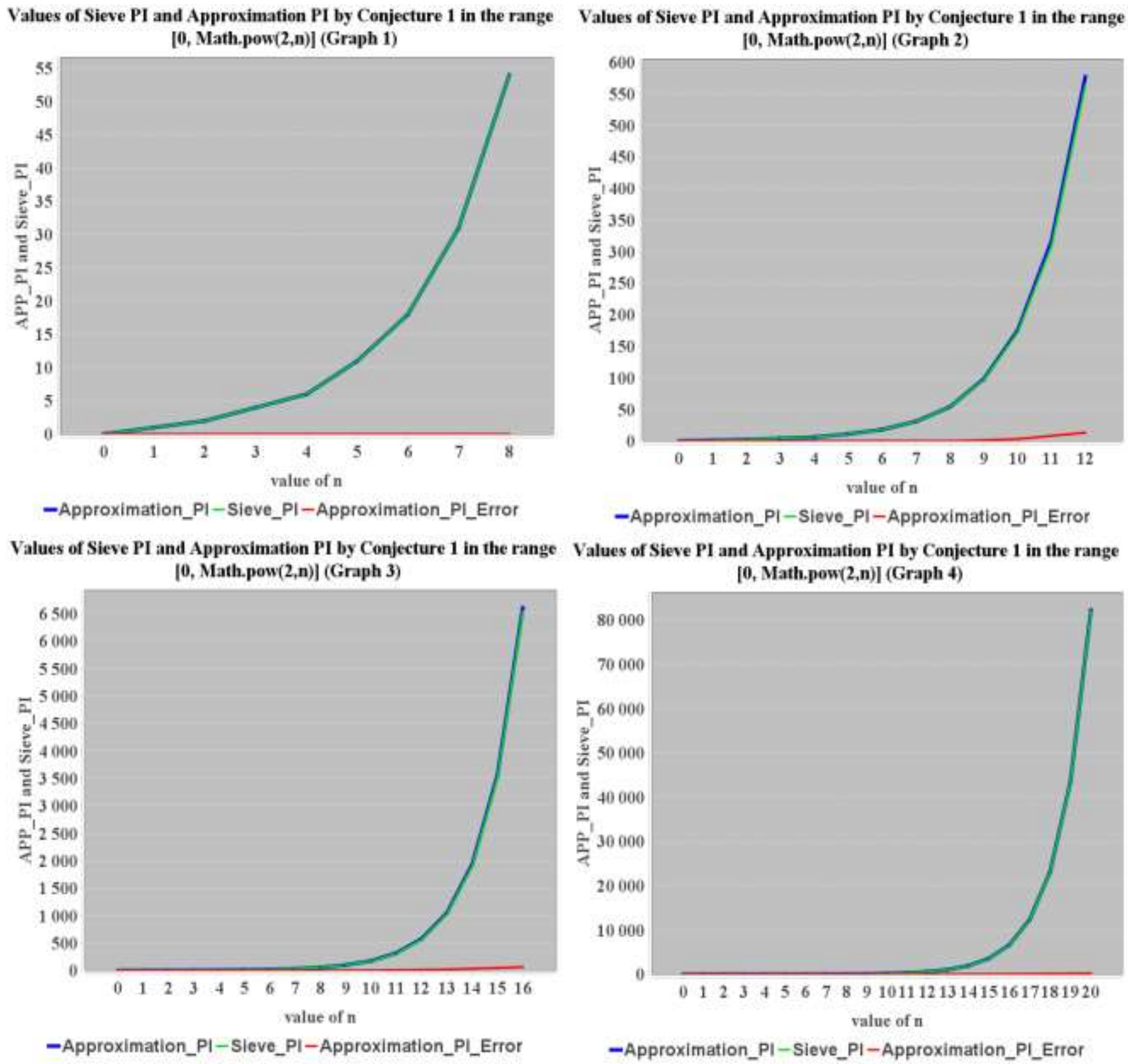

Fig. 11 Comparison graphs between using Conjecture 1 and Sieve algorithm to count the amount of prime numbers in the range $[\![0, 2^n]\!]$

From the shown results in Figure 11, we deduce that counting prime numbers by using Conjecture 1 in each range $[\![0, 2^{(n)}]\!]$ is highly approximating the actual number of prime numbers identified by using the sieve algorithm, whereas the error difference between them is negligible by comparison to the actual number of primes.

