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Original Article

Solving nth Order Differential Equations and Polynomial Equations of nth Degree by Using Unified Formulas Composed of Radical Expressions

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Abstract - *This paper presents new theorems solving differential equations of nth-order, where the possibility of calculating solutions nearly in parallel is considered. These theorems are based on an engineering methodology that forwards the concept of solutions architecting according to an engineering approach, where the process of developing expressions and sub-expressions of solutions is based on requirements engineering, analysis, design, and then developing the complete algebraic formulas of solutions to be scalable and projectable on any orders or degrees of equations. The new engineering methodology in this paper is initially developed to solve nth degree polynomial equations in general forms while using specific new unified formulas composed of radical expressions, which allow calculating the roots nearly in parallel. Then, this paper forwards this engineering methodology by using the roots of nth degree polynomial equations in general forms to solve differential equations of nth order. This methodology presents specific logic, statements, conditions, mathematical expressions, and new unified formulas that allow calculating the solutions of nth degree polynomials and nth-order differential equations. In addition, this paper presents the results of applying this engineered methodology to differential and polynomial equations of fourth degree, fifth degree, and sixth degree. This methodology is built on precise details that provide step-by-step logic and formulas to calculate the solutions, which allow concretizing multiple theorems, formulating the algebraic expressions of all solutions for different orders and degrees of equations, where the possibility of calculating the values of these solutions nearly in parallel while relying on distributed structures of terms.*

Keywords - *Differential equations***,** *New engineered methodology, Polynomial equations, Roots parallel calculations, Solutions architecting, Solving nth degree polynomials, Solving nth order differential equations.*

1. Introduction

In the fields of mathematics, differential equations are specific forms of equations expressed by relying on derivatives, with at least one unknown function and its derivatives.

Differential equations are widely used in physics, economics, biology, automation, industries, and engineering, because these differential equations formulate the relations between quantitative functions and their rates of change, which allow studying, analyzing, controlling, and even predicting the values and evolutions of these functions.

These differential equations were introduced as a part of Calculus, which was invented in the 17^{th} century by Isaac Newton [1-3] and Gottfried Leibniz [2]. Since then, there have been many books and research papers tackling the solvability of differential equations and their classifications according to the properties of equations.

Among the classes of differential equations, we have ordinary differentials [4], partial differentials [5], linear differentials [6], non-linear differentials [7], homogeneous differentials [8], and heterogeneous differentials [9]. In addition, there are other classes of differential equations that vary depending on context and properties of equations [10-12].

Ordinary differentials are a specific form of differential equations with exactly one unknown function and its derivatives based on one unknown variable, whereas partial differentials are more complex forms of differential equations handling multiple variables by at least one unknown function and its derivatives.

Linear differentials are a category of differential equations that are linear in the unknown function and its derivatives, so they can be written in the form $\left\{\left(\sum_{i=0}^{i=u} a_i(x) * y^{(i)}\right) = b(x)\right\}$. The expressions $a_0(x),\dots,\ a_n(x)$ and $b(x)$ are arbitrary differentiable functions that do not need to be linear, whereas $y',\dots,\ y^{(n)}$ are the consecutive derivatives of an unknown function *y* in term of the variable *x*.

Nonlinear differentials are more complex forms of differential equations where the unknown function (or its derivatives) does not appear in a linear way. This means the equation cannot be expressed as a sum of terms where each term is a constant (or a function of independent variables) multiplied by either the unknown function of the dependent variable or one of its derivatives.

A differential equation can be referred to as homogeneous in two scenarios. The first prospect is when we describe a first-order differential equation as homogeneous if we can express it as follows: $\{f(x,y)dy = g(x,y)dx\}$ where *f* and *g* are homogeneous functions of the same degree of *x* and *y*. The second prospect is when we describe a differential equation as being homogeneous if it is a homogeneous function of the unknown function and its derivatives. In simpler terms, this means that if we scale the input variables by a constant, the output of the function scales by a power of that constant. $\{f(tx_1, tx_2, \dots, tx_m) = t^k f(x_1, x_2, \dots, x_m)\}$.

Heterogeneous differentials are differential equations where the right-hand side of the equation is not equal to zero, in case we dedicate this side of the equation to express only the independent variables. In simpler terms, it is a differential equation that includes a non-zero function of the independent variable(s) on one side of the equation or containing constant terms that are not multiplied by the function of the dependent variable or its derivatives.

Among the most common methods to solve differential equations is converting them into polynomial forms of equations, then using the roots of these polynomials to express the solutions of differential equations. This method is built on the inductive relation between the order of differential equations and the degrees of polynomial equations.

In order to help solve the nth-order differential equations, we need to start by converting them to polynomial forms of nth degree, then using the roots of these polynomials to express the solutions of corresponding differentials, which usually leads to using numerical analysis algorithms to find these roots, especially when the equations do not have symmetries and the nth degree is higher than four.

Solving nth order differential equations and nth degree polynomials has been challenging for centuries for mathematicians, scientists, and researchers, particularly when looking for algebraic terms that may help express the roots of equations. This encountered challenge is due to the complexity of calculations that can outstrip the previsions of the human mind, especially when orders and degrees of equations are making them transcend above the quartic form.

The complexity of mathematical calculations during the attempts to solve polynomial equations of high degrees is mainly due to using radicals while adopting particular approaches. In addition, reaching a point of having highly complex outcomes where the form of the resulting equation is far from foreseeing a simplification or reduction, will lead to conducting exhaustive changes on the used approach or even replacing it by adopting a different one.

Adopting a specific approach to solve particular degrees or orders of equations leads to limitations in the resulting forms of equations, where simplifications and reductions are harder to conduct by comparison to the starting point. In addition, conducted calculations may expand at a high rate when searching for the solution while trying different approaches that may require restarting calculations from scratch. Furthermore, the induced complexity of the resulting forms of equations may lead to adopting a narrowed solution to be used on a specific form of polynomials or differential equations that have particular conditions stated by the values of included coefficients.

As a result, discovering unified solution formulas for polynomials and differential equations in general forms and in complete forms is more challenging when relying on a research methodology where calculations and logic may need to be restarted from scratch when the approach is drastically modified or even replaced.

Therefore, we rely on an engineering methodology where we construct the appropriate approach bit by bit based on patterns and characteristics that should be met. Then, we use this built approach to architect the adequate roots and to structure their involved terms and sub-terms. Then, we forward logic and calculations toward engineering the mathematical formulas of all roots, in order to allow the calculation of all expected solutions nearly in parallel.

Relying on this engineering methodology to solve polynomials and differential equations avoids us restarting the logic and calculations from scratch, and allows us to keep relying on the exact approach and results of calculations while conducting only slight modifications, when necessary, toward reaching the final forms of unified formulas. In addition, this engineering methodology allows us to project the exact approach and extend the same logic toward solving higher orders of differentials and higher degrees of polynomials.

In this engineering methodology, the axis of focus is building and architecting the necessary formulas according to a scalable logic, where we start from requirements engineering toward designing the starting point, the path, the destination, and the structure of the expected final results. As a result, the calculations and adopted reasoning follow a pre-designed path toward structuring the unified formulas of nested roots.

The advantage of this paper is presenting specific theorems listing algebraic formulas designed to solve fourth degree, fifth degree, and sixth degree polynomial equations in general forms by using radical expressions, where the possibility of calculating the values of all roots is nearly in parallel. Then, this paper presents theorems toward solving differential equations of fourth order, fifth order, and sixth order while providing the same convenience of calculating solutions nearly in parallel.

This paper also presents the engineered requirements and techniques that lead to architect the results of proposed theorems, while allowing to scale the roots of nth degree polynomial equations toward calculating the solutions of nth order differential equations.

This paper is a principal step in our work of solving nth order differential equations and nth degree polynomials based on projecting the presented methods and results in this paper on other equations where orders and degrees are higher than six, which will be presented in other articles.

The proposed concept in this article about architecting solutions according to a distributed structure of terms can extend itself to different problems in geometry, number theory, and algebra in general, because this concept introduces an engineering methodology based on identifying patterns and characteristics that allow forwarding calculations and expressions toward specific converging points where the results are built step-by-step and not only searched.

This engineering methodology was first used to develop the solutions of quartic polynomial equations in general forms [13], by building the unified formulas of the four roots of any quartic equation, which allow calculating the four solutions nearly simultaneously. Then, the same engineering methodology was scaled to solve quantic equations [14] in complete forms by proposing the necessary unified formulas of roots to calculate the five solutions nearly in parallel. In addition, this engineering methodology was used to solve sixth-degree polynomial equations [15] in complete form while providing the possibility of calculating the six solutions nearly in parallel. Therefore, this paper allows us to scale this engineering methodology and its results on nth order differential equations and nth degree polynomials.

In addition, this paper presents new theorems developed to solve nth order differential equations and nth degree polynomial equations while providing necessary logic, conditions, parameters, and formulas to calculate the solutions nearly in parallel by extending the proposed methodologies.

Furthermore, this paper presents new additional solutions for nth order differential equations that allow interconnecting many arbitrary points, which opens the way to scale up the range of using these differential equations in business analytics, data analytics, predictive analysis, and systems control.

Because the content of this paper is original, and it embeds many new proposed formulas, mathematical expressions, and theorems which are built on extendable logic, this paper will focus on presenting these theorems and their formulas in a scaling manner, while relying on relevant proofs from the papers [13], [14], and [15].

The contents of this paper are structured as follows: Section 2 presents the used engineering methodology and its developed requirements and techniques to solve nth order differential equations. Section 3 presents the methodology used to solve nth degree polynomial equations. Section 4 presents six theorems developed to solve fourth-degree polynomial equations, allowing for the determination of the number of complex solutions. Section 5 presents developed theorems and formulas to solve fourth-order differential equations, allowing for determining the amount of expected complex functions among the solutions. Section 6 presents two theorems developed to solve fifth-degree polynomial equations. Section 7, presenting developed theorems and formulas to solve fifth-order differential equations. Section 8, presenting two theorems developed to solve sixth-degree polynomial equations. Section 9, presenting developed theorems and formulas to solve sixth-order differential equations. Section 10, presenting a developed theorem to solve nth degree polynomial equations. Section 11, presenting new developed theorems and formulas to solve nth order differential equations. Finally, Section 12 is for the conclusion.

2. Methodology to Solve nth Order Differential Equations

This section presents the requirements, the techniques, and the formulas to solve nth order differential Equations according to an organized method, step by step, while relying on the proposed methodology in this paper to solve nth degree polynomial equations.

In addition, this section presents new unified solutions for nth order differential equations that allow interconnecting many arbitrary points, which allow scaling up their use in business analytics, data analytics, and predictive analysis by mapping variables and datums of collected data to each other while tracking macro transitions and micro variations among them.

1. Expressing a differential equation of nth order according to the form $\sum c_i f_{(x)}^{(i)} = C$ where $f_{(x)}^{(i)}$ is the derivation of order (i), whereas c_i and C are arbitrary values.
2. Supposing the function $f_{(x)}$ is expressed as follows: $f_{(x)} = e^{sx+a} + b$.
3. Expressing the derivative of order (i) of the function $f_{(x)}$ as follows $f_{(x)}^{(k)} = s^i e^{sx+a}$, where $(i > 0)$.
4. Converting the nth order differential equation to be presented as an nth degree polynomial form, which is expressed as follows: $e^{sx+a} * \sum c_i s^i = C - bc_0$.
5. In order to identify the value of the variable (b), we consider the equation $(bc_0 + e^{sx+a} * \sum c_i s^i = C)$ at the point of calculated root $(s = s_k)$.
6. Solving the equation $(b * c_0 = C)$ in order to identify the value of the variable (b), which is as follows: $\left[b = \frac{C}{c_0}\right]$ where $c_0 \neq 0$.
7. Solving the nth degree polynomial equation $\sum c_i s^i = 0$ in order to calculate *n* roots, which we can present in the form of the group $\{s_1; \dots; s_n\}$.
8. Using the proposed engineered methodology to solve the nth degree polynomial equation $\sum c_i s^i = 0$, or using numerical analysis.
9. In order to calculate all the roots nearly in parallel for the polynomial equation $\sum c_i s^i = 0$, we use the proposed engineering methodology to solve nth degree polynomial equations.
10. We identify an initialization condition for $f_{(x)}$ where $(x = 0)$.
11. The initialization condition for $f_{(x)}$ where $(x = 0)$ should be presented as an arbitrary value, which is to be expressed as follows: $f_{(x=0)} = I_0$.
12. The value of the variable (a) should be calculated by relying on the initialization condition.
13. The value of the variable (a) is to be identified by using the expression $a = \log(I_0 - b)$, which allows calculating the value of the variable (a) as follows: $a = \log\left(I_0 - \frac{C}{c_0}\right)$ where $c_0 \neq 0$.
14. The solution of the nth order differential equation should be expressed as $\mathrm{DS_k} = e^{s_k x + \log\left(I_0 - \frac{C}{c_0}\right)} + \frac{C}{c_0}$, where s_k is a calculated root for the nth degree polynomial equation $\sum c_i s^i = 0$.
15. When using the proposed engineering methodology to solve nth degree polynomials, we became able to calculate all the roots $\{s_1; \dots; s_n\}$ nearly in parallel, which allows calculating all the solutions $\{DS_1; \dots; DS_n\}$ of an nth order differential equation nearly in parallel.
16. When having a differential equation of the nth order $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, we can identify T arbitrary values $\{I_k \mid k \in [\![0, T-1]\!] \; and \; T \in [\![1, n]\!]\}$, where each arbitrary value I_k is identified at a specific referencing point x_k by using the expression $f(x_k) = I_k$.
17. If there is an amount of T different roots $\{s_{p_1}; s_{p_2}; \dots; s_{p_T}\}$ for the nth degree polynomial equation $(\sum c_i s^i = C)$ corresponding to the nth order differential equation $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, then we can identify T arbitrary values $\{I_k \mid k \in [\![0, T-1]\!] \; and \; T \in [\![1, n]\!]\}$, where each arbitrary value I_k is identified at a specific referencing point x_k which is logically relevant to the root s_{p_k} in terms of distribution, behavior, or ratio of change.

18. After identifying T arbitrary values $\{I_k \mid k \in [\![0, T-1]\!] \; and \; T \in [\![1, n]\!]\}$, where each arbitrary value I_k is identified at a specific referencing point x_k by using the expression $f(x_k) = I_k$, we can interconnect these arbitrary points to each other by using the solutions of the differential equation.
19. If there are two different roots $\{s_a;\ s_{b \neq a}\}$ for the nth degree polynomial equation $(\sum c_i s^i = C)$ corresponding to the nth order differential equation $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, then we can calculate other new solutions for the differential equation, which can be expressed as $\left\{DS'_{n+1} = R'_{(I_1)} e^{s_b x} + \left(R'_{(I_0)} - \frac{C}{c_0} - R'_{(I_1)}\right) e^{xS_a} + \frac{C}{c_0};\ DS'_{n+2} = R'_{(I_1)} e^{xS_a} + \left(R'_{(I_0)} - \frac{C}{c_0} - R_{(I_1)}\right) e^{xS_b} + \frac{C}{c_0}\right\}$.
20. If there are three different roots $\{s_a;\ s_b;\ s_c\}$ for the nth degree polynomial equation $(\sum c_i s^i = C)$ corresponding to the nth order differential equation $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, then we can calculate other new solutions for the differential equation, which can be expressed as $\left\{DS'_{(n+i>n)} = R'_{(I_2)} e^{xS_3} + \left(R'_{(I_1)} - R_{(I_2)}\right) e^{xS_2} + \left(R'_{(I_0)} - \frac{C}{c_0} - R'_{(I_1)}\right) e^{xS_1} + \frac{C}{c_0};\ where\ s_k \in \{s_a;\ s_b;\ s_c\}\right\}$
21. If there is an amount of T different roots $\{s_{p_1};\ s_{p_1};\ \dots;\ s_{p_T}\}$ for the nth degree polynomial equation $(\sum c_i s^i = C)$ corresponding to the nth order differential equation $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, then we can calculate other new solutions for the differential equation, which can be expressed as $\left\{DS'_{(n+i>n)} = R'_{(I_{(T-1)})} e^{xS_T} + \sum_{L=1}^{L=T-2} \left[\left(R'_{(I_L)} - R'_{(I_{(L+1)})}\right) e^{xS_{(L+1)}}\right] + \left(R'_{(I_0)} - \frac{C}{c_0} - R'_{(I_1)}\right) e^{xS_1} + \frac{C}{c_0};\ where\ s_k \in \{s_{p_1};\ s_{p_1};\ \dots;\ s_{p_T}\}\right\}$

3. Engineered Methodology to Solve nth Degree Polynomials

The presented methodology in this paper to solve general forms of nth degree polynomial equations is based on architecting the roots of these equations according to a distributed structure of terms while relying on radical expressions.

In addition, this engineering methodology relies on developing specific patterns into the structure of niched roots in order to help converge calculations, while eliminating degrees of polynomials.

Furthermore, this developed methodology is built on an engineering logic where roots are predesigned before being expressed according to unified mathematical formulas, which support the structuring of expressions for all expected roots of the aimed polynomial equation.

The presented methodology in this paper leads to defining a list of engineered requirements and techniques according to a scaled logic, which helps to develop the necessary unified formulas to calculate the roots of nth degree polynomial equations in general forms, while enabling the calculation of the values of possible roots nearly in parallel.

The results of our engineered requirements, techniques, and formulas according to the developed methodology are described as follows:

1. Roots should be expressed according to a distributed structure of terms $\left\{\sum_{i=0}^{i=u} \mathrm{T_i}\right\}$, which will be multiplied by each other during calculations.
2. Each included term in the distributed structure of roots should be expressed according to the simplest possible radicality.
3. All included terms in the distributed structure of roots should either be constants or be radical expressions.
4. The included constant terms in the distributed structure of roots should allow for eliminating specific parts with specific degrees from a polynomial equation.
5. We adapt a polynomial equation of the nth degree $\left\{\left(\sum_{i=0}^{i=n} a_i X^i\right) = 0\right\}$ where $\{a_n \neq 0\}$ by presenting it as $\left\{\left(\sum_{i=0}^{i=n} \frac{a_i}{a_n} X^i\right) = 0\right\}$.
6. We use the expression. $\left\{X = \frac{-a_{n-1}}{n a_n} + \frac{Y}{n}\right\}$ to eliminate the term of degree $(n-1)$ from a polynomial equation of nth degree $\left\{\left(\sum_{i=0}^{i=n} \frac{a_i}{a_n} X^i\right) = 0\right\}$ when $(n-1)$ is an odd value, or when this elimination simplifies calculations.
7. All included radical terms in each root should have the same radicality, in order to converge the resulting expressions during calculations. Therefore, we choose them to have a radicality of the square root.

