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Original Article

A Study on Approximation of a Conjugate Function using Triple Product Mean

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Abstract - *In this paper, for the first time, we introduce the degree of approximation of* 2π*-periodic functions belonging to the* $Lip(\alpha, r)$ $(0 < \alpha \leq 1, r \geq 1)$ *class by using* $(E, q)T(C, 1)$ *Means of their Conjugate Trigonometric Fourier Series.*

Keywords - *Conjugate trigonometric Fourier Series,* $Lip(\alpha, r)$ *class,* $(E, q)T(C, 1)$ *Means.*

1. Introduction

The studies of the degree of approximation of signals belonging to Lipschitz classes by the product summability techniques by researchers like [3-10] have discussed the degree of approximation of functions by using single, double, and triple product means.

Let $\sum_{n=0}^{\infty} U_n$ be given an infinite series with $\{S_n\}$, the sequence of its nth partial sum of the sequence to sequence transform

$$C_n^1 = \frac{1}{n+1}\sum_{k=0}^{n} S_k \qquad n = 0, 1, 2, \ldots$$

Defines the Castro mean of order 1 of $\{S_n\}$.

The Series $\sum_{n=0}^{\infty} U_n$ is said to be $(C, 1)$ summable to S, if $\lim_{n\to\infty} C_n^1 = S$

The sequence-to-sequence transform

$$t_r^{Eq} = E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} S_k, q > 0, n = 0, 1, 2, \ldots$$

Defines the Euler means of order $q > 0$ of $\{S_n\}$.

Let $\sum a_n$ be given an infinite series with the sequence of partial sums $\{S_n\}$

Let $t_n = \sum_{k=0}^{n} a_{m,v} S_v$, $n = 1, 2 \ldots$

Defines the sequence $\{t_n\}$ of the t men the sequence $\{S_n\}$
If $t_n \to S$, as $n \to \infty$

Then the series $\sum a_n$ is said to be A Summable to S.
The condition for regular A Summability is easily seen to be

(i) $\sup_m \sum_{n=0}^{\infty} |a_{mn}| < H$ Where H is an absolute constant.

(ii) $\lim_{n\to\infty} a_{mn} = 0$

(iii) $\lim_{n\to\infty} \sum_{n=0}^{\infty} a_{mn} = 1$

The sequence-to-sequence transformation

$$T_n = \frac{1}{(1+q)^n} = \sum_{k=0}^{n}\binom{n}{k} q^{n-k} S_k$$

Defines the sequence $\{T_n\}$ of the (E, q) Mean.

$$T_n = \frac{1}{(1+q)^n} = \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k$$

$$= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k \left\{ \sum_{v=0}^{k} a_{k,v} S_v \right\}$$

Let $T = (a_{nk})$ be an Infinite lower triangular matrix satisfying Silverman
To … Conditions of regularly $\sum_{k=0}^{n} a_{n,k} \to 1$ as $n \to \infty$
$a_{nk} = 0$ for $k > n$
$\sum_{k=0}^{\infty} |a_{nk}| \leq N$, N is a finite constant matrix mean of the sequence $\{S_n\}$ is given by

$$t_n^T = \sum_{k=0}^{n} a_{n,n-k} S_{n-k}$$

If $T_n^T \to S$ as $n \to \infty$ then sequence $\{S_n\}$ or Infinite series $\sum_{k=0}^{\infty} u_k$ is said to be summable by matrix means to a finite number S, if the matrix means is summable on the Cesaro means of TC_1 means of the sequence$\{S_n\}$ is given by

$$t_n^{TC_1} = \sum_{k=0}^{n} a_{n(n-k)} \sigma_{n-k}$$

$$= \sum_{k=0}^{n} a_{n,n-k} \frac{1}{n-k+1} \sum_{v=0}^{n-k} S_v$$

$$t_n = \sum_{k=0}^{n} a_{n(n-k)} \sigma_{n-k}$$

$$= \sum_{k=0}^{n} a_{n,n-k} \frac{1}{n-k+1} \sum_{r=0}^{n-k} S_r$$

If $t_n \to S$ as $n \to \infty$ then sequence $\{S_n\}$ or Infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by the matrix-Cesaro mean TC_1 method to S.
Important Particular cases of matrix César means are
(i) $(N, p)n\, C_1$ means, when $a_n, a_{n-k} = \frac{p_k}{p_n}$ where $P_n = \sum_{k=0}^{n} P_k \neq 0$
(ii) $(N, p)n\, C_1$ means, when $a_n, a_{n-k} = \frac{p_{n-k}}{p_n}$ where $P_n = \sum_{k=0}^{n} P_k \neq 0$
(iii) $(N, p)n\, C_1$ means, when $a_n, a_{n-k} = \frac{p_k q_{n-k}}{R_n}$ where $R_n = \sum_{k=0}^{n} P_k\, q_{n-k} \neq 0$

We write $\varphi(t) = f(x+t) + f(x-t) - f(x)$. Fourier series

$$k(n,t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\sin^2(n-k+1)\, t/2}{\sin^2 t/2}$$

$$t_n^{TC_1} = \sum_{k=0}^{n} a_{n,n-k} \frac{1}{n-k+1} \sum_{v=0}^{n-k} S_v$$

$$(E,q)T = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} S_v$$

$$(E,q)TC^1 = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} a_{k,v} \cdot \frac{1}{v+1} \sum_{i=0}^{n-v} S_i$$

Let $f(t)$ be a periodic function with periodic 2π, L-integrable over $(-\pi, \pi)$

The Fourier Series associated with f at any point x is defined by

$$f(x) := \frac{a}{2} + \sum_{n=0}^{\infty} (a_n cosnx + b_n sinnx) \equiv \sum_{n=0}^{\infty} A_n(x) \quad (2)$$