Since the error difference $\left[\pi_{c1}\left(2^{(n)}\right) - \pi_s\left(2^{(n)}\right)\right]$ between approximated amount of prime numbers and actual amount of prime numbers is neglectable by comparison to the actual amount of primes (Figure 11), we deduce that we can use the proposed patterns in this paper as an official way to determine prime numbers whereas searching for specific criteria and testing conditions to eliminate the odd numbers that are generating the error difference in these proposed patterns.

Since the evolution of the number of prime numbers $\left(\pi_{(2^n)}\right)$ in function of the natural number "n" is taking an exponential model as shown in Figure 12, we can approximate this amount $\left(\pi_{(2^n)}\right)$ as presented in (Equation 172), where the error function $\varepsilon_{(n)}$ is presenting a negligible value by comparison to the exponential value $\left(e^{(\alpha n+\beta)}\right)$.

$$\pi_{(2^n)} = e^{(n\alpha+\beta)} - \varepsilon_{(n)} \quad (172)$$

The results shown in Figure 12 in blue color present the evolution of the values $\left(Div\left(\pi_{(2^{n+1})}\right)\right)$ which is expressed as shown in (Equation 173).

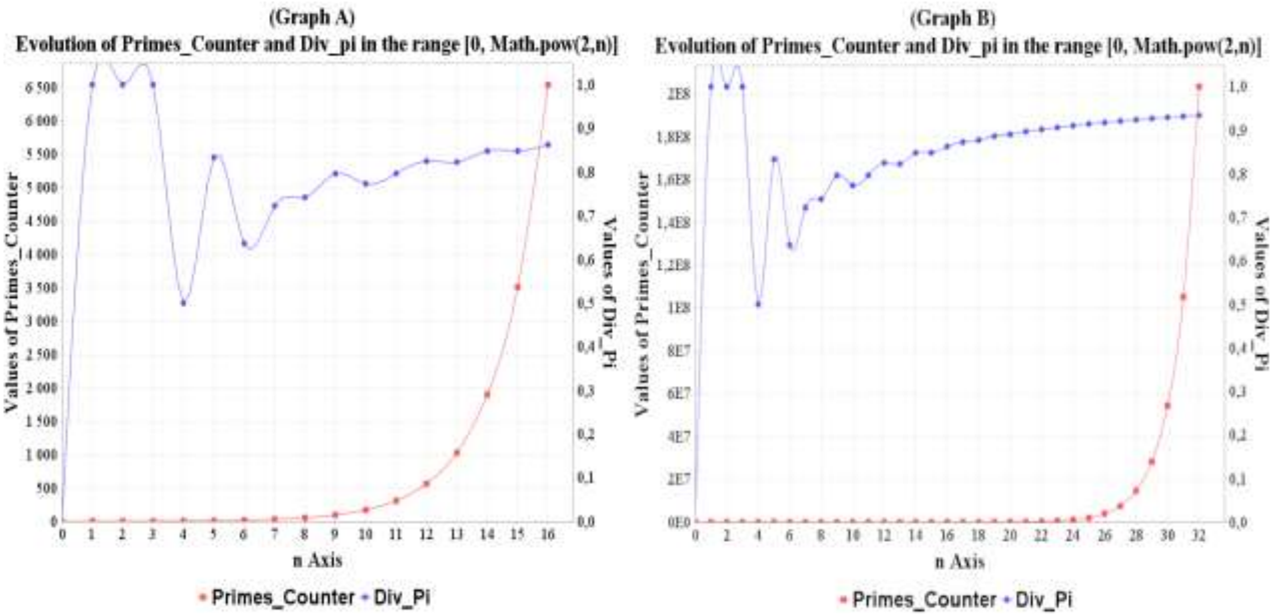

Fig. 12 Illustration graphs showing the evolution of primes counter $\left[\pi_{(2^n)}\right]$ and the coefficient $\left[Div\left(\pi_{(2^{n+1})}\right)\right]$ in the range $[\![0, 2^n]\!]$

