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Original Article

Liouville Product Cordial Labeling of Certain Graphs

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Abstract *- Graph labeling provides a structured approach for assigning numerical values to the elements of a graph in order to study its underlying properties. Motivated by the interaction between number theory and graph theory, this paper introduces a new labeling scheme called Liouville Product Cordial Labeling based on the Liouville function, which assigns the values -1 and 1. In this labeling, the value of each edge is obtained as the product of the Liouville values associated with its end vertices. A graph is said to satisfy Liouville Product Cordial Labeling if the counts of edges labeled -1 and 1 differ by at most one. In this paper, the existence of this newly developed labeling for various families of graphs, such as Wheel, Tortoise graph, Star, Bistar, Twig graph, Sparkler graph, and Lobster graph, has been examined, and the conditions under which such labeling can be constructed have been identified. The proposed approach extends cordial labeling concepts through a multiplicative framework and enriches the study of number-theoretic graph labelings.*

Keywords *- Graph Labeling, Wheel, Tortoise Graph, Twig Graph, Sparkler Graph, and Lobster Graph.*

1. Introduction

Graph labeling is a well-established branch of graph theory concerned with assigning integers to the vertices or edges of a graph according to prescribed conditions [1]. Owing to its strong theoretical foundation and wide applicability, this area has received sustained attention from researchers. Early developments and fundamental concepts of graph labeling are discussed in works such as [2]. Depending on the nature of the imposed restrictions, a graph may admit a wide variety of labeling patterns. Consequently, numerous labeling schemes have been introduced and studied, including cordial, magic, antimagic, graceful, fuzzy, harmonious, and related labeling [8-12]. Labeled graphs have also found applications in areas such as network design, information theory, and mathematical modeling [2,3]. A comprehensive survey of labeling techniques and their developments can be found in the work of Gallian [2].

By incorporating number-theoretic ideas into graph labeling, Asharani et al. proposed Möbius cordial labeling and investigated its properties for several classes of graphs [4]. Motivated by this direction, the Liouville function was later employed to define Liouville Difference Cordial Labeling, and its existence was established for certain acyclic graphs.

The Liouville function takes only two possible values according to whether an integer has an even or odd number of prime factors, making it suitable for constructing balanced edge labelings. Moreover, its inherent multiplicative nature offers a natural way to define product-based edge labels that reflect number-theoretic features within graphs. Although additive and difference-based labelings derived from number-theoretic functions have been widely studied, the multiplicative use of the Liouville function in cordial labeling has not yet been addressed. In particular, a product-oriented cordial labeling scheme based on Liouville values has not been previously reported.

Motivated by this observation, the present study introduces Liouville Product Cordial Labeling, in which edge labels are obtained from the product of the Liouville values of their end vertices. In addition, constructive labeling techniques are presented to establish the existence of this labeling for several standard graph families, including wheel, tortoise, Star, Bistar, Twig, Sparkler, and Lobster Graphs. These findings show that Liouville-based product labeling forms a meaningful extension of cordial labeling within a number-theoretic setting.

Throughout this paper, standard graph-theoretic terminology is used in accordance with Bondy [5].

2. Preliminaries

Let $G = G(V, E)$ denote a simple, finite, connected, and undirected graph. We summarize below the basic definitions that are essential for the present investigation.

Definition 2.1:

A Tortoise graph $T_n, n \geq 3$is a graph obtained from the path $v_1, v_2, v_3, \ldots, \rightleftarrows v_n$ where n n is odd by attaching an edge between v_i and v_{n+1-i} for $i = 1,2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

Definition 2.2:

A Twig $TW(n), n \geq 3$is a tree obtained from a path by attaching exactly two pendant edges to each internal vertex of the path.

Definition 2.3:

A Sparkler, denoted as $P_{(m+n)}$, is a graph obtained from the path P_m by appending n edges to an endpoint. This is a special case of a caterpillar. That is the graph obtained by joining an end vertex of a path to the centre of a star.

Definition 2.4:

The Lobster graph $L_n(2, r)$ is a graph formed from a path on n vertices as a backbone, each vertex in the backbone is joined to two different vertex hands, and each vertex hand is joined to *r* different vertex fingers, each of which has degree one.

Definition 2.5:

The Liouville Lambda function, denoted by $\lambda(n)$ and named after Joseph Liouville, is an important arithmetic function. Its value is 1 if *n* is the product of an even number of prime numbers and -1 if it is the product of an odd number of primes. (1 is given by the empty product). $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of not necessarily distinct prime factors of *n*, with $\Omega(1) = 0.$[8,9]

3. Main Results

Definition 3.1: (Liouville Product Cordial labeling).

Let V and E denote the vertex and edge set of a simple graph $G = (V, E)$, respectively. Let $e_{f*}(k)$ be the number of edges labeled with k. Then, a $1 - 1$ function $f: V \rightarrow N$ is said to be a *Liouville Product Cordial Labeling (LPCL)* of G if the induced edge function $f^*: E \rightarrow \{-1,1\}$ is defined by

$$f * (uv) = \lambda\big(f(u)f(v)\big),$$

with the condition $\left|e_{f*}(-1) - e_{f*}(1)\right| \leq 1$, for all $uv \in E$, where $\lambda(n)$ denotes the Liouville lambda function of the integer n.

Note: *Liouville Product Cordial Graph* is a graph that admits *Liouville Product Cordial Labeling.*

Theorem 3.1: For all $n \geq 4$, there exists a Liouville Product Cordial labelling for the wheel W_n.