8. Each included radical term in the distributed structure of a root $\left\{\sum_{i=0}^{i=u} \mathrm{T_i}\right\}$ should be expressed according to a sum of simple radical terms $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i=0}^{i=u} x_i\right) = \left(\sum_{i=0}^{i=u} \sqrt{y_i}\right)\right\}$, when the degree of the polynomial equation is equal to four.
9. Each included radical term in the distributed structure of a root $\left\{\sum_{i=0}^{i=u} T_i\right\}$ should be expressed according to a multiplication of at least two different sub-terms $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i\neq j} x_i\, x_j\right)\right\}$ when the degree of the polynomial equation surpasses four.
10. When the degree of the polynomial equation surpasses four, each included sub-term $\{x_l\}$ in the distributed structure of a root $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i\neq j} x_i\, x_j\right)\right\}$ should appear in multiple distributed terms in order to allow further factorizations.
11. When the degree of the polynomial equation surpasses four, each included sub-term $\{x_l\}$ in the distributed structure of terms $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i\neq j} x_i\, x_j\right)\right\}$ should be presented according to a radical expression of a cubic root, a quadratic root, or a constant.
12. Combinations among included sub-terms in a root should allow expressing the values of involved coefficients in a polynomial equation.
13. The included sub-terms in the distributed structure of terms should allow neutralizing their contents when they are multiplied by each other in order to have simplified results.
14. The included sub-terms in the distributed structure of terms should allow eliminating radicality when they are raised to the power of higher polynomial degrees.
15. The included sub-terms in the distributed structure of terms should allow for the elimination of radicality when they are multiplied by each other.
16. The included sub-terms in the distributed structure of terms should allow forward calculations to suppress terms that have odd values of polynomial degrees.
17. The included sub-terms in the distributed structure of terms should allow forward calculations to either suppress terms of the highest degrees or suppress terms of the lowest degrees.
18. The distributed structure of terms $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i\neq j} x_i\, x_j\right)\right\}$ should include a sub-term $\{x_1\}$ presented according to a radical expression of the cubic root, where $\left(x_1 = \sqrt{\frac{-b}{3} + \frac{1}{3}\sqrt[3]{-\frac{\mathrm{D}}{2} + \sqrt{\left(\frac{\mathrm{D}}{2}\right)^2 + \left(\frac{\mathrm{C}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{\mathrm{D}}{2} - \sqrt{\left(\frac{\mathrm{D}}{2}\right)^2 + \left(\frac{\mathrm{C}}{3}\right)^3}}}\right)$.
19. The distributed structure of terms should include two sub-terms $\{x_2, x_3\}$ presented according to radical expressions of quadratic roots, where $\left(x_2 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} + \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}}\right)$ and $\left(x_3 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} - \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}}\right)$.
20. In order to eliminate high-degree expressions in a polynomial equation, while allowing calculations to converge, we use a constant value $\{\alpha_1\}$ expressed by using included sub-terms in the distributed structure of the root, where $\{\alpha_1 = \sum x_i^2\}$.
21. In order to eliminate average degree expressions in a polynomial equation, while allowing calculations to converge, we use a constant value $\{\alpha_2\}$ expressed by using included sub-terms in the distributed structure of the root, where $\left\{\alpha_2 = \sum_{i\neq j} x_i^2\, x_j^2\right\}$
22. In order to eliminate low-degree expressions in a polynomial equation, while allowing calculations to converge, we use a constant value $\{\alpha_3\}$ expressed by using included sub-terms in the distributed structure of the root, where $\left\{\alpha_3 = \sum_{i\neq j\neq k} x_i x_j x_k\right\}$
23. In order to eliminate the lowest degree expressions in a polynomial equation, while allowing calculations to converge, we use a constant value $\{\alpha_4\}$ expressed by using included sub-terms in the distributed structure of the root, where $\left\{\alpha_4 = \sum_{i\neq j\neq k\neq l} x_i x_j x_k\, x_l\right\}$.
24. In order to eliminate odd degrees of expressions in a polynomial equation, while allowing calculations to converge, we reformulate the solution $\left\{X = \left(\sum_{i\neq j} x_i\, x_j\right)\right\}$ to be presented as $\{X = (\sum x_i)^2 - \sum x_i^2 = (\sum x_i)^2 - \alpha_1\}$.
25. In order to reduce the degree of expressions in a polynomial equation, while allowing calculations to converge, we reformulate the second-degree form $\left\{X^2 = \left(\sum_{i\neq j} x_i\, x_j\right)^2\right\}$ to be presented as $\{X^2 = \alpha_2 + 2\alpha_3(\sum x_i) + 6\alpha_4\}$.

26. In order to reduce the degrees of complex expressions in a polynomial equation, while allowing calculations to converge, we reformulate the quartic form $\left\{X^4 = \left(\sum_{i\neq j} x_i\, x_j\right)^4\right\}$ to be presented as $\{X^4 = 4(\sum x_i)^2\, \alpha_3^2 + 4\alpha_3(\sum x_i)[\alpha_2 + 6\alpha_4] + [\alpha_2 + 6\alpha_4]^2\}$.
27. We use the proposed constants. $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and the expression $\left\{X = \left(\sum_{i\neq j} x_i\, x_j\right)\right\}$ in order to re-express the polynomial equation $\left\{\left(\sum_{i=0}^{i=n} \frac{a_i}{a_n} X^i\right) = 0\right\}$ to be represented as $\left\{\left(\sum_{i=0}^{i=n} \gamma_i Z^i\right) = 0\right\}$ where $\{Z = (\sum x_i)\}$.
28. When the solution of an nth-degree polynomial equation is expressed as $\left\{X = \left(\sum_{i\neq j} x_i\, x_j\right)\right\}$, we adopt a constant value $\left\{\frac{\Gamma}{\alpha_3} = \frac{\Gamma}{\sum_{i\neq j\neq k} x_i x_j x_k} = V\right\}$ where Γ is expressed as a function of $\{(\sum x_k)\}$; in order to converge calculations during the process of solving equations.
29. The resulting polynomial expression at the final stages of forward calculations should have only one unknown variable expressed by including the use of one sub-term incorporated in the distributed structure of roots, which is to be considered as an unknown variable.
30. The resulting polynomial form at the final stages of forward calculations should have a lower degree than the starting point, or should not have the term of constant value (the term with zero degree). Otherwise, the resulting polynomial form should not have any terms with odd degrees.
31. The included sub-terms $\{x_i\}$ in the distributed structure of a root $\left\{(\sum \mathrm{T_i} = \sum x_i)\ or\ \left(\sum \mathrm{T_i} = \sum_{i\neq j} x_i\, x_j\right)\right\}$ should also be used in the calculation of all other roots by changing the signs of these sub-terms, while exploiting the involved coefficients in the polynomial equation.
32. Reusing the included sub-terms in the structure of a root while only changing their signs should allow calculating the values of different roots $\{Solution_k = \sum \pm \mathrm{T_i}\}$ nearly in parallel.

4. Solving Fourth Degree Polynomial Equations

This section presents the developed theorems and formulas to solve fourth-degree polynomial equations by using the proposed engineering methodology in this paper to solve nth degree polynomial equations in general forms and in complete forms.

4.1. First proposed theorem for fourth-degree polynomials

This section presents the first developed theorem to solve fourth-degree polynomial equations that are expressed according to the form: $x^4 + cx^2 + dx + e = 0$, by converting this fourth-degree equation into the form of a third-degree polynomial, which we can express as follows: $x_0^3 + \frac{c}{2}x_0^2 + \frac{c^2-4e}{16}x_0 - \frac{d^2}{64} = 0$. The proof of this theorem is detailed in [13].

Theorem 1

A fourth-degree polynomial equation under the expression (Equation 1), where coefficients belong to the group of numbers $\mathbb{R}$, has four solutions.

$$x^4 + cx^2 + dx + e = 0 \quad (1)$$

$$-2\left[\sqrt{x_0}^2 + \sqrt{x_1}^2 + \sqrt{x_0}^2\right] = c \quad (2)$$

$$\text{For } d \le 0\text{: } -8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2} = d \quad (3)$$

$$\text{For } d \ge 0\text{: } 8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2} = d \quad (4)$$

$$x_{0,1} = \frac{-B}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} + \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} - \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} \mid \left\{B = \frac{c}{2}\ ;\ D^{;} = \frac{-27d^2-2c^3+72ce}{64}\ ;\ C^{;} = -\frac{3c^2+36e}{16}\right\} \quad (5)$$

$$x_{0,1} = \frac{-B}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} + \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} - \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} \mid \left\{B = \frac{c}{2}\ ;\ D^{;} = \frac{-2c^3+72ce}{64}\ ;\ C^{;} = -\frac{3c^2+36e}{16}\right\} \quad (6)$$

If $d < 0$, and by using the expressions of $x_{0,1}$ in (Equation 5), c in (Equation 2) and d in (Equation 3), the four solutions for (Equation 1) are as shown in (Equation 7), (Equation 8), (Equation 9) and (Equation 10).

$$\text{Solution 1: } S_{1,1} = \sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{7}$$

$$\text{Solution 2: } S_{1,2} = -\sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{8}$$

$$\text{Solution 3: } S_{1,3} = -\sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{9}$$

$$\text{Solution 4: } S_{1,4} = \sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{10}$$

If $d > 0$, and by using the expressions of $x_{0,1}$ in (Equation 5), c in (Equation 2) and d in (Equation 4), the four solutions for (Equation 1) are as shown in (Equation 11), (Equation 12), (Equation 13) and (Equation 14).

$$\text{Solution 1: } S_{2,1} = -\sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{11}$$

$$\text{Solution 2: } S_{2,2} = -\sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{12}$$

$$\text{Solution 3: } S_{2,3} = \sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \qquad 13)$$

$$\text{Solution 4: } S_{2,4} = \sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \qquad 14)$$

If $d = 0$, and by using the expressions of $x_{0,1}$ in (Equation 6) and c in (Equation 2), the four solutions for (Equation 1) are as shown in (Equation 15), (Equation 16), (Equation 17), and (Equation 18).

$$\text{Solution 1: } S_{3,1} = \sqrt{x_{0,1}} + \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \tag{15}$$

$$\text{Solution 2: } S_{3,2} = -\sqrt{x_{0,1}} - \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \tag{16}$$

$$\text{Solution 3: } S_{3,3} = -\sqrt{x_{0,1}} + \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \tag{17}$$

$$\text{Solution 4: } S_{3,4} = \sqrt{x_{0,1}} - \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \quad (18)$$

4.2. Second Proposed Theorem for Fourth-Degree Polynomials

This section presents the second developed theorem for fourth-degree polynomials that are expressed according to the form: $x^4 + cx^2 + dx + e = 0$. This theorem identifies whether this fourth-degree equation accepts complex solutions with imaginary parts different from zero. The proof of this theorem is detailed in [13].

Theorem 2

Considering the fourth-degree polynomial equation $x^4 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$. If $e \neq 0$ and $c > 0$, then this fourth-degree polynomial equation accepts at least two complex solutions with imaginary parts different from zero.

4.3. Third proposed theorem for fourth-degree polynomials

This section presents the third developed theorem to solve fourth-degree polynomial equations that are expressed according to the form: $ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a \neq 0$, by converting this fourth-degree equation into the form of a third-degree equation, which we can express as follows: $y_0^3 + \frac{P}{2}\, y_0^2 + \frac{P^2-4R}{16}\, y_0 - \frac{Q^2}{64} = 0$. The proof of this theorem is detailed in [13].

Theorem 3

A fourth-degree polynomial equation under the expressed form in (Equation 19), where coefficients belong to the group of numbers $\mathbb{R}$ and $a \neq 0$, has four solutions.

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \; with \; a \neq 0 \quad (19)$$

$$P = -6\left(\frac{b}{a}\right)^2 + \frac{16c}{a}; Q = 8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}; R = -3\left(\frac{b}{a}\right)^4 + \frac{16cb^2}{a^3} - \frac{64db}{a^2} + \frac{256e}{a} \quad (20)$$

$$y_{0,1} = -\frac{P^;}{3} + \frac{1}{3}\sqrt[3]{-\frac{R^;}{2} + \sqrt{\left(\frac{R^;}{2}\right)^2 + \left(\frac{Q^;}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{R^;}{2} + \sqrt{\left(\frac{R^;}{2}\right)^2 + \left(\frac{Q^;}{3}\right)^3}} \;\Big|\; \left\{P^; = \frac{P}{2}\; ;\; R^; = \frac{-27Q^2 - 2P^3 + 72PR}{64}\; ;\; Q^; = -\frac{3P^2 + 36R}{16}\right\} \quad (21)$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) < 0$, and by using $y_{0,1}$ in (Equation 21), P in (Equation 20) and Q in (Equation 20); the four solutions for (Equation 19) are as shown in (Equation 22), (Equation 23), (Equation 24) and (Equation 25).

$$\text{Solution 1: } S_{1,1} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (22)$$

$$\text{Solution 2: } S_{1,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (23)$$

$$\text{Solution 3: } S_{1,3} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (24)$$

$$\text{Solution 4: } S_{1,4} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (25)$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) > 0$, and by using $y_{0,1}$ in (Equation 21), P in (Equation 20), and Q in (Equation 20); the four solutions for (Equation 19) are as shown in (Equation 26), (Equation 27), (Equation 28), and (Equation 29).

$$\text{Solution 1: } S_{2,1} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (26)$$

$$\text{Solution 2: } S_{2,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (27)$$

$$\text{Solution 3: } S_{2,3} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (28)$$

$$\text{Solution 4: } S_{2,4} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (29)$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) = 0$, and by using $y_{0,1}$ in (Equation 21) and P in (Equation 20); the four solutions for (Equation 19) are as shown in (Equation 30), (Equation 31), (Equation 32), and (Equation 33).

$$\text{Solution 1: } S_{3,1} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \quad (30)$$

$$\text{Solution 2: } S_{3,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \quad (31)$$

$$\text{Solution 3: } S_{3,3} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \quad (32)$$

$$\text{Solution 4: } S_{3,4} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \quad (33)$$

4.4. Fourth proposed theorem for fourth-degree polynomials

This section presents the fourth developed theorem for fourth-degree polynomials that are expressed according to the form: $ax^4 + bx^2 + cx^2 + dx + e = 0$ where $a \neq 0$. This theorem identifies whether this fourth-degree equation accepts at least two complex solutions with imaginary parts different from zero by relying on the value $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$. The proof of this theorem is detailed in [13].

Theorem 4

Considering the polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$; then, this fourth-degree polynomial equation accepts at least two complex solutions, where the imaginary parts are different from zero and dependent on the group of coefficients $\{a, b, c\}$.

4.5. Fifth proposed theorem for fourth-degree polynomials

This section presents the fifth developed theorem for fourth-degree polynomials that are expressed according to the form: $ax^4 + bx^2 + cx^2 + dx + e = 0$ where $a \neq 0$ and $e \neq 0$. This theorem identifies whether this fourth-degree equation accepts at least two complex solutions with imaginary parts different from zero by relying on the value $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$. The proof of this theorem is detailed in [13].

Theorem 5

Considering the polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, this fourth-degree polynomial equation accepts at least two complex solutions, where the imaginary parts are different from zero and dependent on the group of coefficients $\{c, d, e\}$.

3.6. Sixth proposed theorem for fourth-degree polynomials

This section presents the sixth developed theorem for fourth-degree polynomials that are expressed according to the form: $ax^4 + bx^2 + cx^2 + dx + e = 0$ where $a \neq 0$ and $e \neq 0$. This theorem identifies whether this fourth-degree equation accepts four complex solutions with imaginary parts different from zero by relying on the values $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$. The proofs of this theorem are detailed in [13].

Theorem 6

Considering the polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, this fourth-degree polynomial equation accepts four complex solutions with imaginary parts different from zero.

5. Solving Fourth-Order Differential Equations

This section presents the developed theorems and formulas to solve fourth-order differential equations by using the proposed methodologies in this paper to solve nth order differential equations and nth degree polynomial equations.

5.1. First proposed theorem for fourth-order differential equations

This section presents the first developed theorem to solve fourth-order differential equations that are expressed according to the form: $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where $a \neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the fourth-order differential equation into an equivalent polynomial form of fourth degree, where we use the presented theorems to solve fourth-degree equations in this paper.

Theorem 7

A fourth-order differential equation under the expressed form in (Equation 34), where coefficients belong to the group of numbers $\mathbb{R}$ and $a \neq 0$, has multiple solutions presented as $f(x)$, which we can express according to the exponential form shown in (Equation 35).

$$a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K \text{ where a} \neq 0 \tag{34}$$

$$f(x) = e^{sx+u} + v \tag{35}$$

The value of *v*, which is included in the solution *f(x)* shown in (Equation 35), is supposed to be an arbitrary value. We can calculate the arbitrary value of *v* by using the shown expression in (Equation 36).

$$v = \frac{K}{e} \tag{36}$$

The value of *u,* which is included in the solution *f(x)* shown in (Equation 35), is supposed to be an arbitrary value. We can calculate the value of *u* while relying on a condition for the initialization value I_0 which is to be identified at the point $x = 0$. Therefore, we can use the expression $f(x = 0) = I_0$ in order to identify the arbitrary value of *u* as shown in (Equation 37).

$$u = log\left(I_0 - \frac{K}{e}\right) \tag{37}$$

By supposing that the solution of the fourth-order differential equation is expressed according to the exponential form shown in (Equation 35), we can convert this differential equation into the form of a fourth-degree equation as shown in (Equation 38), where we can use the proposed solutions in Theorem 3 for fourth-degree polynomial equations in complete forms.

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \; with \; a \neq 0 \tag{38}$$

$$P = -6\left(\frac{b}{a}\right)^2 + \frac{16c}{a}; Q = 8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}; R = -3\left(\frac{b}{a}\right)^4 + \frac{16cb^2}{a^3} - \frac{64db}{a^2} + \frac{256e}{a} \tag{39}$$

$$y_{0,1} = -\frac{P^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} + \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} + \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} \mid \left\{P^{;} = \frac{P}{2}\ ;\ R^{;} = \frac{-27Q^2-2P^3+72PR}{64}\ ;\ Q^{;} = -\frac{3P^2+36R}{16}\right\} \quad (40)$$

$$y_{0,2} = -\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} \quad (41)$$

$$y_{0,3} = -\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} \quad (42)$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) < 0$, and by using $y_{0,1}$ in (Equation 40), $y_{0,2}$ in (Equation 41), $y_{0,3}$ in (Equation 42), P in (Equation 39) and Q in (Equation 39); the four solutions for the fourth-order differential equation show in (Equation 34) are as expressed in (Equation 43), (Equation 44), (Equation 45) and (Equation 46).

Solution 1: $DS_{1,1}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{y_{0,2}}+\frac{1}{4}\sqrt{y_{0,3}}\right]x}$ (43)

Solution 2: $DS_{1,2}(x) = \frac{K}{e} + \left(I_0 - \frac{k}{e}\right) e^{\left[-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{y_{0,2}}+\frac{1}{4}\sqrt{y_{0,3}}\right]x}$ (44)

Solution 3: $DS_{1,3}(x) = \frac{K}{e} + \left(I_0 - \frac{k}{e}\right) e^{\left[-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{y_{0,2}}-\frac{1}{4}\sqrt{y_{0,3}}\right]x}$ (45)

Solution 4: $DS_{1,4}(x) = \frac{K}{e} + \left(I_0 - \frac{k}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{y_{0,2}}-\frac{1}{4}\sqrt{y_{0,3}}\right]x}$ (46)

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) > 0$, and by using $y_{0,1}$ in (Equation 40), $y_{0,2}$ in (Equation 41), $y_{0,3}$ in (Equation 42), P in (Equation 39) and Q in (Equation 39); the four solutions for the fourth-order differential equation show in (Equation 34) are as expressed in (Equation 47), (Equation 48), (Equation 49) and (Equation 50).