$$Div\left(\pi_{(2^{n+1})}\right) = \frac{\pi_{(2^{n+1})} - \pi_{(2^n)}}{\pi_{(2^n)}} \quad (173)$$

According to the shown results by red color in Figure 12, we find that the amount of prime numbers follows specific patterns when we count them in each range expressed as $[\![1, 2^{n+1}]\!]$, where the blue value of $Div\left(\pi_{(2^{n+1})}\right)$ is converging toward a stable linear level when "n" goes to higher values.

The blue function of $Div\left(\pi_{(2^{n+1})}\right)$ Figure 12 has oscillating patterns at the starting values before converging toward a stable linear level when "n" goes to higher values, where the oscillations attenuate, which inspires considering the inclusion of harmonic functions, not only to analyze the distribution of prime numbers but also to identify the precise values of possible primes.

The shown oscillations by $Div\left(\pi_{(2^{n+1})}\right)$ in Figure 12, the inclusion of harmonic functions with specific frequencies in expressing the values of $Div\left(\pi_{(2^{n+1})}\right)$, which is encouraging toward extending the results of this paper while considering the Zeta function of Riemann and related harmonics to the Riemann hypothesis [29-30].

As long as we keep counting prime numbers toward infinity while using the function $\pi_{(2^n)}$, the evolution keeps following an exponential behavior as shown in Figure 11 and Figure 12, whereas the value of $Div\left(\pi_{(2^{n+1})}\right)$ (Equation 173) keeps converging toward a stable linear level with a specific limit, which we can approximate as shown in (Equation 174).

$$\lim_{n\to\infty} Div\left(\pi_{(2^{n+1})}\right) \approx (e^{\alpha} - 1) \tag{174}$$

Since the function $\pi_{(2^{n+1})}$ is having an exponential evolution, as shown in Figures 11 and 12, which we can express as $\left(\pi_{(2^{n+1})} = e^{((n+1)\alpha+\beta)} - \varepsilon_{(n)}\right)$, therefore, we can calculate the space occupied under the evolution of $\pi_{(2^{n+1})}$ in Figures 11 and 12 by using the integral shown in (Equation 175).

$$\int_0^{n+1} \pi_{(2^i)} di = \int_0^{n+1} e^{(i\alpha+\beta)} di - \int_0^{n+1} \varepsilon_{(i)} di \Rightarrow \int_0^{n+1} \pi_{(2^i)} di = \frac{1}{\alpha}\left[e^{(i\alpha+\beta)}\right]_0^{n+1} - \int_0^{n+1} \varepsilon_{(i)} di$$

$$\Rightarrow \int_0^{n+1} \pi_{(2^i)} di = \frac{1}{\alpha}\left(e^{((n+1)\alpha+\beta)} - e^{\beta}\right) - \int_0^{n+1} \varepsilon_{(i)} di \tag{175}$$

Based on the presented expression in (Equation 175), we deduce that we can express the value of $e^{((n+1)\alpha+\beta)}$ as shown in (Equation. 176).

$$e^{((n+1)\alpha+\beta)} = \alpha \int_0^{n+1} \pi_{(2^i)} di + \alpha \int_0^{n+1} \varepsilon_{(i)} di + e^{\beta} \tag{176}$$

Since the value of $\pi_{(2^{n+2})}$ is based on using the exponential function, we deduce that the integral $\alpha \int_0^{n+1} \pi_{(2^i)} di$ can be used to calculate the value of $\pi_{(2^{n+2})}$, which can also allow us to present $\pi_{(2^{n+2})}$ in terms of $\sum_{i=1}^{n+1} \pi_{(2^i)} \Delta_i$.