Let $S_n(f;x)$ be the nth partial sum

The $L_\infty - norm$ of a function $f: R \to R$ is defined by

$$||f||_\infty = \sup\{|f(x)|: x \in R\}$$

and the $L_v - norm$ is defined by $||f||_v = \left(\int_0^{2\pi}|f(x)|^v\right)^{\frac{1}{v}}, v \geq 1$

The degree of approximation of a function $f: R \rightarrow R$ by a triangular polynomial $P_n(x)$ of degree n under the norm. $||\cdot||_\infty$ is defined by [1]

$$||P_n - f||_\infty = \sup\{P_n(x) - f(x)|: x \in R\}$$

and the degree of approximation $E_n(f)$ of function $f \in L_v$ is given by

$$E_n(f) = \min_{P_n}||P_n - f||_v$$

This method of approximation is called the triangular Fourier series
A function $f \in Lip(\alpha)$ If

$$|f(x+t) - f(x)| = 0\,(f|t|^\alpha)\; 0 < \alpha \leq 1$$

We will use the following Notation throughout this paper.

$$\varphi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$k(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n}\frac{a_{n,n-k}}{n-k+1}\frac{\sin^2(n-k+1)\,t/2}{\sin^2 t/2}$$

Further, the $(E,q)T(C,1)$ is assumed to be regular.

2. Known Theorem

Dealing with the degree of approximation by the product (E,q), A mean of the Fourier series [2] produced the following theorem

Let $A = (a_{m\times n})_{\infty\times\infty}$ be a regular matrix. If f is 2π –periodic function of Class Lipα, then the degree of approximation by the Product (E,q) summability means that its Fourier series is given by $||T_n - f||_\infty = 0\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1$
Where $T_n = (E,q)T(C,1)$

3. Main Theorem

Let $f: R \rightarrow R$ be 2π periodic Lebesgue integrable Lipα function in $(-\pi,\pi)$. Then, the degree of approximation of the function F by the Euler-Matrix, Cesaro means of the Fourier series is given by

$$||T_n - f||_\infty = 0\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1 \qquad \text{Where } T_n = (E,q)T(C,1)$$

A part of the following are important lemmas to prove the assumed theorem

Lemma (1)

$$k(n,t) = \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sum_{v=0}^{k}a_{k,k-v}\,\frac{1}{n-k+1}\sum_{i=0}^{k-v}a_{k,k-v}\left\{\frac{sin\left(i+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}$$

For $0 \leq t \leq \frac{1}{n+1}$, we have $\sin nt \leq n\sin t$

$$\sum_{i=0}^{k-v}\frac{(2i+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\;\frac{1}{k-v+1}(k-v+1)$$

$$\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\sum_{v=0}^{k}a_{k,k-v}\right|$$

$$\leq \frac{N}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|$$

$$= \frac{N}{2\pi(1+q)^n}(1+q)^n$$
$$= \frac{N}{2\pi} = O(1)$$

This proves the lemma.

Lemma (2)

For $\frac{1}{n+1} \leq t \leq \pi$, We have by Jordan's Lemma $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$, $\sin nt \leq 1$

$$k(n,t) = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} a_{k,k-v} \frac{1}{n-k+1} \sum_{i=0}^{k-v} a_{k,k-v} \left\{ \frac{\sin\left(i+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}$$
$$\leq \_\_\_\_\_ \sum_{i=0}^{k-v} \frac{1}{t/\pi}$$
$$\leq \frac{1}{2\pi(1+q)^n t} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} a_{k,k-v} \right|$$
$$= \frac{N}{0\left(\frac{1}{t}\right)}$$

This proves the lemma.

4. Proof of the Main Theorem

Using Riemann-Lévesque theorem, we have for the n^{th} partial sum. $S_n(f;x)$. The Fourier Series is given by

$$S_n(f;x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \varphi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} a_{k,k-v} \cdot \frac{1}{k-v+1} \sum_{i=0}^{k-v} \frac{\sin\left(i+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \cdot dt$$
$$= \int_0^{\pi} \varphi(t) K_n(t) \cdot dt$$
$$= \left\{ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right\} \varphi(t) K_n(t) \cdot dt$$
$$= I_1 + I_2 \text{ say}$$

$$|I_1| = \frac{1}{2\pi(1+q)^n} \left| \int_0^{\frac{1}{n+1}} \varphi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} a_{k,k-v} \frac{1}{n-k+1} \sum_{i=0}^{k-v} a_{k,k-v} \left\{ \frac{\sin\left(i+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} dt \right|$$
$$\leq O(1) \int_0^{\frac{1}{n+1}} |\varphi(t)| \cdot dt \quad \text{using Lemma (1)}$$
$$= O(1) \int_0^{\frac{1}{n+1}} |t^{\alpha}| \cdot dt$$
$$= O(1) \left[\frac{1}{(\alpha+1)(n+1)^{\alpha+1}} \right]$$
$$= O\left[\frac{1}{(n+1)^{\alpha+1}} \right]$$

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} |\phi(t)||k_n(t)| dt$$
$$= \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| o\left(\frac{1}{t}\right) dt, \quad \text{using Lemma (2)}$$
$$= \int_{\frac{1}{n+1}}^{\pi} |t^{\alpha}| o\left(\frac{1}{t}\right) dt,$$
$$= O\left[\frac{1}{(n+1)^{\alpha}} \right]$$

Then from (I_1) and (I_2), we have

$$\left| t^{E^q t C^1} - f(x) \right| = O\left[\frac{1}{(n+1)^{\alpha}} \right], 0 < \alpha < 1$$

This completes the proof of the main theorem.

5. Conclusion

The result established here is a general form of the $(E,q)T$ mean. When C=1.

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