Solution 1: $DS_{2,1}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{y_{0,2}}-\frac{1}{4}\sqrt{y_{0,3}}\right]x}$ (47)

Solution 2: $DS_{2,2}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{y_{0,2}}+\frac{1}{4}\sqrt{y_{0,3}}\right]x}$ (48)

Solution 3: $DS_{2,3}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{y_{0,2}}+\frac{1}{4}\sqrt{y_{0,3}}\right]x}$ (49)

Solution 4: $DS_{2,4}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{y_{0,2}}-\frac{1}{4}\sqrt{y_{0,3}}\right]x}$ (50)

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) = 0$, and by using $y_{0,1}$ in (Equation 40) and P in (Equation 39); the four solutions for the fourth-order differential equation show in (Equation 34) are as expressed in (Equation 51), (Equation 52), (Equation 53), and (Equation 54).

Solution 1: $DS_{3,1}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)}\right]x}$ (51)

Solution 2: $DS_{3,2}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)}\right]x}$ (52)

Solution 3: $DS_{3,3}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)}\right]x}$ (53)

Solution 4: $DS_{3,4}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)}\right]x}$ (54)

5.2. Second Proposed Theorem for Fourth-Order Differential Equations

This section presents the second developed theorem to solve fourth-order differential equations that are expressed according to the form: $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where $a \neq 0$. This theorem identifies whether this fourth-order differential equation accepts at least two complex functions as solutions, where the imaginary parts are different from zero, by relying on the value $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$. The proof of this theorem relies on the proof of Theorem 4 of this paper, which is detailed in [13].

Theorem 8

Considering the differential equation $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$; then, this fourth-order differential equation accepts at least two complex functions as solutions, where the imaginary parts of these functions are different from zero and principally dependent on the group of coefficients $\{a, b, c\}$.

5.3. Third proposed theorem for fourth-order differential equations

This section presents the third developed theorem to solve fourth-order differential equations that are expressed according to the form: $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where $a \neq 0$ and $e \neq 0$. This theorem identifies whether this fourth-order differential equation accepts at least two complex functions as solutions where the imaginary parts are different from zero by relying on the value $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$. The proof of this theorem relies on the proof of Theorem 5 of this paper, which is detailed in [13].

Theorem 9

Considering the differential equation $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, this fourth-order differential equation accepts at least two complex functions as solutions, where the imaginary parts of these functions are different from zero and principally dependent on the group of coefficients $\{c, d, e\}$.

5.4. Fourth proposed theorem for fourth-order differential equations

This section presents the fourth developed theorem to solve fourth-order differential equations that are expressed according to the form: $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where $a \neq 0$ and $e \neq 0$. This theorem identifies whether this fourth-order differential equation accepts four complex functions as solutions, where the imaginary parts are different from zero, by relying on the values $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$. The proof of this theorem relies on the proofs of Theorem 6 of this paper, which are detailed in [13].

Theorem 10

Considering the differential equation $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, this fourth-order differential equation accepts four complex functions as solutions, where the imaginary parts of these functions are different from zero.

6. Solving Fifth Degree Polynomial Equations

This section presents the developed theorems and formulas to solve fifth-degree polynomial equations by using the proposed engineering methodology in this paper to solve nth degree polynomial equations.

6.1. First proposed theorem for fifth-degree polynomials

This section presents the first developed theorem to solve fifth-degree polynomial equations that are expressed according to the form: $\mathrm{A}w^5 + \mathrm{B}w^4 + \mathrm{C}w^3 + \mathrm{D}w^2 + \mathrm{E}w + \mathrm{F} = 0$ where $\mathrm{A} \neq 0$, by converting this quantic equation into the form of a fourth degree equation, which we can express as follows: $z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0$. The proof of this theorem is detailed in [14].

$$\mathrm{A}w^5 + \mathrm{B}w^4 + \mathrm{C}w^3 + \mathrm{D}w^2 + \mathrm{E}w + \mathrm{F} = 0 \text{ with } \mathrm{A} \neq 0 \tag{55}$$

$$x^5 + cx^3 + dx^2 + ex + f = 0 \tag{56}$$

$$c = -10\frac{\mathrm{B}^2}{\mathrm{A}^2} + 25\frac{\mathrm{C}}{\mathrm{A}} \tag{57}$$

$$d = 20\frac{\mathrm{B}^3}{\mathrm{A}^3} - 75\frac{\mathrm{CB}}{\mathrm{A}^2} + 125\frac{\mathrm{D}}{\mathrm{A}} \tag{58}$$

$$e = -15\frac{\mathrm{B}^4}{\mathrm{A}^4} + 75\frac{\mathrm{CB}^2}{\mathrm{A}^3} - 250\frac{\mathrm{DB}}{\mathrm{A}^2} + 625\frac{\mathrm{E}}{\mathrm{A}} \tag{59}$$

$$f = 4\frac{\mathrm{B}^5}{\mathrm{A}^5} - 25\frac{\mathrm{CB}^3}{\mathrm{A}^4} + 125\frac{\mathrm{DB}^2}{\mathrm{A}^3} - 625\frac{\mathrm{EB}}{\mathrm{A}^2} + 3125\frac{\mathrm{F}}{\mathrm{A}} \tag{60}$$

$$z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0 \tag{61}$$

Theorem 11

After reducing the form of fifth degree polynomial shown in (Equation 55) to the presented form in (Equation 56) where coefficients are expressed in (Equation 57), (Equation 58), (Equation 59) and (Equation 60); the fifth degree polynomial equation shown in (Equation 56), where coefficients belong to the group of numbers $\mathbb{R}$, can be reduced to a fourth degree polynomial equation, which may be expressed as shown in (Equation 61). The reduction from a quantic polynomial to a quartic polynomial is conducted by supposing $= x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for the fourth-degree polynomial equation in (Equation 61) by using Theorem 3 to solve quartic polynomials and by relying on the expression $x_3 = -\frac{\Gamma_3}{4}$. The variable Γ_3 is the solution for the polynomial equation shown in (Equation 62), whereas the coefficients of this equation are shown in (Equation 63), (Equation 64), and (Equation 65). The coefficients Γ_2, Γ_1 and Γ_0 of the quartic equation (Equation 61) are determined by using calculated values of Γ_3 and using the shown expressions in (Equation 67), (Equation 68), and (Equation 69). As a result, we have eight calculated values as potential solutions for the fifth-degree polynomial equation shown in (Equation 56), where many of them are only redundancies of others, because there are only five official solutions to determine.

The eight solutions to calculate for the quantum equation (Equation 56) are shown in the groups (Equation 70) and (Equation 71).

The proposed five values as official solutions for the fifth-degree polynomial equation shown in (Equation 56) are as presented in (Equation 72), (Equation 73), (Equation 74), (Equation 75), and (Equation 76).

The proposed five values as official solutions for the fifth-degree polynomial equation shown in (Equation 55) are as presented in (Equation 77), (Equation 78), (Equation 79), (Equation 80), and (Equation 81).

The group of expressions $\{\Gamma_{3,1}\ ;\ \Gamma_{3,2}\ ; -\Gamma_{3,1}\ ;\ -\Gamma_{3,2}\}$ are the identified four values of the variable Γ_3 by using the presented expressions in (Equation 66), which are calculated as solutions for the fourth-degree polynomial equation shown in (Equation 62).

We use the expression $\left\{\alpha_{(1,\Gamma_{3,i})} = \frac{\Gamma_{3,i}^4 - (16d)^2/(e - c^2/4)}{4\Gamma_{3,i}^2}\right\}$ in order to simplify calculations, which allows obtaining the quartic equation shown in (Equation 62).

The group of expressions $\left\{S'_{(\Gamma_{3,1},1)}\ ;\ S'_{(\Gamma_{3,1},2)}\ ;\ S'_{(\Gamma_{3,1},3)}\ ;\ S'_{(\Gamma_{3,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation 61) by using Theorem 3 to calculate these four roots nearly in parallel.

$$\lambda_2\Gamma_3^4 + \lambda_1\Gamma_3^2 + \lambda_0 = 0 \tag{62}$$

$$\lambda_2 = 1024 \frac{(e-\frac{c^2}{4})}{16d}; \tag{63}$$

$$\lambda_1 = 512c - 40 \frac{(16d)^2}{e-\frac{c^2}{4}}; \tag{64}$$

$$\lambda_0 = -128 \frac{(16d)^2}{\left(e-\frac{c^2}{4}\right)^2} [f - \frac{cd}{2}]; \tag{65}$$

$$\Gamma_3 = \pm\sqrt{-\frac{P^;}{2} \pm \sqrt{\left(\frac{P^;}{2}\right)^2 - 4Q^;}} \mid P^; = \frac{\lambda_1}{\lambda_2} \ and\ Q^; = \frac{\lambda_0}{\lambda_2} \tag{66}$$

$$\Gamma_2 = \frac{(16d)^2}{2\Gamma_3^2\left(e-\frac{c^2}{4}\right)} \tag{67}$$

$$\Gamma_1 = -\frac{1}{4}\Gamma_3^3 + \frac{3(16d)^2}{16\left(e-\frac{c^2}{4}\right)\Gamma_3} \tag{68}$$

$$\Gamma_0 = -\frac{1}{16}\Gamma_3^4 - \frac{(16d)^2}{32\left(e-\frac{c^2}{4}\right)} + \frac{\left(f-\frac{cd}{2}\right)(16d)^2}{2\Gamma_3^2\left(e-\frac{c^2}{4}\right)^2} + \frac{(16d)^4}{16\Gamma_3^4\left(e-\frac{c^2}{4}\right)^2} \tag{69}$$

$$N_{\{\Gamma_{3,1}\}} = \left\{\frac{1}{2}\left[S^{;2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}\right]\right\} \tag{70}$$

$$N_{\{\Gamma_{3,2}\}} = \left\{\frac{1}{2}\left[S^{;2}_{(\Gamma_{3,2},1)} - \alpha_{(1,\Gamma_{3,2})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,2},2)} - \alpha_{(1,\Gamma_{3,2})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,2},3)} - \alpha_{(1,\Gamma_{3,2})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,2},4)} - \alpha_{(1,\Gamma_{3,2})}\right]\right\} \tag{71}$$

$$s_1 = \frac{1}{2}[S^{;2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}] \tag{72}$$

$$s_2 = \frac{1}{2}[S^{;2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}] \tag{73}$$

$$s_3 = \frac{1}{2}[S^{;2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}] \tag{74}$$

$$s_4 = \frac{1}{2}[S^{;2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}] \tag{75}$$

$$s_5 = -\frac{f}{s_1 s_2 s_3 s_4} \tag{76}$$

Solution 1: $S_1 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}\right]$ (77)

Solution 2: $S_2 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}\right]$ (78)

Solution 3: $S_3 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}\right]$ (79)

Solution 4: $S_4 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}\right]$ (80)

Solution 5: $S_5 = -\frac{F}{AS_1S_2S_3S_4}$ (81)

6.2. Second proposed theorem for fifth-degree polynomials

This section presents the second developed theorem to solve fifth-degree polynomial equations that are expressed according to the form: $Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F = 0$, where $A \neq 0$, by converting this quantic equation into the form of a fourth degree equation, which we can express as follows: $z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0$. The proof of this theorem is detailed in [14]. The axis of difference in this theorem is conducting calculations on the fifth-degree polynomial shown in (Equation 82) without eliminating the fourth-degree part by avoiding the use of the expression $x = \frac{-b+y}{5}$.

$$x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \tag{82}$$

$$b = \frac{\mathrm{B}}{\mathrm{A}}; c = \frac{\mathrm{C}}{\mathrm{A}}; d = \frac{\mathrm{D}}{\mathrm{A}}; e = \frac{\mathrm{E}}{\mathrm{A}}; f = \frac{\mathrm{F}}{\mathrm{A}}.$$

Theorem 12

The fifth-degree polynomial equation shown in (Equation 82) is reducible to the quartic equation shown in (Equation 83), where the coefficients belong to the group of numbers $\mathbb{R}$ without the need to eliminate the fourth-degree part. The reduction from fifth degree to fourth degree is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for the fourth-degree polynomial equation shown in (Equation 83) by using Theorem 3. The variable Υ_3 is the solution for the polynomial equation shown in (Equation 87) by using the expression of the quadratic solution, whereas $x_3 = -\frac{\Upsilon_3}{4}$. The coefficients of the shown polynomial in (Equation 87) are as expressed in (Equation 88), (Equation 89), and (Equation 90). The value of Υ_3 is equal to $\Upsilon_{3,1}$, which is presented in (Equation 91). The coefficients Υ_2, Υ_1 and Υ_0 are determined by using the calculated value of Υ_3 in (Equation 91) and using the shown expressions in (Equation 84), (Equation 85), and (Equation 86).

The five proposed solutions for the polynomial equation (Equation 82) are as shown in (Equation 92), (Equation 93), (Equation 94), (Equation 95), and (Equation 96).

We use the expression $\left\{\alpha_{(1,\Upsilon_{3,1})} = \frac{\Upsilon_{3,1}^4 - 8b\Upsilon_{3,1}^2 - [16(d-bc)]^2/(e-\frac{c^2}{4})}{4\Upsilon_{3,1}^2}\right\}$ in order to simplify calculations, which allows obtaining the quartic equation shown in (Equation 87).

The group of expressions $\left\{\xi_{(\Upsilon_{3,1},1)}\ ;\ \xi_{(\Upsilon_{3,1},2)}\ ;\ \xi_{(\Upsilon_{3,1},3)}\ ;\ \xi_{(\Upsilon_{3,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation 83) by using Theorem 3 to calculate these four roots nearly in parallel.

$$z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0 \tag{83}$$

$$\Upsilon_2 = \frac{[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)} + 4b \tag{84}$$

$$\Upsilon_1 = -\frac{1}{4}\Upsilon_3^3 + \frac{3[16(d-bc)]^2}{16\left(e-\frac{c^2}{4}\right)\Upsilon_3} + 4b\Upsilon_3 \tag{85}$$

$$\Upsilon_0 = -\frac{1}{16}\Upsilon_3^4 + b\Upsilon_3^2 - \frac{[16(d-bc)]^2}{32\left(e-\frac{c^2}{4}\right)} + \frac{\left(f-\frac{cd}{2}+\frac{bc^2}{4}\right)[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)^2} + \frac{b[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)} + \frac{[16(d-bc)]^4}{16\Upsilon_3^4\left(e-\frac{c^2}{4}\right)^2} \tag{86}$$

$$\beta_2\Upsilon_3^4 + \beta_1\Upsilon_3^2 + \beta_0 = 0 \tag{87}$$

$$\beta_2 = 1024\frac{\left(e-\frac{c^2}{4}\right)}{16(d-bc)} \tag{88}$$

$$\beta_1 = 512c - 40\frac{[16(d-bc)]^2}{e-\frac{c^2}{4}} + 1024b^2 \tag{89}$$

$$\beta_0 = -128\frac{[16(d-bc)]^2}{\left(e-\frac{c^2}{4}\right)^2}[f-\frac{cd}{2}+\frac{bc^2}{4}] + 128b\frac{[16(d-bc)]^2}{\left(e-\frac{c^2}{4}\right)} \quad (90)$$

$$\Gamma_{3,1} = \pm\sqrt{-\frac{M^{;}}{2} \pm \sqrt{\left(\frac{M^{;}}{2}\right)^2 - 4N^{;}}} \mid M^{;} = \frac{\beta_1}{\beta_2} \text{ and } N^{;} = \frac{\beta_0}{\beta_2} \quad (91)$$

$$\text{Solution 1: } S_1 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right] \quad (92)$$

$$\text{Solution 2: } S_2 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},2)} - \alpha_{(1,\Upsilon_{3,1})}\right] \quad (93)$$

$$\text{Solution 3: } S_3 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},3)} - \alpha_{(1,\Upsilon_{3,1})}\right] \quad (94)$$

$$\text{Solution 4: } S_4 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},4)} - \alpha_{(1,\Upsilon_{3,1})}\right] \quad (95)$$

$$\text{Solution 5: } S_5 = -\frac{f}{S_1S_2S_3S_4} \quad (96)$$

7. Solving Fifth-Order Differential Equations

This section presents the developed theorems and formulas to solve fifth-order differential equations by using the proposed methodologies in this paper to solve nth order differential equations and nth degree polynomial equations.

7.1. First proposed theorem for the fifth-order differential equation

This section presents the first developed theorem to solve fifth-order differential equations that are expressed according to the form: $A*g^{(5)}(x)+B*g^{(4)}(x)+C*g^{(3)}(x)+D*g^{(2)}(x)+E*g^{(1)}(x)+F*g^{(0)}(x)=K$ where $A\neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the fifth-order differential equation into an equivalent polynomial form of fifth degree, where we use the presented theorems in this paper to solve fifth-degree equations.

Theorem 13

A fifth-order differential equation under the expressed form in (Equation 97), where coefficients belong to the group of numbers $\mathbb{R}$ and $A\neq 0$, has multiple solutions presented as $g_{(x)}$ which we can express according to the exponential form shown in (Equation 98).

$$A*g^{(5)}(x)+B*g^{(4)}(x)+C*g^{(3)}(x)+D*g^{(2)}(x)+E*g^{(1)}(x)+F*g^{(0)}(x)=K \text{ with } A\neq 0 \quad (97)$$

$$g_{(x)} = e^{sx+u}+v \quad (98)$$

The value of *v*, which is included in the solution *g(x)* shown in (Equation 98), is considered an arbitrary value. We can calculate the arbitrary value of *v* by using the shown expression in (Equation 99).

$$v = \frac{K}{F} \quad (99)$$

The value of *u,* which is included in the solution *g(x)* shown in (Equation 98), is considered an arbitrary value. We can calculate the arbitrary value of *u* while relying on the condition of the initialization value I_0 which is to be identified at the point $x=0$. Therefore, we can use the expression $g(x=0)=I_0$ in order to identify the arbitrary value of *u* as shown in (Equation 100).

$$u = log\left(I_0 - \frac{K}{F}\right) \quad (100)$$

By supposing that the solution of the fifth-order differential equation is expressed according to the exponential form shown in (Equation 98), we can convert this differential equation into the form of a fifth-degree polynomial equation as shown in (Equation 101), where we can use the proposed solutions in Theorem 11 for fifth-degree polynomial equations in complete forms.

$$\mathrm{A}w^5 + \mathrm{B}w^4 + \mathrm{C}w^3 + \mathrm{D}w^2 + \mathrm{E}w + \mathrm{F} = 0 \text{ with } \mathrm{A} \neq 0 \quad (101)$$

$$x^5 + cx^3 + dx^2 + ex + f = 0 \quad (102)$$

$$c = -10\frac{\mathrm{B}^2}{\mathrm{A}^2} + 25\frac{\mathrm{C}}{\mathrm{A}} \quad (103)$$

$$d = 20\frac{\mathrm{B}^3}{\mathrm{A}^3} - 75\frac{\mathrm{CB}}{\mathrm{A}^2} + 125\frac{\mathrm{D}}{\mathrm{A}} \quad (104)$$

$$e = -15\frac{\mathrm{B}^4}{\mathrm{A}^4} + 75\frac{\mathrm{CB}^2}{\mathrm{A}^3} - 250\frac{\mathrm{DB}}{\mathrm{A}^2} + 625\frac{\mathrm{E}}{\mathrm{A}} \quad (105)$$

$$f = 4\frac{\mathrm{B}^5}{\mathrm{A}^5} - 25\frac{\mathrm{CB}^3}{\mathrm{A}^4} + 125\frac{\mathrm{DB}^2}{\mathrm{A}^3} - 625\frac{\mathrm{EB}}{\mathrm{A}^2} + 3125\frac{\mathrm{F}}{\mathrm{A}} \quad (106)$$

$$z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0 \quad (107)$$

We use Theorem 11 in this paper to solve the fifth-degree polynomial equation shown in (Equation 101).