As a result, we can express the value of $\pi_{(2^{n+2})}$ as shown in (Equation 177).

$$\pi_{(2^{n+2})} = \alpha e^{\alpha} \int_0^{n+1} \pi_{(2^i)} di + \alpha e^{\alpha} \int_0^{n+1} \varepsilon_{(i)} di + e^{\alpha+\beta} - \varepsilon_{(n+2)} \tag{177}$$

The presented expressions in (Equation 176) and (Equation 177) highlight the possibility of using the sum of previous values expressed as $\pi_{(2^i)}$ to approximate the value of $\pi_{(2^{n+2})}$ for 10:10:00 AM prime counting. In addition, the shown expressions in (Equation 176) and (Equation 177) highlight the possibility of approximating $\pi_{(2^i)}$ by using an exponential function along with an error function $\varepsilon_{(i)}$ which have negligible values by comparison to $\pi_{(2^i)}$. Furthermore, the illustrated results in Figure 11 highlight that using the proposed conjecture in this paper (Conjecture 1) to identify possible prime numbers provides high precision in their identification, which allows approximating the number of prime numbers $\pi_{(2^i)}$ with exemplary precision.

8.4. Resulted insights from using the Collatz conjecture on prime numbers

In the previous sub-sections, we relied on using the Collatz conjecture and the unified formula of Theorem 25 in this paper to identify patterns of prime numbers, while analyzing their distribution in the group of natural numbers $\mathbb{N}$.

The resulting insights from the previous sub-section are as follows:

1) The operations of the Collatz conjecture can be re-expressed according to unified formulas, which are presented in Theorem 25, Theorem 8, and Theorem 6 in this paper, along with proofs.
2) All proposed unified formulas in this paper to re-express the operations of the Collatz conjecture are built on the group of numbers $\{2^i, 3^k\}$ where "i" and "k" are natural numbers.
3) The proposed formula in Theorem 25 re-expresses the operation of the Collatz conjecture according to one unified form interconnecting all odd numbers.
4) The proposed Theorems and formulas in this paper can be scaled up by forwarding the Collatz Conjecture and the sub-expressions of the proposed formulas in order to architect specific new formulas that can either test primality or calculate successive series of prime numbers.
5) We used the sub-expressions and variables included in the presented unified formula in Theorem 25 to identify possible prime numbers according to specific patterns.
6) We proposed four new conjectures presenting four patterns $\{PATTERN_n\ ;\ PATTERN'_q\ ;\ PATTERN'_p\ ;\ PATTERN'_v\}$ to identify possible prime numbers by using Collatz operations and the proposed unified formula in Theorem 25.