After reducing the form of fifth degree polynomial shown in (Equation 101) to the presented form in (Equation 102) where coefficients are expressed in (Equation 103), (Equation 104), (Equation 105) and (Equation 106); the fifth degree polynomial equation shown in (Equation 102), where coefficients belong to the group of numbers $\mathbb{R}$, can be reduced to a fourth degree polynomial equation, which may be expressed as shown in (Equation 107). The reduction from a quantic polynomial to a quartic polynomial is conducted by supposing $= x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for the fourth degree polynomial equation in (Equation 107) by using Theorem 3 to solve quartic polynomials and by relying on the expression $x_3 = -\frac{\Gamma_3}{4}$. The variable Γ_3 is the solution for the polynomial equation shown in (Equation 108), whereas the coefficients of this equation are shown in (Equation 109), (Equation 110), and (Equation 111). The coefficients Γ_2, Γ_1 and Γ_0 of the quartic equation (Equation 107) are determined by using calculated values of Γ_3 and using the shown expressions in (Equation 113), (Equation 114), and (Equation 115).

The proposed five values as official solutions for the fifth-degree polynomial equation shown in (Equation 101) are as presented in (Equation 116), (Equation 117), (Equation 118), (Equation 119), and (Equation 120).

The proposed five functions as official solutions for the fifth-order differential equation shown in (Equation 97) are as presented in (Equation 121), (Equation 122), (Equation 123), (Equation 124), and (Equation 125).

The group of expressions $\{\Gamma_{3,1}\ ;\ \Gamma_{3,2}\ ; -\Gamma_{3,1}\ ;\ -\Gamma_{3,2}\}$ are the identified four values of the variable Γ_3 by using the presented expressions in (Equation 112), which are calculated as solutions for the fourth-degree polynomial equation shown in (Equation 108).

We use the expression $\left\{\alpha_{(1,\Gamma_{3,i})} = \frac{\Gamma_{3,i}^4 - (16d)^2/(e - c^2/4)}{4\Gamma_{3,i}^2}\right\}$ in order to simplify calculations, which allows obtaining the quartic equation shown in (Equation 108).

The group of expressions $\left\{S'_{(\Gamma_{3,1},1)}\ ;\ S'_{(\Gamma_{3,1},2)}\ ;\ S'_{(\Gamma_{3,1},3)}\ ;\ S'_{(\Gamma_{3,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation 107) by using Theorem 3 to calculate these four roots nearly in parallel.

$$\lambda_2\Gamma_3^4 + \lambda_1\Gamma_3^2 + \lambda_0 = 0 \quad (108)$$

$$\lambda_2 = 1024\frac{(e - \frac{c^2}{4})}{16d}; \quad (109)$$

$$\lambda_1 = 512c - 40\frac{(16d)^2}{e - \frac{c^2}{4}}; \quad (110)$$

$$\lambda_0 = -128\frac{(16d)^2}{\left(e-\frac{c^2}{4}\right)^2}[f - \frac{cd}{2}]; \quad (111)$$

$$\Gamma_3 = \pm\sqrt{-\frac{P^{;}}{2} \pm \sqrt{\left(\frac{P^{;}}{2}\right)^2 - 4Q^{;}}} \mid P^{;} = \frac{\lambda_1}{\lambda_2} \ \ and\ Q^{;} = \frac{\lambda_0}{\lambda_2} \quad (112)$$

$$\Gamma_2 = \frac{(16d)^2}{2\Gamma_3^2\left(e-\frac{c^2}{4}\right)} \quad (113)$$

$$\Gamma_1 = -\frac{1}{4}\Gamma_3^3 + \frac{3(16d)^2}{16\left(e-\frac{c^2}{4}\right)\Gamma_3} \quad (114)$$

$$\Gamma_0 = -\frac{1}{16}\Gamma_3^4 - \frac{(16d)^2}{32\left(e-\frac{c^2}{4}\right)} + \frac{\left(f-\frac{cd}{2}\right)(16d)^2}{2\Gamma_3^2\left(e-\frac{c^2}{4}\right)^2} + \frac{(16d)^4}{16\Gamma_3^4\left(e-\frac{c^2}{4}\right)^2} \quad (115)$$

$$S_1 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}\right] \quad (116)$$

$$S_2 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}\right] \quad (117)$$

$$S_3 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}\right] \quad (118)$$

$$S_4 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}\right] \quad (119)$$

$$S_5 = -\frac{F}{AS_1S_2S_3S_4} \quad (120)$$

Solution 1: $DS_1 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\left[-\frac{B}{5A}+\frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}\right]\right]x}$ (121)

Solution 2: $DS_2 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\left[-\frac{B}{5A}+\frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}\right]\right]x}$ (122)

Solution 3: $DS_3 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\left[-\frac{B}{5A}+\frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}\right]\right]x}$ (123)

Solution 4: $DS_4 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\left[-\frac{B}{5A}+\frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}\right]\right]x}$ (124)

Solution 5: $DS_5 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\left[\frac{-F}{AS_1S_2S_3S_4}\right]x}$ (125)

7.2. Second proposed theorem for the fifth-order differential equation

This section presents the second developed theorem to solve fifth-order differential equations that are expressed according to the form: $A * g^{(5)}(x) + B * g^{(4)}(x) + C * g^{(3)}(x) + D * g^{(2)}(x) + E * g^{(1)}(x) + F * g^{(0)}(x) = K$ where $A \neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the fifth-order differential equation into an equivalent polynomial form of fifth degree, where we use the presented theorems to solve fifth-degree equations in this paper. The advantage of this second theorem is avoiding the elimination of the fourth-order part from the equation.

Theorem 14

A fifth-order differential equation under the expressed form in (Equation 126), where coefficients belong to the group of numbers ℝ and $A \neq 0$, has multiple solutions presented as $g(x)$, which we can express according to the exponential form shown in (Equation 127).

$$A * g^{(5)}(x) + B * g^{(4)}(x) + C * g^{(3)}(x) + D * g^{(2)}(x) + E * g^{(1)}(x) + F * g^{(0)}(x) = K \text{ with } A \neq 0 \tag{126}$$

$$g(x) = e^{sx+u} + v \tag{127}$$

The value of *v*, which is included in the solution *g(x)* shown in (Equation 127), is considered an arbitrary value. We can calculate the arbitrary value of *v* by using the shown expression in (Equation 128).

$$v = \frac{K}{F} \tag{128}$$

The value of *u,* which is included in the solution *g(x)* shown in (Equation 127), is considered an arbitrary value. We can calculate the arbitrary value of *u* while relying on the condition of the initialization value I_0 which is to be identified at the point $x = 0$. Therefore, we can use the expression $g(x = 0) = I_0$ in order to identify the arbitrary value of *u* as shown in (Equation 129).

$$u = log\left(I_0 - \frac{K}{F}\right) \tag{129}$$

By supposing that the solution of the fifth order differential equation is expressed according to the exponential form shown in (Equation 127); we can convert this differential equation into the form of a fifth degree polynomial equation as shown in (Equation 130), where we can use the proposed solutions in Theorem 12 for fifth degree polynomial equations in complete forms without eliminating the fourth degree part by avoiding the use of the expression $x = (-b + y)/5$.

$$x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \tag{130}$$

$$b = \frac{\mathrm{B}}{\mathrm{A}}; c = \frac{\mathrm{C}}{\mathrm{A}}; d = \frac{\mathrm{D}}{\mathrm{A}}; e = \frac{\mathrm{E}}{\mathrm{A}}; f = \frac{\mathrm{F}}{\mathrm{A}}.$$

We use Theorem 12 in this paper to solve the fifth-degree polynomial equation shown in (Equation 130).

The fifth-degree polynomial equation shown in (Equation 130) is reducible to the quartic equation shown in (Equation 131), where the coefficients belong to the group of numbers ℝ without the need to eliminate the fourth-degree part. The reduction from fifth degree to fourth degree is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for the fourth-degree polynomial equation shown in (Equation 131) by using Theorem 3. The variable Υ_3 is the solution for the polynomial equation shown in (Equation 135) by using the expression of the quadratic solution, whereas $x_3 = -\frac{\Upsilon_3}{4}$. The coefficients of the shown polynomial in (Equation 135) are as expressed in (Equation 136), (Equation 137), and (Equation 138). The value of Υ_3 is equal to $\Upsilon_{3,1}$, which is presented in (Equation 139). The coefficients Υ_2, Υ_1 and Υ_0 are determined by using the calculated value of Υ_3 in (Equation 139) and using the shown expressions in (Equation 132), (Equation 133), and (Equation 134).

The five solutions for the polynomial equation (Equation 130) are as shown in (Equation 140), (Equation 141), (Equation 142), (Equation 143), and (Equation 144).

The five proposed solutions for the fifth-order differential equation (Equation 126) are as shown in (Equation 145), (Equation 146), (Equation 147), (Equation 148), and (Equation 149).

We use the expression $\left\{\alpha_{(1,\Upsilon_{3,1})} = \frac{\Upsilon_{3,1}^4 - 8b\Upsilon_{3,1}^2 - [16(d-bc)]^2/(e-\frac{c^2}{4})}{4\Upsilon_{3,1}^2}\right\}$ in order to simplify calculations, which allows obtaining the quartic equation shown in (Equation 135).

The group of expressions $\left\{\xi_{(\Upsilon_{3,1},1)}\ ;\ \xi_{(\Upsilon_{3,1},2)}\ ;\ \xi_{(\Upsilon_{3,1},3)}\ ;\ \xi_{(\Upsilon_{3,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation 131) by using Theorem 3 to calculate these four roots nearly in parallel.

$$z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0 \tag{131}$$

$$\Upsilon_2 = \frac{[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)} + 4b \tag{132}$$

$$\Upsilon_1 = -\frac{1}{4}\Upsilon_3^3 + \frac{3[16(d-bc)]^2}{16\left(e-\frac{c^2}{4}\right)\Upsilon_3} + 4b\Upsilon_3 \tag{133}$$

$$\Upsilon_0 = -\frac{1}{16}\Upsilon_3^4 + b\Upsilon_3^2 - \frac{[16(d-bc)]^2}{32\left(e-\frac{c^2}{4}\right)} + \frac{\left(f-\frac{cd}{2}+\frac{bc^2}{4}\right)[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)^2} + \frac{b[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)} + \frac{[16(d-bc)]^4}{16\Upsilon_3^4\left(e-\frac{c^2}{4}\right)^2} \tag{134}$$

$$\beta_2\Upsilon_3^4 + \beta_1\Upsilon_3^2 + \beta_0 = 0 \tag{135}$$

$$\beta_2 = 1024\frac{\left(e-\frac{c^2}{4}\right)}{16(d-bc)} \tag{136}$$

$$\beta_1 = 512c - 40\frac{[16(d-bc)]^2}{e-\frac{c^2}{4}} + 1024b^2 \tag{137}$$

$$\beta_0 = -128\frac{[16(d-bc)]^2}{\left(e-\frac{c^2}{4}\right)^2}\left[f - \frac{cd}{2} + \frac{bc^2}{4}\right] + 128b\frac{[16(d-bc)]^2}{\left(e-\frac{c^2}{4}\right)} \tag{138}$$

$$\Gamma_{3,1} = \sqrt{-\frac{M^{;}}{2} + \sqrt{\left(\frac{M^{;}}{2}\right)^2 - 4N^{;}}} \mid M^{;} = \frac{\beta_1}{\beta_2} \ and \ N^{;} = \frac{\beta_0}{\beta_2} \tag{139}$$

$$S_1 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right] \tag{140}$$

$$S_2 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},2)} - \alpha_{(1,\Upsilon_{3,1})}\right] \tag{141}$$

$$S_3 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},3)} - \alpha_{(1,\Upsilon_{3,1})}\right] \tag{142}$$

$$S_4 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},4)} - \alpha_{(1,\Upsilon_{3,1})}\right] \tag{143}$$

$$S_5 = -\frac{f}{S_1S_2S_3S_4} \tag{144}$$

Solution 1: $DS_1 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right]x}$ (145)

Solution 2: $DS_2 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right]x}$ (146)

Solution 3: $DS_3 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right]x}$ (147)

Solution 4: $DS_4 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right]x}$ (148)

Solution 5: $DS_5 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right)e^{\left[\frac{-f}{S_1S_2S_3S_4}\right]x}$ (149)

8. Solving Sixth Degree Polynomial Equations

This section presents the developed theorems and formulas to solve sixth-degree polynomial equations by using the proposed engineering methodology in this paper to solve nth degree polynomial equations.

8.1. First proposed Theorem for Sixth-Degree Polynomials

This section presents the first developed theorem to solve sixth-degree polynomial equations that are expressed according to the form: $\mathrm{A}x^6 + \mathrm{B}x^5 + \mathrm{C}x^4 + \mathrm{D}x^3 + \mathrm{E}x^2 + \mathrm{F}x + \mathrm{G} = 0$, where $\mathrm{A} \neq 0$ and $\mathrm{B} \neq 0$, by converting the sixth degree polynomial into the form of a fourth degree, which we can express as follows: $z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0$. The proof of this theorem is detailed in [15].

$$\mathrm{A}x^6 + \mathrm{B}x^5 + \mathrm{C}x^4 + \mathrm{D}x^3 + \mathrm{E}x^2 + \mathrm{F}x + \mathrm{G} = 0 \; with \; \mathrm{A} \neq 0 \; and \; \mathrm{B} \neq 0 \tag{150}$$

$$x^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0 \; with \; b \neq 0 \tag{151}$$

$$b = \frac{\mathrm{B}}{\mathrm{A}};\; c = \frac{\mathrm{C}}{\mathrm{A}};\; d = \frac{\mathrm{D}}{\mathrm{A}};\; e = \frac{\mathrm{E}}{\mathrm{A}};\; f = \frac{\mathrm{F}}{\mathrm{A}};\; g = \frac{\mathrm{G}}{\mathrm{A}}; \tag{152}$$

$$z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0 \tag{153}$$

Theorem 15

After reducing the form of sixth degree polynomial shown in (Equation 150) to the presented form in (Equation 151) where coefficients are as expressed in (Equation 152); the sixth-degree polynomial equation shown in (Equation 151), where coefficients belong to the group of numbers $\mathbb{R}$, can be reduced to a fourth-degree polynomial equation, which may be expressed as shown in (Equation 153). The reduction from a sixth-degree polynomial to a quartic polynomial is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for the fourth-degree polynomial equation in (Equation 153) by using Theorem 3 and relying on the expression $x_3 = -\frac{\Gamma_3}{4}$. The variable Γ_3 is defined as shown in (Equation 154), where α_3 is presented in (Equation 155) and Γ_4 is the solution for the polynomial equation (Equation 156), which relies on the coefficients (Equation 157), (Equation 158), (Equation 159), and (Equation 160). The shown coefficients in (Equation 157), (Equation 158), (Equation 159), and (Equation 160) are expressed by using the constant V, which is presented in (Equation 161). The coefficients Γ_3, Γ_2, Γ_1 and Γ_0 of the quartic equation (Equation 153), which is used to calculate z, are determined by using the shown expressions in (Equation 154), (Equation 163), (Equation 164), and (Equation 165) while using calculated values of Γ_4 and V. As a result, we have twelve calculated values as potential solutions for the sixth-degree polynomial equation, as shown in (Equation 151), where many of them are only redundancies of others, because there are only six official solutions to determine.

The twelve solutions to calculate for the sixth-degree equation (Equation 151) are as shown in the groups (Equation 166), (Equation 167), and (Equation 168). The proposed six values as official solutions for the sixth-degree polynomial equation shown in (Equation 151) are as presented in (Equation 169), (Equation 170), (Equation 171), (Equation 172), (Equation 173), and (Equation 174).

The group of expressions $\{\pm\Gamma_{4,1}\;;\;\pm\Gamma_{4,2}\;;\;\pm\Gamma_{4,3}\}$ are the identified values of the variable Γ_4, which are calculated as solutions for the sixth-degree polynomial equation shown in (Equation 156) by using the solution of third-degree equations, which is presented in (Equation 162).

We use the expressions $\left\{ \alpha_{(1,\Gamma_{4,i})} = \frac{\Gamma_{4,i}^4 + \frac{32\Gamma_{4,i}^4}{V^2b^2} - \frac{8\Gamma_{4,i}^4}{Vb} + \frac{12d\Gamma_{4,i}^2}{b^2} - \frac{8c\Gamma_{4,i}^2}{b} - \frac{V^2\left(f - \frac{d^2}{4b}\right)}{b}}{4\Gamma_{4,i}^2} \right\}$ in order to simplify calculations, which allows obtaining the shown equation in (Equation 156).

The value of V shown in (Equation 161) is used to simplify the expression of the formulas during calculations, where $\frac{\Upsilon_4}{\alpha_3} = V$.

The group of expressions $\left\{\dot{S}_{(\Gamma_{4,1},1)}\;;\;\dot{S}_{(\Gamma_{4,1},2)}\;;\;\dot{S}_{(\Gamma_{4,1},3)}\;;\;\dot{S}_{(\Gamma_{4,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation 153) by using Theorem 3 to calculate these four roots nearly in parallel.

$$\Gamma_3 = \frac{4\alpha_3}{b} + \Gamma_4 \tag{154}$$

$$\alpha_3 = -\frac{\frac{4\Gamma_4\left(f-\frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2}+\frac{40d^2}{b^3}-\frac{64cd}{b^2}+\frac{64e}{b}} \tag{155}$$

$$\lambda_3\Gamma_4^6 + \lambda_2\Gamma_4^4 + \lambda_1\Gamma_4^2 + \lambda_0 = 0 \tag{156}$$

$$\lambda_3 = -\frac{40960}{V^4b^4} + \frac{16384}{V^3b^3} - \frac{1536}{V^2b^2} \tag{157}$$

$$\lambda_2 = -\frac{24576d}{V^2b^4} + \frac{16384c}{V^2b^3} + \frac{3072d}{Vb^3} - \frac{2048c}{Vb^2} + \frac{1024}{V} \tag{158}$$

$$\lambda_1 = -\frac{512d}{b} + \frac{1536f}{b^3} + \frac{28V^2f}{b} - \frac{7V^2d^2}{b^2} + \frac{96Vf}{b^2} - \frac{168d^2V}{b^3} + \frac{192cdV}{b^2} - \frac{192Ve}{b} - \frac{3456d^2}{b^4} + \frac{4096cd}{b^3} - \frac{1024e}{b^2} - \frac{1024c^2}{b^2} \tag{159}$$

$$\lambda_0 = -\frac{64V^2d^3}{b^4} + \frac{64cd^2V^2}{b^3} - \frac{64eV^2d}{b^2} + \frac{128V^2g}{b} + \frac{192V^2df}{b^3} - \frac{128V^2cf}{b^2} \tag{160}$$

$$V = -\frac{\frac{32f}{b^2}+\frac{40d^2}{b^3}-\frac{64cd}{b^2}+\frac{64e}{b}}{4\frac{\left(f-\frac{d^2}{4b}\right)}{b}} \tag{161}$$

$$\Gamma_{4,1}^2 = \frac{-b^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2}+\sqrt{\left(\frac{D^{;}}{2}\right)^2+\left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2}-\sqrt{\left(\frac{D^{;}}{2}\right)^2+\left(\frac{C^{;}}{3}\right)^3}} \mid \left\{b^{;} = \frac{\lambda_2}{\lambda_3}, c^{;} = \frac{\lambda_1}{\lambda_3} \text{ and } d^{;} = \frac{\lambda_0}{\lambda_3}\right\}\left\{D^{;} = 27d^{;} + 2b^{;3} - 9c^{;}\, b^{;} \text{ and } C^{;} = 9c^{;} - 3b^{;2}\right\} \tag{162}$$

$$\Gamma_2 = \frac{8\Gamma_4^2}{Vb} - \frac{6d}{b^2} + \frac{4c}{b} + \frac{\left(f-\frac{d^2}{4b}\right)V^2}{2b\Gamma_4^2} - \frac{8\Gamma_4^2}{V^2b^2} \tag{163}$$

$$\Gamma_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Gamma_4} - \frac{6d\Gamma_4}{b^2} + \frac{4c\Gamma_4}{b} - \frac{dcV}{\mathrm{b}^2\Gamma_4} + \frac{eV}{b\Gamma_4} - \frac{\Gamma_4^3}{4} - \frac{8\Gamma_4^3}{V^2b^2} + \frac{f-\frac{d^2}{4b}}{4\Gamma_4 b}V^2 \tag{164}$$

$$\Gamma_0 = \frac{\Gamma_4^4}{2Vb} - \frac{V^2d^3}{16b^4\Gamma_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Gamma_4^2}{4b^2} + \frac{c\Gamma_4^2}{2b} + \frac{cd^2V^2}{8\mathrm{b}^3\Gamma_4^2} - \frac{cdV}{2b^2} + \frac{eV}{2b} - \frac{eV^2d}{4b^2\Gamma_4^2} + \frac{gV^2}{2b\Gamma_4^2} - \left(\frac{\Gamma_4^2}{4} + V^2\frac{f-\frac{d^2}{4b}}{4b\Gamma_4^2}\right)\left(\frac{\Gamma_4^2}{4} + \frac{8\Gamma_4^2}{V^2b^2} - \frac{2\Gamma_4^2}{VB} + \frac{3d}{b^2} - \frac{2c}{b} - \frac{\left(f-\frac{d^2}{4b}\right)}{4b\Gamma_4^2}V^2\right) \tag{165}$$

$$N_{\{\Gamma_{4,1}\}} = \left\{\frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,1},1)} - \alpha_{(1,\Gamma_{4,1})}\right], \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,1},2)} - \alpha_{(1,\Gamma_{4,1})}\right], \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,1},3)} - \alpha_{(1,\Gamma_{4,1})}\right], \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,1},4)} - \alpha_{(1,\Gamma_{4,1})}\right]\right\} \tag{166}$$

$$N_{\{\Gamma_{4,2}\}} = \left\{\frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,2},1)} - \alpha_{(1,\Gamma_{4,2})}\right], \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,2},2)} - \alpha_{(1,\Gamma_{4,2})}\right], \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,2},3)} - \alpha_{(1,\Gamma_{4,2})}\right], \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,2},4)} - \alpha_{(1,\Gamma_{4,2})}\right]\right\} \tag{167}$$

$$N_{\{\Gamma_{4,3}\}} = \left\{\frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,3},1)} - \alpha_{(1,\Gamma_{4,3})}\right], \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,3},2)} - \alpha_{(1,\Gamma_{4,3})}\right], \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,3},3)} - \alpha_{(1,\Gamma_{4,3})}\right], \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,3},4)} - \alpha_{(1,\Gamma_{4,3})}\right]\right\} \tag{168}$$

$$S_1 = \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma_{4,1},1)} - \alpha_{(1,\Gamma_{4,1})}\right] \tag{169}$$

$$S_2 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},2)} - \alpha_{(1,\Gamma_{4,1})}\right] \quad (170)$$

$$S_3 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},3)} - \alpha_{(1,\Gamma_{4,1})}\right] \quad (171)$$

$$S_4 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},4)} - \alpha_{(1,\Gamma_{4,1})}\right] \quad (172)$$

$$S_5 = -\frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (173)$$

$$S_6 = -\frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (174)$$

8.2. Second proposed Theorem for Sixth-Degree Polynomials

This section presents the second developed theorem to solve sixth-degree polynomial equations that are expressed according to the form: $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0$ with $A \neq 0$, whereas the coefficient of the fifth-degree part is equal to zero. Solving this sixth-degree equation is based on using the expression $w = \sqrt{\frac{-C}{15A}} + x$ to induce a fifth degree part, then converting the result into the form of a fourth degree equation, which we can express as follows: $z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0$. The proof of this theorem is detailed in [15].

$$x^6 + bx^5 + dx^3 + ex^2 + fx + g = 0 \quad (175)$$

$$b = 6\sqrt{\frac{-C}{15A}} \quad (176)$$

$$d = \frac{8C}{3A}\sqrt{\frac{-C}{15A}} + \frac{D}{A} \quad (177)$$

$$e = \frac{-C^2}{3A^2} + \frac{3D}{A}\sqrt{\frac{-C}{15A}} + \frac{E}{A} \quad (178)$$

$$f = -\frac{18C^2}{5A^2}\sqrt{\frac{-C}{15A}} - \frac{DC}{5A^2} + \frac{2E}{A}\sqrt{\frac{-C}{15A}} + \frac{F}{A} \quad (179)$$

$$g = \frac{-16C^3}{3375A^3} - \frac{DC}{15A^2}\sqrt{\frac{-C}{15A}} - \frac{EC}{15A^2} + \frac{F}{A}\sqrt{\frac{-C}{15A}} + \frac{G}{A} \quad (180)$$

Theorem 16

In order to reduce the sixth-degree polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0 \; with \; A \neq 0$ to the quartic equation shown in (Equation 181), where coefficients belong to the group of numbers $\mathbb{R}$, we first replace w with $\left(w = \sqrt{\frac{-C}{15A}} + x\right)$ in order to obtain the fifth degree equation shown in (Equation 175), where the coefficients are as expressed in (Equation 176), (Equation 177), (Equation 178), (Equation 179), and (Equation 180). Then, the reduction from sixth degree to fourth degree is conducted by supposing $x = (x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for the fourth-degree polynomial equation in (Equation 181) by using Theorem 3 and relying on the expression $x_3 = -\frac{\Upsilon_3}{4}$. The variable Υ_3 is defined as shown in (Equation 182), where α_3 is presented in (Equation 186) and Υ_4 is the solution for the polynomial equation (Equation 187), which relies on the coefficients (Equation 188), (Equation 189), (Equation 190), and (Equation 191). The shown coefficients in (Equation 188), (Equation 189), (Equation 190), and (Equation 191) are expressed by using the constant V, which is defined in (Equation 192). The coefficients Υ_3, Υ_2, Υ_1 and Υ_0 of the quartic equation (Equation 181) are determined by using the calculated value of Υ_4 and using the shown expressions in (Equation 182), (Equation 183), (Equation 184), and (Equation 185).

The six proposed solutions for the polynomial equation $x^6 + bx^5 + dx^3 + ex^2 + fx + g = 0$ shown in (Equation 175) are as shown in (Equation 194), (Equation 195), (Equation 196), (Equation 197), (Equation 198), and (Equation 199).

The six proposed solutions for the polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0 \; with \; A \neq 0$ are as shown in (Equation 200), (Equation 201), (Equation 202), (Equation 203), (Equation 204), and (Equation 205).

The group of expressions $\{\pm\Gamma_{4,1} \; ; \; \pm\Gamma_{4,2} \; ; \; \pm\Gamma_{4,3}\}$ are the identified values of the variable Γ_4, which are calculated as solutions for the sixth-degree polynomial equation presented in (Equation 187) by using the solution of third-degree equations shown in (Equation 193).

We use the expressions $\left\{ \alpha_1 = \dfrac{\Upsilon_4^4 + \frac{32\Upsilon_4^4}{V^2b^2} - \frac{8\Upsilon_4^4}{Vb} + \frac{12d\Upsilon_4^2}{b^2} - \frac{V^2\left(f - \frac{d^2}{4b}\right)}{b}}{4\Upsilon_4^2} \right\}$ in order to simplify calculations, which allows obtaining the quartic equation shown in (Equation 187).

The value of V shown in (Equation 192) is used to simplify the expression of the formulas during calculations, where $\frac{\Upsilon_4}{\alpha_3} = V$.

The group of expressions $\left\{\xi_{(\Upsilon_{4,1},1)} \; ; \; \xi_{(\Upsilon_{4,1},2)} \; ; \; \xi_{(\Upsilon_{4,1},3)} , \xi_{(\Upsilon_{4,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation 181) by using Theorem 3 to calculate these four roots nearly in parallel.

$$z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0 \tag{181}$$

$$\Upsilon_3 = \frac{4\alpha_3}{b} + \Upsilon_4 \tag{182}$$

$$\Upsilon_2 = \frac{8\Upsilon_4^2}{Vb} - \frac{6d}{b^2} + \frac{\left(f - \frac{d^2}{4b}\right)V^2}{2b\Upsilon_4^2} - \frac{8\Upsilon_4^2}{V^2b^2} \tag{183}$$

$$\Upsilon_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Upsilon_4} - \frac{6d\Upsilon_4}{b^2} + \frac{eV}{b\Upsilon_4} - \frac{\Upsilon_4^3}{4} - \frac{8\Upsilon_4^3}{V^2b^2} + \frac{f - \frac{d^2}{4b}}{4\Upsilon_4 b}V^2 \tag{184}$$

$$\Upsilon_0 = \frac{\Upsilon_4^4}{2Vb} - \frac{V^2d^3}{16b^4\Upsilon_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Upsilon_4^2}{4b^2} + \frac{eV}{2b} - \frac{eV^2d}{4b^2\Upsilon_4^2} + \frac{gV^2}{2b\Upsilon_4^2} - \left(\frac{\Upsilon_4^2}{4} + V^2\frac{f - \frac{d^2}{4b}}{4b\Upsilon_4^2}\right)\left(\frac{\Upsilon_4^2}{4} + \frac{8\Upsilon_4^2}{V^2b^2} - \frac{2\Upsilon_4^2}{VB} + \frac{3d}{b^2} - \frac{\left(f - \frac{d^2}{4b}\right)}{4b\Upsilon_4^2}V^2\right) \tag{185}$$

$$\alpha_3 = -\frac{\Upsilon_4 \frac{4\left(f - \frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2} + \frac{40d^2}{b^3} + \frac{64e}{b}} \qquad 186)$$

$$\beta_3\Upsilon_4^6 + \beta_2\Upsilon_4^4 + \beta_1\Upsilon_4^2 + \beta_0 = 0 \tag{187}$$

$$\beta_3 = -\frac{40960}{V^4b^4} + \frac{16384}{V^3b^3} - \frac{1536}{V^2b^2} \tag{188}$$

$$\beta_2 = -\frac{24576d}{V^2b^4} + \frac{3072d}{Vb^3} + \frac{1024}{V} \tag{189}$$

$$\beta_1 = -\frac{512d}{b} + \frac{1536f}{b^3} + \frac{28V^2f}{b} - \frac{7V^2d^2}{b^2} + \frac{96Vf}{b^2} - \frac{168d^2V}{b^3} - \frac{192Ve}{b} - \frac{3456d^2}{b^4} - \frac{1024e}{b^2} \tag{190}$$

$$\beta_0 = -\frac{64V^2d^3}{b^4} - \frac{64eV^2d}{b^2} + \frac{128V^2g}{b} + \frac{192V^2df}{b^3} \tag{191}$$

$$V = -\frac{\frac{32f}{b^2}+\frac{40d^2}{b^3}+\frac{64e}{b}}{\frac{4\left(f-\frac{d^2}{4b}\right)}{b}} \quad (192)$$

$$\Upsilon_{4,1}^2 = \frac{-b^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} + \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} - \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} \mid \left\{b^{;} = \frac{\beta_2}{\beta_3}, c^{;} = \frac{\beta_1}{\beta_3} \text{ and } d^{;} = \frac{\beta_0}{\beta_3}\right\}; \{D\hat{};= 27d^{;} + 2b^{;3} - 9c^{;}b^{;} \text{ and } C^{;} = 9c^{;} - 3b^{;2}\} \quad (193)$$

$$s_1 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},1)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (194)$$

$$s_2 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},2)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (195)$$

$$s_3 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},3)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (196)$$

$$s_4 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (197)$$

$$s_5 = -\frac{b+s_1+s_2+s_3+s_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (198)$$

$$s_6 = -\frac{b+s_1+s_2+s_3+s_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (199)$$

$$\text{Solution 1}: S_1' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},1)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (200)$$

$$\text{Solution 2}: S_2' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},2)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (201)$$

$$\text{Solution 3}: S_3' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},3)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (202)$$

$$\text{Solution 4}: S_4' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (203)$$

$$\text{Solution 5}: S_5' = \sqrt{\frac{-C}{15A}} - \frac{b+s_1+s_2+s_3+s_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (204)$$

$$\text{Solution 6}: S_6' = \sqrt{\frac{-C}{15A}} - \frac{b+s_1+s_2+s_3+s_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (205)$$

9. Solving Sixth-Order Differential Equations

This section presents the developed theorems and formulas to solve sixth-order differential equations by using the proposed methodologies in this paper to solve nth order differential equations and nth degree polynomial equations.

9.1. First proposed theorem for sixth-order differential equations

This section presents the first developed theorem to solve sixth-order differential equations that are expressed according to the form: $A * H^{(6)}(x) + B * H^{(5)}(x) + C * H^{(4)}(x) + D * H^{(3)}(x) + E * H^{(2)}(x) + F * H^{(1)}(x) + G * H^{(0)}(x) = K$ where A ≠ 0, by supposing that the solution is expressed according to an exponential form, then converting the sixth-order differential equation into an equivalent polynomial form of sixth degree, where we use the presented theorems to solve polynomial equations in this paper.

Theorem 17

A sixth-order differential equation under the expressed form in (Equation 206), where coefficients belong to the group of numbers $\mathbb{R}$ and $A \neq 0$, has multiple solutions presented as $H(x)$, which we can express according to the exponential form shown in (Equation 207).

$$A * H^{(6)}(x) + B * H^{(5)}(x) + C * H^{(4)}(x) + D * H^{(3)}(x) + E * H^{(2)}(x) + F * H^{(1)}(x) + G * H^{(0)}(x) = K \text{ with } A \neq 0 \quad (206)$$

$$H_{(x)} = e^{sx+u} + v \quad (207)$$

The value of *v*, which is included in the solution *H(x)* shown in (Equation 207), is considered an arbitrary value. We can calculate the arbitrary value of *v* by using the shown expression in (Equation 208).

$$v = \frac{K}{G} \quad (208)$$

The value of *u,* which is included in the solution *H(x)* shown in (Equation 207), is considered an arbitrary value. We can calculate the arbitrary value of *u* while relying on the condition of the initialization value I_0 which is to be identified at the point $x = 0$. Therefore, we can use the expression $H(x = 0) = I_0$ in order to identify the arbitrary value of *u* as shown in (Equation 209).

$$u = log\left(I_0 - \frac{K}{G}\right) \quad (209)$$

By supposing that the solution of the sixth-order differential equation is expressed according to the exponential form shown in (Equation 207), we can convert this differential equation into the form of a sixth-degree polynomial equation as shown in (Equation 210), where we can use the proposed solutions in Theorem 15 for sixth-degree polynomial equations in general forms.

$$\mathrm{A}x^6 + \mathrm{B}x^5 + \mathrm{C}x^4 + \mathrm{D}x^3 + \mathrm{E}x^2 + \mathrm{F}x + \mathrm{G} = 0 \; with \; \mathrm{A} \neq 0 \; and \; \mathrm{B} \neq 0 \quad (210)$$

$$x^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0 \; with \; b \neq 0 \quad (211)$$

$$b = \frac{\mathrm{B}}{\mathrm{A}};\; c = \frac{\mathrm{C}}{\mathrm{A}};\; d = \frac{\mathrm{D}}{\mathrm{A}};\; e = \frac{\mathrm{E}}{\mathrm{A}};\; f = \frac{\mathrm{F}}{\mathrm{A}};\; g = \frac{\mathrm{G}}{\mathrm{A}}; \quad (212)$$

$$z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0 \quad (213)$$

We use Theorem 15 in this paper to solve the sixth-degree polynomial shown in (Equation 210).

After reducing the form of sixth degree polynomial shown in (Equation 210) to the presented form in (Equation 211) where coefficients are as expressed in (Equation 212); the sixth-degree polynomial equation shown in (Equation 211), where coefficients belong to the group of numbers $\mathbb{R}$, can be reduced to a fourth-degree polynomial equation, which may be expressed as shown in (Equation 213). The reduction from a sixth-degree polynomial to a quartic polynomial is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for the fourth-degree polynomial equation in (Equation 213) by using Theorem 3 and relying on the expression $x_3 = -\frac{\Gamma_3}{4}$. The variable Γ_3 is defined as shown in (Equation 214), where α_3 is presented in (Equation 215) and Γ_4 is the solution for the polynomial equation (Equation 216), which relies on the coefficients (Equation 217), (Equation 218), (Equation 219), and (Equation 220). The shown coefficients in (Equation 217), (Equation 218), (Equation 219), and (Equation 220) are expressed by using the constant V, which is presented in (Equation 221). The coefficients Γ_3, Γ_2, Γ_1 and Γ_0 of the quartic equation (Equation 213), which is used to calculate z, are determined by using the shown expressions in (Equation 214), (Equation 223), (Equation 224), and (Equation 225) while using calculated values of Γ_4 and V.

The proposed six values as official solutions for the sixth-degree polynomial equation shown in (Equation 211) are as presented in (Equation 226), (Equation 227), (Equation 228), (Equation 229), (Equation 230), and (Equation 231).

The proposed six functions as official solutions for the sixth-order differential equation shown in (Equation 206) are as presented in (Equation 232), (Equation 233), (Equation 234), (Equation 235), (Equation 236), and (Equation 237).

The group of expressions $\{\pm\Gamma_{4,1}\ ;\ \pm\Gamma_{4,2}\ ;\ \pm\Gamma_{4,3}\}$ are the identified values of the variable Γ_4, which are calculated as solutions for the sixth-degree polynomial equation presented in (Equation 216) by using the solution of the third-degree equations shown in (Equation 222).