7) The proposed four patterns $\{PATTERN_n\ ;\ PATTERN'_q\ ;\ PATTERN'_p\ ;\ PATTERN'_v\}.T$he four proposed conjectures are based on re-expressing the pattern $\{2^{n-1}\ MOD\ [n]\}$ to determine whether an odd number "s" can be a prime or not.
8) The four proposed patterns $\{PATTERN_n\ ;\ PATTERN'_q\ ;\ PATTERN'_p\ ;\ PATTERN'_v\}$ are expressed according to a distributed architecture of terms, which allows for computing them while using more reduction patterns and more parallel threads.
9) The four proposed patterns $\{PATTERN_n\ ;\ PATTERN'_q\ ;\ PATTERN'_p\ ;\ PATTERN'_v\}$ are expressed according to a distributed architecture of terms, where each term has a lower exponential value than the pattern $\{2^{n-1}\ MOD\ [n]\}$, which allows them to occupy less memory space as presented in the four proposed conjectures.
10) Computing the four proposed patterns $\{PATTERN_n\ ;\ PATTERN'_q\ ;\ PATTERN'_p\ ;\ PATTERN'_v\}$. Using more reduction patterns and more parallel threads can optimize the exploitation of memory space and processing time.
11) We used the proposed pattern $PATTERN_n$ in Conjecture 1, to count prime numbers $\left(\pi_{c1}(2^L)\right)$ in different ranges expressed as $[\![0, 2^L]\!]$ where $L \in \{\mathbb{N} - \{0\}\}$.
12) Counting prime numbers by using $\left(\pi_{c1}(2^L)\right)$, while relying on the proposed conjecture (Conjecture 1), is highly approximating the actual number of prime numbers in Figure 11, which we calculated by using the sieve algorithm $\left(\pi_s(2^L)\right)$.
13) The error difference $\left(\pi_{c1}(2^L) - \pi_s(2^L)\right)$ between counting prime numbers while using Conjecture 1 and counting them by using the sieve algorithm is negligible by comparison to the actual number of primes, as shown in Figure 11.
14) We can use the proposed patterns in this paper as an official way to determine primes, whereas searching for specific criteria and testing conditions in order to eliminate the odd numbers that generate the error difference $\left(\pi_{c1}(2^L) - \pi_s(2^L)\right)$ in proposed patterns.
15) The number of prime numbers using the function $\left(\pi(2^n)\right)$ is having an exponential behavior with a negligible error, as shown in Figures 11 and 12.
16) The number of prime numbers by using the function $\left(\pi(2^n)\right)$ can be formulated by relying on an exponential function while including the use of a negligible error as follows: $\pi_{(2^n)} = e^{(n\alpha+\beta)} - \varepsilon_{(n)}$.
17) The exponential evolution of the counted amounts of prime numbers when using the function $\left(\pi(2^n)\right)$ are allowing us to calculate the function $Div\left(\pi_{(2^{n+1})}\right) = \frac{\pi_{(2^{n+1})} - \pi_{(2^n)}}{\pi_{(2^n)}}$ in order to analyze the distribution of primes.
18) The function $Div\left(\pi_{(2^{n+1})}\right)$ is converging toward a stable linear level when the value of "n" is scaling up toward higher values, as shown in Figure 12.
19) The shown convergence of the function $Div\left(\pi_{(2^{n+1})}\right)$ in Figure 12, when the values of "n" are increasing, it leads to expressing the stable linear level of convergence as follows: $\lim_{n\to\infty} Div\left(\pi_{(2^{n+1})}\right) \approx (e^\alpha - 1)$.
20) The shown function $Div\left(\pi_{(2^{n+1})}\right)$ in Figure 12 shows oscillating evolutions at the starting values, whereas converging toward a more stable linear level when the values of "n" are increasing.
21) The shown oscillations by the function $Div\left(\pi_{(2^{n+1})}\right)$ in Figure 12, this can be explained by the inclusion of harmonic functions with different frequencies.
22) The inclusion of harmonic functions in expressing the function $Div\left(\pi_{(2^{n+1})}\right)$ can be forwarded to analyze the distribution of prime numbers while deploying these harmonics, or even be used to identify specific values of primes or other specific patterns interconnecting them.
23) The inclusion of harmonic functions with different frequencies in expressing the function $Div\left(\pi_{(2^{n+1})}\right)$ is leading toward extending the results of this paper while considering the Zeta function of Riemann and related harmonics to the Riemann hypothesis.
24) Expressing the number of prime numbers $\left(\pi(2^{n+1})\right)$ by using an exponential function $\left(e^{((n+1)\alpha+\beta)} - \varepsilon_{(n+1)}\right)$ is allowing us to calculate the space under this function by using the integral.
25) Calculating the space under the exponential function $\left(e^{((n+1)\alpha+\beta)} - \varepsilon_{(n+1)}\right)$ is allowing us to include the use of integral as follows: $\int_0^{n+1} \pi_{(2^i)} di = \frac{1}{\alpha}\left(e^{((n+1)\alpha+\beta)} - e^\beta\right) - \int_0^{n+1} \varepsilon_{(i)} di$.