We use the expressions $\left\{\alpha_{(1,\Gamma_{4,i})} = \frac{\Gamma_{4,i}^4 + \frac{32\Gamma_{4,i}^4}{V^2b^2} - \frac{8\Gamma_{4,i}^4}{Vb} + \frac{12d\Gamma_{4,i}^2}{b^2} - \frac{8c\Gamma_{4,i}^2}{b} - \frac{V^2\left(f-\frac{d^2}{4b}\right)}{b}}{4\Gamma_{4,i}^2}\right\}$ in order to simplify calculations, which allows obtaining the shown equation in (Equation 216).

The value of V shown in (Equation 221) is used to simplify the expression of the formulas during calculations, where $\frac{\Upsilon_4}{\alpha_3} = V$.

The group of expressions $\left\{\dot{S}_{(\Gamma_{4,1},1)}\ ;\ \dot{S}_{(\Gamma_{4,1},2)}\ ;\ \dot{S}_{(\Gamma_{4,1},3)}\ ;\ \dot{S}_{(\Gamma_{4,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation 213) by using Theorem 3 to calculate these four roots nearly in parallel.

$$\Gamma_3 = \frac{4\alpha_3}{b} + \Gamma_4 \tag{214}$$

$$\alpha_3 = -\frac{\frac{4\Gamma_4\left(f-\frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2} + \frac{40d^2}{b^3} - \frac{64cd}{b^2} + \frac{64e}{b}} \tag{215}$$

$$\lambda_3\Gamma_4^6 + \lambda_2\Gamma_4^4 + \lambda_1\Gamma_4^2 + \lambda_0 = 0 \tag{216}$$

$$\lambda_3 = -\frac{40960}{V^4b^4} + \frac{16384}{V^3b^3} - \frac{1536}{V^2b^2} \tag{217}$$

$$\lambda_2 = -\frac{24576d}{V^2b^4} + \frac{16384c}{V^2b^3} + \frac{3072d}{Vb^3} - \frac{2048c}{Vb^2} + \frac{1024}{V} \tag{218}$$

$$\lambda_1 = -\frac{512d}{b} + \frac{1536f}{b^3} + \frac{28V^2f}{b} - \frac{7V^2d^2}{b^2} + \frac{96Vf}{b^2} - \frac{168d^2V}{b^3} + \frac{192cdV}{b^2} - \frac{192Ve}{b} - \frac{3456d^2}{b^4} + \frac{4096cd}{b^3} - \frac{1024e}{b^2} - \frac{1024c^2}{b^2} \tag{219}$$

$$\lambda_0 = -\frac{64V^2d^3}{b^4} + \frac{64cd^2V^2}{b^3} - \frac{64eV^2d}{b^2} + \frac{128V^2g}{b} + \frac{192V^2df}{b^3} - \frac{128V^2cf}{b^2} \tag{220}$$

$$V = -\frac{\frac{32f}{b^2} + \frac{40d^2}{b^3} - \frac{64cd}{b^2} + \frac{64e}{b}}{4\frac{\left(f-\frac{d^2}{4b}\right)}{b}} \tag{221}$$

$$\Gamma_{4,1}^2 = \frac{-b^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} + \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} - \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} \mid \left\{b^{;} = \frac{\lambda_2}{\lambda_3}, c^{;} = \frac{\lambda_1}{\lambda_3}\ and\ d^{;} = \frac{\lambda_0}{\lambda_3}\right\}\left\{D^{;} = 27d^{;} + 2b^{;3} - 9c^{;}b^{;}\ and\ C^{;} = 9c^{;} - 3b^{;2}\right\} \tag{222}$$

$$\Gamma_2 = \frac{8\Gamma_4^2}{Vb} - \frac{6d}{b^2} + \frac{4c}{b} + \frac{\left(f-\frac{d^2}{4b}\right)V^2}{2b\Gamma_4^2} - \frac{8\Gamma_4^2}{V^2b^2} \tag{223}$$

$$\Gamma_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Gamma_4} - \frac{6d\Gamma_4}{b^2} + \frac{4c\Gamma_4}{b} - \frac{dcV}{b^2\Gamma_4} + \frac{eV}{b\Gamma_4} - \frac{\Gamma_4^3}{4} - \frac{8\Gamma_4^3}{V^2b^2} + \frac{f-\frac{d^2}{4b}}{4\Gamma_4 b}V^2 \tag{224}$$

$$\Gamma_0 = \frac{\Gamma_4^4}{2Vb} - \frac{V^2d^3}{16b^4\Gamma_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Gamma_4^2}{4b^2} + \frac{c\Gamma_4^2}{2b} + \frac{cd^2V^2}{8b^3\Gamma_4^2} - \frac{cdV}{2b^2} + \frac{eV}{2b} - \frac{eV^2d}{4b^2\Gamma_4^2} + \frac{gV^2}{2b\Gamma_4^2}$$

$$-\left(\frac{\Gamma_4^2}{4}+V^2\frac{f-\frac{d^2}{4b}}{4b\Gamma_4^2}\right)\left(\frac{\Gamma_4^2}{4}+\frac{8\Gamma_4^2}{V^2b^2}-\frac{2\Gamma_4^2}{VB}+\frac{3d}{b^2}-\frac{2c}{b}-\frac{\left(f-\frac{d^2}{4b}\right)}{4b\Gamma_4^2}V^2\right) \tag{225}$$

$$S_1=\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},1)}-\alpha_{(1,\Gamma_{4,1})}\right] \tag{226}$$

$$S_2=\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},2)}-\alpha_{(1,\Gamma_{4,1})}\right] \tag{227}$$

$$S_3=\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},3)}-\alpha_{(1,\Gamma_{4,1})}\right] \tag{228}$$

$$S_4=\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},4)}-\alpha_{(1,\Gamma_{4,1})}\right] \tag{229}$$

$$S_5=-\frac{b+S_1+S_2+S_3+S_4}{2}-\sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2-\frac{g}{S_1S_2S_3S_4}} \tag{230}$$

$$S_6=-\frac{b+S_1+S_2+S_3+S_4}{2}+\sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2-\frac{g}{S_1S_2S_3S_4}} \tag{231}$$

$$\text{Solution 1: } DS_1=\frac{K}{G}+\left(I_0-\frac{K}{G}\right)e^{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},1)}-\alpha_{(1,\Gamma_{4,1})}\right]x} \tag{232}$$

$$\text{Solution 2: } DS_2=\frac{K}{G}+\left(I_0-\frac{K}{G}\right)e^{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},2)}-\alpha_{(1,\Gamma_{4,1})}\right]x} \tag{233}$$

$$\text{Solution 3: } DS_3=\frac{K}{G}+\left(I_0-\frac{K}{G}\right)e^{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},3)}-\alpha_{(1,\Gamma_{4,1})}\right]x} \tag{234}$$

$$\text{Solution 4: } DS_4=\frac{K}{G}+\left(I_0-\frac{K}{G}\right)e^{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1}41)}-\alpha_{(1,\Gamma_{4,1})}\right]x} \tag{235}$$

$$\text{Solution 5: } DS_5=\frac{K}{G}+\left(I_0-\frac{K}{G}\right)e^{\left[-\frac{b+S_1+S_2+S_3+S_4}{2}-\sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2-\frac{g}{S_1S_2S_3S_4}}\right]x} \tag{236}$$

$$\text{Solution 6: } DS_6=\frac{K}{G}+\left(I_0-\frac{K}{G}\right)e^{\left[-\frac{b+S_1+S_2+S_3+S_4}{2}+\sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2-\frac{g}{S_1S_2S_3S_4}}\right]x} \tag{237}$$

9.2. Second proposed Theorem for Sixth-Order Differential Equation

This section presents the second developed theorem to solve sixth-order differential equations that are expressed according to the form: $A*H^{(6)}(x)+C*H^{(4)}(x)+D*H^{(3)}(x)+E*H^{(2)}(x)+F*H^{(1)}(x)+G*H^{(0)}(x)=K$ where $A\neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the sixth-order differential equation into an equivalent polynomial form of sixth degree, where we use the presented theorems to solve polynomial equations in this paper. The axis of difference in this form of sixth-order differential equation has a value of zero for the coefficient of the fifth-order part.

Theorem 18

A sixth-order differential equation under the expressed form in (Equation 238), where coefficients belong to the group of numbers $\mathbb{R}$ and $A\neq 0$, has multiple solutions presented as $H(x)$, which we can express according to the exponential form shown in (Equation 239).

$$A*H^{(6)}(x)+C*H^{(4)}(x)+D*H^{(3)}(x)+E*H^{(2)}(x)+F*H^{(1)}(x)+G*H^{(0)}(x)=K \text{ with } A\neq 0 \tag{238}$$

$$H(x) = e^{sx+u} + v \tag{239}$$

The value of *v*, which is included in the solution *H(x)* shown in (Equation 239), is considered an arbitrary value. We can calculate the arbitrary value of *v* by using the shown expression in (Equation 240).

$$v = \frac{K}{G} \tag{240}$$

The value of u, which is included in the solution *H(x)* shown in (Equation 239), is considered an arbitrary value. We can calculate the arbitrary value of u while relying on the condition of the initialization value I_0 which is to be identified at the point $x = 0$. Therefore, we can use the expression $H(x = 0) = I_0$ in order to identify the arbitrary value of u as shown in (Equation 241).

$$u = log\left(I_0 - \frac{K}{G}\right) \tag{241}$$

By supposing that the solution of the sixth-order differential equation is expressed according to the exponential form shown in (Equation 239), we can convert this differential equation into the form of a sixth-degree polynomial equation as shown in (Equation 242), where we can use the proposed solutions in Theorem 16 for sixth-degree polynomial equations in general forms.

$$Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0 \; with \; A \neq 0 \tag{242}$$

$$x^6 + bx^5 + dx^3 + ex^2 + fx + g = 0 \tag{243}$$

$$b = 6\sqrt{\frac{-C}{15A}} \tag{244}$$

$$d = \frac{8C}{3A}\sqrt{\frac{-C}{15A}} + \frac{D}{A} \tag{245}$$

$$e = \frac{-C^2}{3A^2} + \frac{3D}{A}\sqrt{\frac{-C}{15A}} + \frac{E}{A} \tag{246}$$

$$f = -\frac{18C^2}{5A^2}\sqrt{\frac{-C}{15A}} - \frac{DC}{5A^2} + \frac{2E}{A}\sqrt{\frac{-C}{15A}} + \frac{F}{A} \tag{247}$$

$$g = \frac{-16C^3}{3375A^3} - \frac{DC}{15A^2}\sqrt{\frac{-C}{15A}} - \frac{EC}{15A^2} + \frac{F}{A}\sqrt{\frac{-C}{15A}} + \frac{G}{A} \tag{248}$$

We use Theorem 16 in this paper to solve the sixth-degree polynomial shown in (Equation 242).

In order to reduce the sixth-degree polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0 \; with \; A \neq 0$ to the quartic equation shown in (Equation 249), where coefficients belong to the group of numbers ℝ, we first replace w with $w = \sqrt{\frac{-C}{15A}} + x$ in order to obtain the shown equation in (Equation 243), where coefficients are presented in (Equation 244), (Equation 245), (Equation 246), (Equation 247), and (Equation 248). Then, the reduction from sixth degree to fourth degree is conducted by supposing $x = (x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for the fourth-degree polynomial equation in (Equation 249) by using Theorem 3 and relying on the expression $x_3 = -\frac{\Upsilon_3}{4}$. The variable Υ_3 is defined as shown in (Equation 250), where α_3 is presented in (Equation 254) and Υ_4 is the solution for the polynomial equation (Equation 255), which relies on the coefficients (Equation 256), (Equation 257), (Equation 258), and (Equation 259). The shown coefficients in (Equation 256), (Equation 257), (Equation 258), and (Equation 259) are expressed by using the constant V, which is defined in (Equation 260). The coefficients Υ_3, Υ_2, Υ_1 and Υ_0 of the quartic equation (Equation 249) are determined by using the calculated value of Υ_4 and using the shown expressions in (Equation 250), (Equation 251), (Equation 252), and (Equation 253).

The six proposed solutions for the polynomial equation $x^6 + bx^5 + dx^3 + ex^2 + fx + g = 0$ shown in (Equation 243) are as shown in (Equation 262), (Equation 263), (Equation 264), (Equation 265), (Equation 266), and (Equation 267).

The six proposed solutions for the polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0\ with\ A \neq\ 0$ are as shown in (Equation 268), (Equation 269), (Equation 270), (Equation 271), (Equation 272), and (Equation 273).

The six solutions for the sixth-order differential equation are as shown in (Equation 274), (Equation 275), (Equation 276), (Equation 277), (Equation 278), and (Equation 279).

The group of expressions $\{\pm\Gamma_{4,1}\ ;\ \pm\Gamma_{4,2}\ ;\ \pm\Gamma_{4,3}\}$ are the identified values of the variable Γ_4, which are calculated as solutions for the sixth-degree polynomial equation presented in (Equation 255) by using the solution of the third-degree equations shown in (Equation 261).

We use the expressions $\left\{\alpha_1 = \frac{\Upsilon_4^4 + \frac{32\Upsilon_4^4}{V^2b^2} - \frac{8\Upsilon_4^4}{Vb} + \frac{12d\Upsilon_4^2}{b^2} - \frac{V^2\left(f-\frac{d^2}{4b}\right)}{b}}{4\Upsilon_4^2}\right\}$ in order to simplify calculations, which allows obtaining the sixth-degree equation shown in (Equation 255).

The value of V shown in (Equation 260) is used to simplify the expression of the formulas during calculations, where $\frac{\Upsilon_4}{\alpha_3} = V$.

The group of expressions $\left\{\xi_{(\Upsilon_{4,1},1)}\ ;\ \xi_{(\Upsilon_{4,1},2)}\ ;\ \xi_{(\Upsilon_{4,1},3)}\ ;\ \xi_{(\Upsilon_{4,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation 249) by using Theorem 3 to calculate these four roots nearly in parallel.

$$z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0 \tag{249}$$

$$\Upsilon_3 = \frac{4\alpha_3}{b} + \Upsilon_4 \tag{250}$$

$$\Upsilon_2 = \frac{8\Upsilon_4^2}{Vb} - \frac{6d}{b^2} + \frac{\left(f-\frac{d^2}{4b}\right)V^2}{2b\Upsilon_4^2} - \frac{8\Upsilon_4^2}{V^2b^2} \tag{251}$$

$$\Upsilon_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Upsilon_4} - \frac{6d\Upsilon_4}{b^2} + \frac{eV}{b\Upsilon_4} - \frac{\Upsilon_4^3}{4} - \frac{8\Upsilon_4^3}{V^2b^2} + \frac{f-\frac{d^2}{4b}}{4\Upsilon_4 b}V^2 \tag{252}$$

$$\Upsilon_0 = \frac{\Upsilon_4^4}{2Vb} - \frac{V^2d^3}{16b^4\Upsilon_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Upsilon_4^2}{4b^2} + \frac{eV}{2b} - \frac{eV^2d}{4b^2\Upsilon_4^2} + \frac{gV^2}{2b\Upsilon_4^2} - \left(\frac{\Upsilon_4^2}{4} + V^2\frac{f-\frac{d^2}{4b}}{4b\Upsilon_4^2}\right)\left(\frac{\Upsilon_4^2}{4} + \frac{8\Upsilon_4^2}{V^2b^2} - \frac{2\Upsilon_4^2}{VB} + \frac{3d}{b^2} - \frac{\left(f-\frac{d^2}{4b}\right)}{4b\Upsilon_4^2}V^2\right) \tag{253}$$

$$\alpha_3 = -\frac{\Upsilon_4 \frac{4\left(f-\frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2} + \frac{40d^2}{b^3} + \frac{64e}{b}} \tag{254}$$

$$\beta_3\Upsilon_4^6 + \beta_2\Upsilon_4^4 + \beta_1\Upsilon_4^2 + \beta_0 = 0 \tag{255}$$

$$\beta_3 = -\frac{40960}{V^4b^4} + \frac{16384}{V^3b^3} - \frac{1536}{V^2b^2} \tag{256}$$

$$\beta_2 = -\frac{24576d}{V^2b^4} + \frac{3072d}{Vb^3} + \frac{1024}{V} \tag{257}$$

$$\beta_1 = -\frac{512d}{b} + \frac{1536f}{b^3} + \frac{28V^2f}{b} - \frac{7V^2d^2}{b^2} + \frac{96Vf}{b^2} - \frac{168d^2V}{b^3} - \frac{192Ve}{b} - \frac{3456d^2}{b^4} - \frac{1024e}{b^2} \tag{258}$$

$$\beta_0 = -\frac{64V^2d^3}{b^4} - \frac{64eV^2d}{b^2} + \frac{128V^2g}{b} + \frac{192V^2df}{b^3} \tag{259}$$

$$V = -\frac{\frac{32f}{b^2} + \frac{40d^2}{b^3} + \frac{64e}{b}}{\frac{4\left(f - \frac{d^2}{4b}\right)}{b}} \tag{260}$$

$$\Upsilon_{4,1}^2 = \frac{-b^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} + \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} - \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} \mid \left\{b^{;} = \frac{\beta_2}{\beta_3}, c^{;} = \frac{\beta_1}{\beta_3} \ and \ d^{;} = \frac{\beta_0}{\beta_3}\right\} ; \{D^{\wedge}; = 27d^{;} + 2b^{;3} - 9c^{;} b^{;} \ and \ C^{;} = 9c^{;} - 3b^{;2}\} \tag{261}$$

$$s_1 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},1)} - \alpha_{(1,\Upsilon_{4,1})}] \tag{262}$$

$$s_2 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},2)} - \alpha_{(1,\Upsilon_{4,1})}] \tag{263}$$

$$s_3 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},3)} - \alpha_{(1,\Upsilon_{4,1})}] \tag{264}$$

$$s_4 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} - \alpha_{(1,\Upsilon_{4,1})}] \tag{265}$$

$$s_5 = -\frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \tag{266}$$

$$s_6 = -\frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \tag{267}$$

$$S_1' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},1)} - \alpha_{(1,\Upsilon_{4,1})}] \tag{268}$$

$$S_2' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},2)} - \alpha_{(1,\Upsilon_{4,1})}] \tag{269}$$

$$S_3' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},3)} - \alpha_{(1,\Upsilon_{4,1})}] \tag{270}$$

$$S_4' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} - \alpha_{(1,\Upsilon_{4,1})}] \tag{271}$$

$$S_5' = \sqrt{\frac{-C}{15A}} - \frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \tag{272}$$

$$S_6' = \sqrt{\frac{-C}{15A}} - \frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \tag{273}$$

Solution 1: $$DS_1' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},1)} - \alpha_{(1,\Upsilon_{4,1})}]\right]x} \tag{274}$$

Solution 2: $$DS_2' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},2)} - \alpha_{(1,\Upsilon_{4,1})}]\right]x} \tag{275}$$

Solution 3: $$DS_3' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},3)} - \alpha_{(1,\Upsilon_{4,1})}]\right]x} \tag{276}$$

Solution 4: $$DS_4' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1,4})} - \alpha_{(1,\Upsilon_{4,1})}]\right]x} \quad (277)$$

Solution 5: $$DS_5' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} - \frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}}\right]x} \quad (278)$$

Solution 6: $$DS_6' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} - \frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}}\right]x} \quad (279)$$

10. Solving nth Degree Polynomial Equations

This section presents the developed theorem and formulas to solve nth degree polynomial equations by using the proposed engineering methodology in this paper.

10.1. Proposed theorem for nth Degree Polynomials

This subsection presents the developed theorem to solve nth degree polynomial equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_N \neq 0$, by converting this nth degree polynomial into the form of a reduced polynomial equation with a lower degree, which we can express as follows: $\left\{\left(\sum_{i=0}^{i=M} \Gamma_i Z^i\right) = 0\right\}$ where $N>M$.

$$\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\} \; with \; A_n \neq 0 \quad (280)$$

$$\left\{\left(\sum_{i=0}^{i=N} \frac{A_i}{A_n} X^i\right) = 0\right\} \; with \; A_n \neq 0 \quad (281)$$

$$if\,(N \geq 7 \; and \; N \equiv 1 \; MOD[2]) \Longrightarrow \left\{\left(\sum_{j=0}^{j=n} \frac{a_j}{a_n} X^j\right) = 0\right\} \; where \; \{(n = N+1); \; (a_n \neq 0); \; (a_0 = 0) \; and \; (a_{j+1} = A_{j\geq 0})\} \quad (282)$$

$$if\;[N < 7 \; or \; (N \equiv 0 \; MOD[2])] \Longrightarrow \left\{\left(\sum_{j=0}^{j=n} \frac{a_j}{a_n} X^j\right) = 0\right\} \; where \; \{(n = N) \; and \; (a_n \neq 0) \; and \; (a_j = A_{j\geq 0})\} \quad (283)$$

$$\left\{X = \frac{-a_{(n-1)}}{n a_n} + \frac{x}{n}\right\} \; to \; eleminate \; the \; part \; with \; the \; degree \; (n-1) \quad (284)$$

$$\left\{X = \sqrt{\frac{-2a_{(n-2)}}{n(n-1)a_n}} + x\right\} \; to \; create \; the \; part \; with \; the \; degree \; (n-1) \quad (285)$$

$$\{X = x\} \; to \; keep \; the \; same \; form \; of \; polynomial \quad (286)$$

$$\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\} \; with \; b_n = 1 \quad (287)$$

$$\left\{x = \sum_{i=0}^{i=u} T_i = \sum_{i=0}^{i=u} x_i\right\} \; if \; the \; degree \; of \; polynomial \; equation \; is \; n = 4 \quad (288)$$

$$\left\{x = \sum_{i=0}^{i=u} T_i = \sum_{i\neq j} x_i \, x_j\right\} \; if \; the \; degree \; of \; polynomial \; equation \; is \; n \geq 5 \quad (289)$$

$$\left\{\sum_{i=0}^{i=n'} \Gamma_i \, z^i = 0\right\} \; where \; (n' < n) \; and \; (\Gamma_{n'} = 1) \quad (290)$$

$$z = \sum_{i=0}^{i=u'} x_i = \sum_{i=0}^{i=u'} \sqrt{y_i} \quad (291)$$

$$x_1 = \sqrt{\frac{-b}{3} + \frac{1}{3}\sqrt[3]{-\frac{D}{2} + \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{C}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D}{2} - \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{C}{3}\right)^3}}} \quad (292)$$

$$x_2 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} + \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}} \quad (293)$$

$$x_3 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} - \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}} \quad (294)$$

$$\{\alpha_1 = \sum x_i^2\}\,;\, \{\alpha_2 = \sum_{i\neq j} x_i^2\, x_j^2\}\,;\, \{\alpha_3 = \sum_{i\neq j\neq k} x_i x_j x_k\}\,;\, \{\alpha_4 = \sum_{i\neq j\neq k\neq l} x_i x_j x_k\, x_l\} \quad (295)$$

$$\{X = (\sum x_i)^2 - \alpha_1\}\,;\, \{X^2 = \alpha_2 + 2\alpha_3(\sum x_i) + 6\alpha_4\}\,;\, \{X^4 = 4(\sum x_i)^2\, \alpha_3^2 + 4\alpha_3(\sum x_i)[\alpha_2 + 6\alpha_4] + [\alpha_2 + 6\alpha_4]^2\} \quad (296)$$

$$\left\{\sum_{i=0}^{i=v'} \lambda_i\, \Gamma^{2i} = 0\right\} where \left(v' \leq \frac{n}{2}\right) \quad (297)$$

$$\left\{V = \frac{\Gamma}{\alpha_3}\right\} \quad (298)$$

$$\lambda_i = g_i(V) \quad (299)$$

$$\Gamma_i = f_i(\Gamma, V) \quad (300)$$

$$Group\ K = \left\{\dot{S}_{\left(\Gamma_{\Gamma_k},1\right)}\,;\, \dot{S}_{\left(\Gamma_{\Gamma_k},2\right)}; \ldots; \dot{S}_{\left(\Gamma_{\Gamma_k},n'\right)}\right\} \quad (301)$$

$$\left\{S_1 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,1)} - \alpha_{(1,\Gamma)}\right]\,;\, S_2 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,2)} - \alpha_{(1,\Gamma)}\right]; \ldots;\, S_{n'} = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,n')} - \alpha_{(1,\Gamma)}\right]\right\} \quad (302)$$

$$\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x - S_j)} = 0 \quad (303)$$

$$\{S_{(n'+1)}\,;\, S_{(n'+2)}; \ldots;\, S_{(n)}\} \quad (304)$$

$$\{S'_1 = S_1\,;\, S'_2 = S_2\,; \ldots;\, S'_{n'} = S_{n'}; \ldots;\, S'_n = S_n\} \quad (305)$$

$$\left\{S'_1 = \frac{-a_{n-1}}{na_n} + \frac{S_1}{n}\,;\, S'_2 = \frac{-a_{n-1}}{na_n} + \frac{S_2}{n}\,; \ldots;\, S'_{n'} = \frac{-a_{n-1}}{na_n} + \frac{S_{n'}}{n}\,; \ldots;\, S'_n = \frac{-a_{n-1}}{na_n} + \frac{S_n}{n}\right\} \quad (306)$$

$$\left\{S'_1 = \sqrt{\frac{-a_{n-1}}{n(n-2)a_n}} + S_1\,;\, S'_2 = \sqrt{\frac{-a_{n-1}}{n(n-2)a_n}} + S_2\,; \ldots;\, S'_{n'} = \sqrt{\frac{-a_{n-1}}{n(n-2)a_n}} + S_{n'}\,; \ldots;\, S'_n = \sqrt{\frac{-a_{n-1}}{n(n-2)a_n}} + S_n\right\} \quad (307)$$

Theorem 19

1. We consider the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq 0$ and $N \geq 4$ and all coefficients belong to the group of numbers $\mathbb{R}$ as shown in (Equation 280).
2. We first adapt the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ shown in (Equation 280) to be as presented in (Equation 281) by dividing it by the coefficient A_n where $A_n \neq 0$.

3. If the degree N of the polynomial equation $\left\{\left(\sum_{i=0}^{i=N}\frac{A_i}{A_n}X^i\right)=0\right\}$ is an odd number, and if it is equal to or greater than seven, then we can multiply this polynomial form by *X* to obtain the polynomial equation $\left\{\left(\sum_{j=0}^{j=n}\frac{a_j}{a_n}X^j\right)=0\right\}$ shown in (Equation 282), where $\{(n=N+1);\ (a_n\neq 0);\ (a_0=0)\ and\ (a_{j+1}=A_{j\geq 0})\}$.
4. If the degree N of the polynomial equation $\left\{\left(\sum_{i=0}^{i=N}\frac{A_i}{A_n}X^i\right)=0\right\}$ is less than seven, or if it is an even number, then we can adapt this polynomial form to be presented as $\left\{\left(\sum_{j=0}^{j=n}\frac{a_j}{a_n}X^j\right)=0\right\}$ where $\{(n=N) and\ (a_n\neq 0)\ and\ (a_j=A_{j\geq 0})\}$ as presented in (Equation 283)
5. If the degree *n* of the polynomial equation $\left\{\left(\sum_{j=0}^{j=n}\frac{a_j}{a_n}X^j\right)=0\right\}$ is even, then there is the possibility to eliminate the part of the degree *(n-1)* by using the expression $\left\{X=\frac{-a_{(n-1)}}{na_n}+\frac{x}{n}\right\}$ shown in (Equation 284).
6. It is optional to create the part of the degree *(n-1)* in the polynomial equation $\left\{\left(\sum_{j=0}^{j=n}\frac{a_j}{a_n}X^j\right)=0\right\}$ by using the expression $\left\{X=\sqrt{\frac{-2a_{(n-2)}}{n(n-1)a_n}}+x\right\}$ shown in (Equation 285), which also allows eliminating the part with degree *(n-2)*.
7. If we do not use the expression $\left\{X=\frac{-a_{(n-1)}}{na_n}+\frac{x}{n}\right\}$ nor the expression $\left\{X=\sqrt{\frac{-2a_{(n-2)}}{n(n-1)a_n}}+x\right\}$, then we rely on the use of the expression $\{X=x\}$ as shown in (Equation 286), in order to reach the presented form in (Equation 287).
8. We adapt the nth degree polynomial equation $\left\{\left(\sum_{j=0}^{j=n}\frac{a_j}{a_n}X^j\right)=0\right\}$ to be presented as $\left\{\left(\sum_{i=0}^{i=n}b_ix^i\right)=0\right\}$ where $b_n=1$ and all coefficients belong to the group of numbers ℝ as shown in (Equation 287).
9. Considering the resulting nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=n}b_ix^i\right)=0\right\}$ where $b_n=1$ and all coefficients belong to the group of numbers ℝ, as shown in (Equation 287); we can reduce this polynomial equation to a polynomial of degree $\left\{\sum_{i=0}^{i=n'}\Gamma_i z^i=0\right\}$ where $(n'<n)\ and\ (\Gamma_{n'}=1)$ as shown in (Equation 290).
10. The reduction of the polynomial equation from the nth degree to the inferior degree n' is conducted by supposing $\left\{x=\sum_{i=0}^{i=u}T_i=\sum_{i=0}^{i=u}x_i\right\}$ when $n=4$ as shown in (Equation 288), or supposing $\left\{x=\sum_{i=0}^{i=u}T_i=\sum_{i\neq j}x_i\,x_j\right\}$ when $n\geq 5$ as presented in (Equation 289), whereas supposing the expression $\left\{z=\sum_{i=0}^{i=u'}x_i\right\}$ shown in (Equation 291) is the solution for the polynomial equation of degree n' shown in (Equation 290) by relying on the expression $x_{u'}=-\frac{\Gamma_{(u'-1)}}{u'}$ which will eventually lead to using the solutions of quartic equations.
11. The value of x_1 is expressed according to the solution of third-degree polynomial equations, where $x_1=\sqrt{\frac{-b}{3}+\frac{1}{3}\sqrt[3]{-\frac{D}{2}+\sqrt{\left(\frac{D}{2}\right)^2+\left(\frac{C}{3}\right)^3}}+\frac{1}{3}\sqrt[3]{-\frac{D}{2}-\sqrt{\left(\frac{D}{2}\right)^2+\left(\frac{C}{3}\right)^3}}}$ as presented in (Equation 292)
12. The value of x_2 is expressed according to the solution of quadratic polynomial equations, where $x_2=\sqrt{-\frac{\frac{P}{2}+x_1}{2}+\sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2-\frac{Q^2}{64x_1}}}$ as presented in (Equation 293)

13. The value of x_3 is expressed according to the solution of quadratic polynomial equations, where $x_3 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} - \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}}$ as presented in (Equation 294)
14. We rely on using the constant values $\{\alpha_1 = \sum x_i^2\}$; $\{\alpha_2 = \sum_{i\neq j} x_i^2\, x_j^2\}$; $\{\alpha_3 = \sum_{i\neq j\neq k} x_i x_j x_k\}$ *and* $\{\alpha_4 = \sum_{i\neq j\neq k\neq l} x_i x_j x_k\, x_l\}$, which are shown in (Equation 295), in order to converge calculations toward reducing the degree of the polynomial.
15. We rely on using the expressions $\{X = (\sum x_i)^2 - \alpha_1\}$; $\{X^2 = \alpha_2 + 2\alpha_3(\sum x_i) + 6\alpha_4\}$ *and* $\{X^4 = 4(\sum x_i)^2\, \alpha_3^2 + 4\alpha_3(\sum x_i)[\alpha_2 + 6\alpha_4] + [\alpha_2 + 6\alpha_4]^2\}$, which are shown in (Equatio. 296), in order to converge calculations toward having simplified forms.
16. The variable Γ is the solution for the polynomial equation (Equation 297), which relies on the coefficients $\{\lambda_0;\ \lambda_1; \ldots;\ \lambda_u\}$.
17. The value of V shown in (Equation 298) is used to simplify the expression of the formulas during calculations, where $\frac{\Gamma}{\alpha_3} = V$.
18. Each coefficient $\{\lambda_i\}$ is among the group $\{\lambda_0;\ \lambda_1; \ldots;\ \lambda_u\}$ and it is calculated according to the shown expression in (Equation 299) by relying only on the coefficients $\{b_0; b_1; \ldots; b_n\}$ and the calculated constant value of V, which is presented in (Equation 298).
19. The polynomial equation $\left\{\sum_{i=0}^{i=v'} \lambda_i\, \Gamma^{2i} = 0\right\}$ where $\left(v' \leq \frac{n}{2}\right)$ has $(2 * v')$ roots which we can express as $\{\pm\Gamma_{r_1};\ \pm\Gamma_{r_2}; \ldots;\ \pm\Gamma_{r_{v'}}\}$.
20. We can select one root among the group $\{\pm\Gamma_{r_1};\ \pm\Gamma_{r_2}; \ldots;\ \pm\Gamma_{r_{v'}}\}$ to be considered as the principal root value Γ.
21. Each coefficient $\{\Gamma_i\}$ is among the group $\{\Gamma_0;\ \Gamma_1; \ldots;\ \Gamma_{u'}\}$ and it is calculated according to the shown expression in (Equation 300) by relying only on the coefficients $\{b_0; b_1; \ldots; b_n\}$ and the calculated values of Γ and V.
22. We use the expressions $\{\alpha_{(1,\Gamma)} = \mathrm{L}(V, \Gamma)\}$ shown in (Equation 301) to calculate the value of α_1 only by using the coefficients $\{b_0; b_1; \ldots; b_n\}$ and the calculated values of Γ and V, which allow simplifying calculations toward obtaining the shown equation in (Equation 297).
23. The group of roots $\{\pm\Gamma_{r_1};\ \pm\Gamma_{r_2}; \ldots;\ \pm\Gamma_{r_{v'}}\}$ identified for the polynomial equation $\left\{\sum_{i=0}^{i=v'} \lambda_i\, \Gamma^{2i} = 0\right\}$ will allow us to calculate an amount of $(2 * v')$ groups of roots for the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_i\, z^i = 0\right\}$ where each group of roots will consist of n' roots as shown in (Equation 301)
24. Each group of roots for the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_i\, z^i = 0\right\}$ is calculated while relying on a specific value of root $\{\pm\Gamma_{r_k}\}$ for the polynomial equation $\left\{\sum_{i=0}^{i=v'} \lambda_i\, \Gamma^{2i} = 0\right\}$; whereas all groups of roots of the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_i\, z^i = 0\right\}$ will have redundancies among them.
25. In order to identify all roots, we can eliminate the redundancies of values among calculated groups of roots for the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_i\, z^i = 0\right\}$ where each group of roots is calculated by using a different value of $\{\pm\Gamma_{r_k}\}$ among the identified group of roots for the polynomial equation $\left\{\sum_{i=0}^{i=v'} \lambda_i\, \Gamma^{2i} = 0\right\}$
26. We calculate the group of roots $\{\dot{S}_{(\Gamma,1)};\ \dot{S}_{(\Gamma,2)}; \ldots;\ \dot{S}_{(\Gamma,n')}\}$ to be the solutions of the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_i\, z^i = 0\right\}$ shown in (Equation 290)
27. We calculate the group of roots $\{\dot{S}_{(\Gamma,1)};\ \dot{S}_{(\Gamma,2)}; \ldots;\ \dot{S}_{(\Gamma,n')}\}$ nearly in parallel to be the solutions of the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_i\, z^i = 0\right\}$ by changing the signs of the included subterms in one solution $\left\{\dot{S}_{(\Gamma,k)} = \sum \pm \mathrm{T}_i\right\}$

28. We calculate a group of n' roots $\left\{S_1 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,1)} - \alpha_{(1,\Gamma)}\right];\ S_2 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,2)} - \alpha_{(1,\Gamma)}\right]; \dots;\ S_{n'} = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,n')} - \alpha_{(1,\Gamma)}\right]\right\}$ as expressed in (Equation 302) to solve the polynomial equation $\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\}$, which is presented in (Equation 287)
29. We can calculate the rest of the roots for the polynomial equation $\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\}$ by solving the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$ shown in (Equation 303) while relying on the calculated group of roots $\left\{S_1 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,1)} - \alpha_{(1,\Gamma)}\right];\ S_2 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,2)} - \alpha_{(1,\Gamma)}\right]; \dots;\ S_{n'} = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,n')} - \alpha_{(1,\Gamma)}\right]\right\}$ which is presented in (Equation 302)
30. We solve the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$, which has a degree of $(n-n')$, in order to identify the group of roots $\{S_{(n'+1)};\ S_{(n'+2)}; \dots;\ S_{(n)}\}$ shown in (Equation 304).
31. We solve the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$ shown in (Equation 303) by using quadratic terms if this polynomial equation is expressed according to a second-degree form.
32. We solve the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$ shown in (Equation 303) by using cubic terms if this polynomial equation is expressed according to a third-degree form.
33. We solve the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$ shown in (Equation 303) by repeating the same engineered methodology to solve nth degree polynomial equations if the degree of the equation is $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$ is equal to or higher than four.
34. By identifying the group of roots $\{S_{(1)};\ S_{(2)}; \dots;\ S_{(n')}\}$ and the group of roots $\{S_{(n'+1)};\ S_{(n'+2)}; \dots;\ S_{(n)}\}$, we will have all the n roots for the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\}$ shown in (Equation 287)
35. The group of roots for the polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ shown in (Equation 280) will be $\{S'_{(1)};\ S'_{(2)}; \dots;\ S'_{(n)}\}$ where $\left\{S'_{(i)} = S_{(i)}\right\}$ as presented in (Equation 305) in case we used the expression $(X = x)$ shown in (Equation 286)
36. The group of roots for the polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ shown in (Equation 280) will be $\{S'_{(1)};\ S'_{(2)}; \dots;\ S'_{(n)}\}$ where $\left\{S'_{(i)} = \frac{-a_{n-1}}{na_n} + \frac{1}{n}S_{(i)}\right\}$ as presented in (Equation 306), in case we used the expression $\left\{X = \frac{-a_{(n-1)}}{na_n} + \frac{x}{n}\right\}$ shown in (Equation 284)
37. The group of roots for the polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ shown in (Equation 280) will be $\{S'_{(1)};\ S'_{(2)}; \dots;\ S'_{(n)}\}$ where $\left\{S'_{(i)} = \sqrt{\frac{-2a_{n-2}}{n(n-1)a_n}} + S_{(i)}\right\}$ as presented in (Equation 307), in case we used the expression $\left\{X = \sqrt{\frac{-2a_{(n-2)}}{n(n-1)a_n}} + x\right\}$ shown in (Equation 285)

11. Solving nth Order Differential Equations

This section presents the developed theorems and formulas to solve nth order differential equations by using the proposed methodologies in this paper.