26) Using an integral to calculate the space under the function $\left(e^{((n+1)\alpha+\beta)} - \varepsilon_{(n+1)}\right)$ is opening the way to calculate the value of $\left(\pi(2^{n+2})\right)$ as follows: $\pi_{(2^{n+2})} = \alpha e^{\alpha} \int_0^{n+1} \pi_{(2^i)} di + \alpha e^{\alpha} \int_0^{n+1} \varepsilon_{(i)} di + e^{\alpha+\beta} - \varepsilon_{(n+2)}$.
27) Using an integral to calculate the space under the function $\left(e^{((n+1)\alpha+\beta)} - \varepsilon_{(n+1)}\right)$ is opening the way to calculate the value of $\left(\pi(2^{n+2})\right)$ while relying on the precedent values of $\left(\pi(2^i)\right)$ whereas $i \in [\![1, n+1]\!]$.
28) Using an integral to calculate the space under the function $\left(e^{((n+1)\alpha+\beta)} - \varepsilon_{(n+1)}\right)$ is opening the way to calculate the value of $\left(\pi(2^{n+2})\right)$ while relying on the precedent values of error $\left(\varepsilon_{(i)}\right)$ whereas $i \in [\![1, n+1]\!]$.
29) Using an integral to express the value of $\left(\pi(2^{n+2})\right)$ is opening the way to present the value of $\left(\pi(2^{n+2})\right)$ in terms of $\sum_{i=1}^{n+1} \pi_{(2^i)} \Delta_i$.
30) Using an integral to express the value of $\left(\pi(2^{n+2})\right)$ is opening the way to present the value of $\left(\pi(2^{n+2})\right)$ in terms of the errors $\sum_{i=1}^{n+1} \varepsilon_{(i)} \Delta_i$.

9. Conclusion

This paper presents algebraic proofs of the correctness of the Collatz conjecture by starting with the development of specific unified formulas forwarding the statements and logic of the Collatz conjecture, then building these proofs based on these formulas while relying on recurrence (induction).

This paper provides specific unified formulas linking all odd numbers by using the operations of the Collatz conjecture. Then, it scales up these formulas to express any natural number according to the operations of the Collatz conjecture.

This paper proves that there is no divergence when we keep repeating the operations of the Collatz conjecture on any natural number different from zero. In addition, this paper demonstrates that there is no Collatz loop where the operations of the Collatz conjecture circulate back to the starting number, "n_1" of the loop, except the loop where $(n_1 = 1)$.

The proposed unified formulas in this paper are used to prove that if any odd number, "n_s" is expressed according to these formulas, then this odd number is converging toward the value "1" when we keep repeating the operations of the Collatz conjecture on it. In addition, we used these formulas to prove that if any odd number, "n_s" is expressed according to these formulas, then other odd numbers are expressed as "$2n_s + 1$" and "$2n_s + 3$" are also converging to the value "1" when we keep repeating the operations of the Collatz conjecture on them. Furthermore, we used these formulas to prove that if any odd number, "n_s" is expressed according to these formulas, then any even number expressed as "$2^k n_s \mid k \in \{\mathbb{N} - 0\}$" is also converging toward the value "1" when using these operations of the Collatz conjecture.

This paper also uses the operations of the Collatz conjecture while relying on the proposed unified formulas to identify specific patterns revealing possible prime numbers, while analyzing their distribution over the group of natural numbers ℕ, which allowed highlighting different insights on the characteristics of prime numbers.

This paper also presents computational results by providing developed codes in Python and Java programming languages to compute the proposed unified formulas and to illustrate specific graphs that guide the development of proofs and theorems, which also allowed highlighting specific patterns and characteristics of prime numbers.

Conflicts of Interest

The author states there is no conflict of interest.

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