11.1. First proposed Theorem for nth Order Differential Equations

This subsection presents the first developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i * H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the nth order differential equation into an equivalent polynomial form of nth degree, where we use the presented theorems to solve polynomial equations in this paper.

Theorem 20

The nth order differential equation under the expressed form in (Equation 308), where the coefficients belong to the group of numbers $\mathbb{R}$, and $A_N \neq 0$, has multiple solutions presented as $H(x)$, which we can express according to the exponential form shown in (Equation 309).

$$\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K \text{ with } A_N \neq 0 \quad (308)$$

$$H_{(x)} = e^{sx+u} + v \quad (309)$$

The value of *v*, which is included in the solution *H(x)* shown in (Equation 309), is considered an arbitrary value. We can calculate the arbitrary value of *v* by using the shown expression in (Equation 310).

$$v = \frac{K}{A_0} \quad (310)$$

The value of *u,* which is included in the solution *H(x)* shown in (Equation 309), is considered an arbitrary value. We can calculate the arbitrary value of *u* while relying on the condition of the initialization value I_0 which is to be identified at the point $x = 0$. Therefore, we can use the expression $H(x = 0) = I_0$ in order to identify the arbitrary value of *u* as shown in (Equation 311).

$$u = log\left(I_0 - \frac{K}{A_0}\right) \quad (311)$$

By supposing that the solution of the nth order differential equation is expressed according to the exponential form shown in (Equation 309), we can convert this differential equation into the form of an nth degree polynomial equation as shown in (Equation 312), where we can use the proposed solutions in Theorem 20 for nth degree polynomial equations in general forms.

$$\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\} \; with \; A_n \neq 0 \quad (312)$$

We use Theorem 20 to solve the nth degree polynomial equation shown in (Equation 312) in order to calculate all *n* roots nearly in parallel. Otherwise, we can use numerical analysis to calculate all these roots.

After identifying the group of *n* roots $\{S'_{(1)} \,;\, S'_{(2)}; \ldots;\, S'_{(n)}\}$ for the nth degree polynomial equation shown in (Equation 312), we calculate the group of *n* solutions $\{DS''_{(1)} \,;\, DS''_{(2)}; \ldots;\, DS''_{(n)}\}$ for the nth order differential equation by relying on the identified roots and the shown expression in (Equation 309), which allows calculating each solution for the differential equation (Equation 308) as shown in (Equation 313).

$$DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right) e^{S'_i x} + \frac{K}{A_0} \quad (313)$$

11.2. Second proposed Theorem for nth Order Differential Equations

This subsection presents the second developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i * H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem identifies new additional solutions for nth order differential equations by combining the use of two different roots to express the new solutions, which allow interconnecting two arbitrary points $\{(x_0, I_0) \,;\, (x_1, I_1)\}$.

Theorem 21

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq 0$ and all coefficients are from the group of numbers $\mathbb{R}$.

Supposing the group of *n* roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)} \,;\, S'_{(2)}; \ldots;\, S'_{(n)}\}$, whereas the group of *n* solutions for the corresponding nth order differential equation

$\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)}\ ;\ DS'_{(2)};\dots;\ DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right) e^{S'_i x} + \frac{K}{A_0}$.

If there are two different roots $\{S_a\ ;\ S_b\ \}$ among the group of n roots $\{S'_{(1)}\ ;\ S'_{(2)};\dots;\ S'_{(n)}\}$ of the nth degree polynomial equation, then we can use the nth order differential equation to interconnect two arbitrary values $\{I_0\ ;\ I_1\ \}$ identified at two arbitrary points $\{(x_0, I_0)\ ;\ (x_1, I_1)\}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation 314), where the coefficients $\{R'_{(I_0)}\ ;\ R'_{(I_1)}\ \}$ are as expressed in (Equation 315) and (Equation 316).

$$\left\{DS'_{(n+i>n)} = R'_{(I_1)} e^{S_2 x} + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right) e^{x S_1} + \frac{K}{A_0}\ ; Where\ S_k \in \{S_a;\ S_b\}\right\} \tag{314}$$

$$R'_{(I_0)} = \frac{I_0 - \frac{K}{A_0}\left(1 - e^{S_1 x_0}\right) - R'_{(I_1)}\left(e^{S_2 x_0} - e^{S_1 x_0}\right)}{e^{S_1 x_0}} \tag{315}$$

$$R'_{(I_1)} = \frac{I_1 - \frac{K}{A_0}\left(1 - e^{S_1 x_1}\right) - R'_{(I_0)} e^{S_1 x_1}}{e^{S_2 x_1} - e^{S_1 x_1}} \tag{316}$$

11.3. Third proposed Theorem for nth Order Differential Equations

This subsection presents the third developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem identifies new additional solutions for nth order differential equations by combining the use of three different roots to express the new solutions, which allow interconnecting three arbitrary points $\{(x_0, I_0)\ ;\ (x_1, I_1)\ ;\ (x_2, I_2)\}$.

Theorem 22

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq 0$ and all coefficients are from the group of numbers $\mathbb{R}$.

Supposing the group of n roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)}\ ;\ S'_{(2)};\dots;\ S'_{(n)}\}$, whereas the group of n solutions for the corresponding nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)}\ ;\ DS'_{(2)};\dots;\ DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right) e^{S'_i x} + \frac{K}{A_0}$.

If there are three different roots $\{S_a\ ;\ S_b\ ;\ S_c\}$ among the group of n roots $\{S'_{(1)}\ ;\ S'_{(2)};\dots;\ S'_{(n)}\}$ of the nth degree polynomial equation, then we can use the nth order differential equation to interconnect three arbitrary values $\{I_0\ ;\ I_1\ ;\ I_2\}$ identified at three arbitrary points $\{(x_0, I_1)\ ;\ (x_1, I_1)\ ;\ (x_2, I_2)\}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation 317), where the coefficients $\{R'_{(I_0)}\ ;\ R'_{(I_1)}\ ;\ R'_{(I_2)}\}$ are calculated by using the shown expressions in (Equation 318), (Equation 319), and (Equation 320).

$$\left\{DS'_{(n+i>n)} = R'_{(I_2)} e^{x S_3} + \left(R'_{(I_1)} - R'_{(I_2)}\right) e^{x S_2} + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right) e^{x S_1} + \frac{K}{A_0}; where\ S_k \in \{S_a;\ S_b;\ S_c\}\right\} \tag{317}$$

$$R'_{(I_0)} = \frac{I_0 - \frac{K}{A_0}\left(1 - e^{S_1 x_0}\right) - R'_{(I_1)}\left(e^{S_2 x_0} - e^{S_1 x_0}\right) - R'_{(I_2)}\left(e^{S_3 x_0} - e^{S_2 x_0}\right)}{e^{S_1 x_0}} \tag{318}$$

$$R'_{(I_1)} = \frac{I_1 - \frac{K}{A_0}\left(1 - e^{S_1 x_1}\right) - R'_{(I_0)} e^{S_1 x_1} - R'_{(I_2)}\left(e^{S_3 x_1} - e^{S_2 x_1}\right)}{e^{S_2 x_1} - e^{S_1 x_1}} \tag{319}$$

$$R'_{(I_2)} = \frac{I_2 - \frac{K}{A_0}\left(1 - e^{S_1 x_2}\right) - R'_{(I_0)} e^{S_1 x_2} - R'_{(I_1)}\left(e^{S_2 x_2} - e^{S_1 x_2}\right)}{e^{S_3 x_2} - e^{S_2 x_2}} \tag{320}$$

11.4. Fourth proposed Theorem for nth Order Differential Equations

This subsection presents the fourth developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem identifies new additional solutions for nth order differential equations by combining the use of four different roots to express the new solutions, which allows interconnecting four arbitrary points $\{(x_0, I_0);\ (x_1, I_1)\ ;\ (x_2, I_2)\ ;\ (x_3, I_3)\}$.

Theorem 23

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq 0$ and all coefficients are from the group of numbers ℝ.

Supposing the group of *n* roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)}\ ;\ S'_{(2)};\ldots;\ S'_{(n)}\}$, whereas the group of *n* solutions for the corresponding nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)}\ ;\ DS'_{(2)};\ldots;\ DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right)e^{S'_i x} + \frac{K}{A_0}$.

If there are four different roots $\{S_a\ ;\ S_b\ ;\ S_c\ ;\ S_d\}$ among the group of *n* roots $\{S'_{(1)}\ ;\ S'_{(2)};\ldots;\ S'_{(n)}\}$ of the nth degree polynomial equation, then we can use the nth order differential equation to interconnect four arbitrary values $\{I_0\ ;\ I_1\ ;\ I_2\ ;\ I_3\}$ identified at four arbitrary points $\{(x_0, I_1)\ ;\ (x_1, I_1)\ ;\ (x_2, I_2)\ ;\ (x_3, I_3)\ \}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation 321), where the coefficients $\left\{R'_{(I_0)}\ ;\ R'_{(I_1)}\ ;\ R'_{(I_2)}\ ;\ R'_{(I_3)}\right\}$ are calculated by using the shown expressions in (Equation 322), (Equation 323), (Equation 324), and (Equation 325).

$$\left\{DS'_{(n+i>n)} = R'_{(I_3)}e^{xS_4} + \left(R'_{(I_2)} - R'_{(I_3)}\right)e^{xS_3} + \left(R'_{(I_1)} - R'_{(I_2)}\right)e^{xS_2} + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right)e^{xS_1} + \frac{K}{A_0};\, where\ S_k \in \{S_a;\ S_b;\ S_c\ ;\ S_d\}\right\} \tag{321}$$

$$R'_{(I_0)} = \frac{I_0 - \frac{K}{A_0}\left(1 - e^{S_1x_0}\right) - R'_{(I_1)}\left(e^{S_2x_0} - e^{S_1x_0}\right) - R'_{(I_2)}\left(e^{S_3x_0} - e^{S_2x_0}\right) - R'_{(I_3)}\left(e^{S_4x_0} - e^{S_3x_0}\right)}{e^{S_1x_0}} \tag{322}$$

$$R'_{(I_1)} = \frac{I_1 - \frac{K}{A_0}\left(1 - e^{S_1x_1}\right) - R'_{(I_0)}e^{S_1x_1} - R'_{(I_2)}\left(e^{S_3x_1} - e^{S_2x_1}\right) - R'_{(I_3)}\left(e^{S_4x_1} - e^{S_3x_1}\right)}{e^{S_2x_1} - e^{S_1x_1}} \tag{323}$$

$$R'_{(I_2)} = \frac{I_2 - \frac{K}{A_0}\left(1 - e^{S_1x_2}\right) - R'_{(I_0)}e^{S_1x_2} - R'_{(I_1)}\left(e^{S_2x_2} - e^{S_1x_2}\right) - R'_{(I_3)}\left(e^{S_4x_2} - e^{S_3x_2}\right)}{e^{S_3x_2} - e^{S_2x_2}} \tag{324}$$

$$R'_{(I_3)} = \frac{I_3 - \frac{K}{A_0}\left(1 - e^{S_1x_3}\right) - R'_{(I_0)}e^{S_1x_3} - R'_{(I_1)}\left(e^{S_2x_3} - e^{S_1x_3}\right) - R'_{(I_2)}\left(e^{S_3x_3} - e^{S_2x_3}\right)}{e^{S_4x_3} - e^{S_3x_3}} \tag{325}$$

11.5. Fifth proposed Theorem for nth Order Differential Equations

This subsection presents the fifth developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem identifies new additional solutions for nth order differential equations by combining the use of T different roots to express the new solutions, which can allow interconnecting T arbitrary points.

Theorem 24

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq 0$ and all coefficients are from the group of numbers ℝ.

Supposing the group of *n* roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)}\ ;\ S'_{(2)};\ldots;\ S'_{(n)}\}$, whereas the group of *n* solutions for the corresponding nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)}\ ;\ DS'_{(2)};\ldots;\ DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right)e^{S'_i x} + \frac{K}{A_0}$.

If there are T different roots $\{S_{p_1};\ S_{p_2};\ldots;\ S_{p_T}\}$ among the group of *n* roots $\{S'_{(1)}\ ;\ S'_{(2)};\ldots;\ S'_{(n)}\}$ of the nth degree polynomial equation, then we can use the nth order differential equation to interconnect T arbitrary values $\{I_0\ ;\ldots;I_{T-1}\}$ identified at T arbitrary points $\{(x_0, I_0)\ ;\ \ldots\ ;\ (x_{T-1}, I_{T-1})\}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation 326), where the values of the coefficients $\left\{R'_{(I_0)}\ ;\ R'_{(I_1)}\ ;\ \ldots\ ;\ R'_{(I_{T-1})}\right\}$ are calculated by using the shown expressions in (Equation 327), whereas the used parameters $R_{(x_k)}$, $O_{(I_k,\ x_k)}$, $P_{(x_k)}$ are as shown in (Equation 328), (Equation 329), and (Equation 330).

$$\left\{DS'_{(n+i>n)} = R'_{(I_{(T-1)})}e^{xS_T} + \left(\sum_{k=1}^{k=T-2}\left[\left(R'_{(I_k)} - R'_{(I_{(k+1)})}\right)e^{xS_{(k+1)}}\right]\right) + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right)e^{xS_1} + \frac{K}{A_0};where\ S_j \in \{S_{p_1};\ S_{p_1};\dots;\ S_{p_T}\}\right\} \quad (326)$$

$$R'_{(I_k)} = \frac{I_k - \frac{K}{A_0}\left(1-e^{S_1x_k}\right) - R_{(x_k)} + O_{(I_k,\ x_k)}}{P_{(x_k)}} \quad (327)$$

$$R_{(x_k)} = R'_{(I_0)}e^{S_1x_k} + \sum_{j=1}^{j=T-1} R'_{(I_j)}\left(e^{S_{j+1}x_k} - e^{S_jx_k}\right) \quad (328)$$

$$O_{(I_k,\ x_k)} = \begin{cases} R'_{(I_0)}e^{S_1x_k},\ k = 0 \\ R'_{(I_k)}\left(e^{S_{k+1}x_k} - e^{S_kx_k}\right),\ k > 0 \end{cases} \quad (329)$$

$$P_{(x_k)} = \begin{cases} e^{S_1x_k},\ k = 0 \\ \left(e^{S_{k+1}x_k} - e^{S_kx_k}\right),\ k > 0 \end{cases} \quad (330)$$

11.6. Sixth proposed Theorem for nth Order Differential Equations

This subsection presents the sixth developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i * H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem identifies new additional solutions for nth order differential equations by combining the use of T' different roots $\{T' \in [\![2, T]\!]\}$ to express the new solutions, which can allow interconnecting T' arbitrary points $\{(x_{q_1}, I_0)\ ;\dots;\ (x_{q_{T'}}, I_{T'-1})\}$.

Theorem 25

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_iH^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_iX^i\right) = 0\right\}$ where $A_n \neq 0$ and all coefficients are from the group of numbers $\mathbb{R}$.

Supposing the group of *n* roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_iX^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)}\ ;\ S'_{(2)};\dots;\ S'_{(n)}\}$, whereas the group of *n* solutions for the corresponding nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i * H^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)}\ ;\ DS'_{(2)};\dots;\ DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right)e^{S'_ix} + \frac{K}{A_0}$.

If there are T different roots $\{S_{p_1};\ S_{p_2};\dots;\ S_{p_T}\}$ among the group of *n* roots $\{S'_{(1)}\ ;\ S'_{(2)};\dots;\ S'_{(n)}\}$ of the nth degree polynomial equation, then for each value of T' $(2 \leq T' \leq T)$, we can select a specific group of T' different roots $\{S_{q_1};\ S_{q_2};\dots;\ S_{q_{T'}}\}$, then we can use the nth order differential equation to interconnect T' arbitrary values $\{I_0\ ;\dots;I_{T'-1}\}$ identified at T' arbitrary points $\{(x_0, I_0)\ ;\dots;\ (x_{T'-1}, I_{T'-1})\}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation 331), where the values of the coefficients $\left\{R'_{(I_0)}\ ;\ R'_{(I_1)}\ ;\dots;\ R'_{(I_{T'-1})}\right\}$ are calculated by using the shown expressions in (Equation 332), whereas the used parameters $R_{(x_L)}$, $O_{(I_L,\ x_L)}$, $P_{(x_L)}$ are as shown in (Equation 333), (Equation 334), and (Equation 335).

$$\left\{DS'_{(n+i>n)} = R'_{(I_{(T'-1)})}e^{xS_{T'}} + \sum_{L=1}^{L=T'-2}\left[\left(R'_{(I_L)} - R'_{(I_{(L+1)})}\right)e^{xS_{(l+1)}}\right] + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right)e^{xS_1} + \frac{K}{A_0};where\ S_k \in \{S_{q_1};\ S_{q_2};\dots;\ S_{q_{T'}}\}\right\} \quad (331)$$

$$R'_{(I_L)} = \frac{I_L - \frac{K}{A_0}\left(1-e^{S_1x_L}\right) - R_{(x_L)} + O_{(I_L,\ x_L)}}{P_{(x_L)}} \quad (332)$$

$$R_{(x_L)} = R'_{(I_0)}e^{S_1x_L} + \sum_{j=1}^{T'-1} R'_{(I_j)}\left(e^{S_{j+1}x_L} - e^{S_jx_L}\right) \quad (333)$$

$$O_{(I_L,\ x_L)} = \begin{cases} R'_{(I_0)} e^{S_1 x_L}, & L = 0 \\ R'_{(I_L)} (e^{S_{L+1} x_L} - e^{S_L x_L}), & L > 0 \end{cases} \tag{334}$$

$$P_{(x_L)} = \begin{cases} e^{S_1 x_L}, & L = 0 \\ (e^{S_{L+1} x_L} - e^{S_L x_L}), & L > 0 \end{cases} \tag{335}$$

12. Conclusion

This paper presents new engineered methodologies to solve nth order differential equations and nth degree polynomial equations step by step, while providing the necessary logic, expressions, conditions, and formulas to solve these equations.

This paper presents the results of deploying the proposed engineered methodologies into solving fourth degree, fifth degree, and sixth degree polynomial equations in general forms while presenting the results according to specific theorems and formulas.

This paper also presents the results of deploying the proposed engineered methodologies into solving fourth-order, fifth-order, and sixth-order differential equations in general forms while presenting the results of this deployment according to specific theorems and formulas expressing the solutions of these differential equations.

In addition, this paper presents generalized theorems along with specific formulated solutions which we propose to solve nth order differential equations and nth degree polynomial equations in general forms and in complete forms.

Furthermore, this paper presents new theorems expressing new additional solutions for nth order differential equations in order to allow the use of these differential equations and their roots in interconnecting many arbitrary values accorded to specific points, which opens the way toward scaling up the use of these differential equations and their solutions in business analytics, data analytics, predictive analysis, and systems control.

Conflicts of Interest

The author states there is no conflict of interest.

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