Proof: Let $V = \{u_0\} \cup \{v_i; 1 \leq i \leq n-1\}$ and $E = \{u_0 v_i; 1 \leq i \leq n-1\} \cup \{v_i v_{i+1}; 1 \leq i \leq n-2\} \cup \{v_{n-1} v_1\}$ denote the vertex and edge set of the wheel W_n , respectively.

$\Rightarrow |V| = n$ and $|E| = 2(n-1)$

Let the vertex labelling of W_n be defined by the 1-1 function as follows:

$$f(u_0) = 2^2$$

and for all $1 \leq i \leq n-1$,

$$f(v_i) = 2^{2i-1}$$

Now, by the definition of induced edge function (3.1), for every edge in E,

$f^*(u_0 v_i) = -1$, for all $1 \leq i \leq n-1$

$f^*(v_i v_{i+1}) = 1$, for all $1 \leq i \leq n-2$

$$f^*(v_1 v_{n-1}) = 1$$

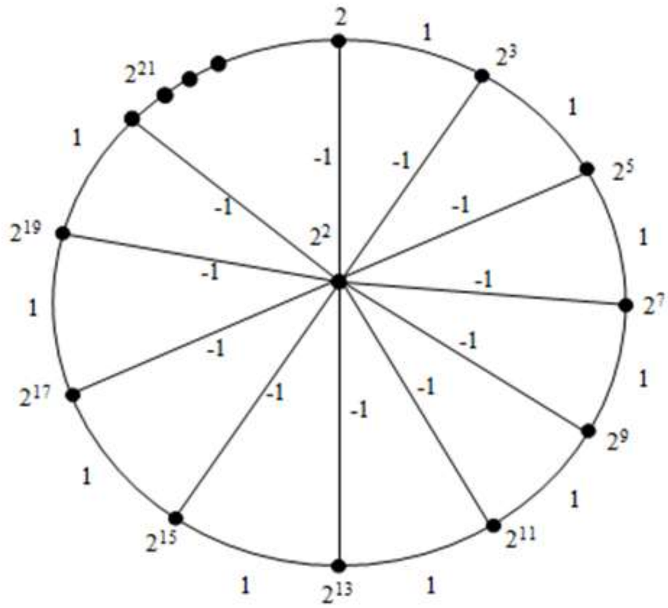

Fig. 1 LPCL of Wheel Graph W_n, When n is Even

In particular, the labeled edges with the labels -1 and 1 are equal, and it is $n-1$ in each case, respectively. i.e., $e_{f^*}(-1) = e_{f^*}(1) = n-1$

Thus, the labeled edges satisfy the condition $\left|e_{f^*}(-1) - e_{f^*}(1)\right| \leq 1$. Hence, there exists a Liouville Product cordial labeling for the wheel W_n.

Theorem 3.2: For all $n \geq 3$, there exists a Liouville Product Cordial labelling in the Tortoise graph T_n.
Proof: Let the vertex set and edge set of the Tortoise graph T_n be denoted by $V = \{u_i; 1 \leq i \leq n\}$ and $E = \{u_i u_{i+1}; 1 \leq i \leq n-1\} \cup \left\{u_i u_{n+1-i}; 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\right\}$ respectively.
$\Rightarrow |V| = n$ and $|E| = \left\lfloor \frac{3n-2}{2} \right\rfloor$

Define the vertex labeling of the graph $\boldsymbol{T_n}$ by the 1-1 function as follows:
for all $1 \leq i \leq n$,

$$f(u_i) = \begin{cases} 2^{\left\lfloor \frac{3i}{2} \right\rfloor} & if\, i \equiv 1(mod\, 2) \\ 2^{\frac{3i}{2}} & if\, i \equiv 0(mod\, 2) \end{cases}$$

Case (i): When $\left\lfloor \frac{n}{2} \right\rfloor$ is odd
By the definition (3.1) of the induced edge function, for every edge in E, the edge function satisfies the following results: for all. $1 \leq i \leq n$,

$$f^*(u_i u_{i+1}) = \begin{cases} 1 & if\, i \equiv 1(mod\, 2) \\ -1 & if\, i \equiv 0(mod\, 2) \end{cases}$$

and

$$f^*(u_i u_{n+1-i}) = \begin{cases} -1 & if\, i \equiv 1(mod\, 2) \\ 1 & if\, i \equiv 0(mod\, 2) \end{cases}$$

In this case, the number of edges with the labels -1 and 1 are $\left\lfloor \frac{3n+2}{4} \right\rfloor$ and $\left\lfloor \frac{3n-2}{4} \right\rfloor$ respectively.
Case (ii): When $\left\lfloor \frac{n}{2} \right\rfloor$ is even
Then, by the definition (3.1) of the induced edge function, for every edge in E, the labeled edges satisfy the following results: for all. $1 \leq i \leq n$,

$$f^*(u_iu_{i+1}) = \begin{cases} 1 & if i \equiv 1(mod\ 2) \\ -1 & if i \equiv 0(mod\ 2) \end{cases}$$

and

$$f^*(u_iu_{n+1-i}) = \begin{cases} 1 & if i \equiv 1(mod\ 2) \\ -1 & if i \equiv 0(mod\ 2) \end{cases}$$

In this particular case, the number of edges with the labels -1 and 1 is equal, and it is $\frac{3n-2}{4}$ in each case, respectively. In both the above cases, it is also evident that the labeled edges satisfy the condition $|e_{f^*}(-1) - e_{f^*}(1)| \leq 1$.
Therefore, there exists a Liouville Product Cordial labeling for the Tortoise graph T_n for all $n \geq 3$.

Theorem 3.3: For all $n \geq 2$, Star S_n is a Liouville Product Cordial graph.
Proof: Let $V = \{u_1, u_2, \ldots, u_n\}$ and $E = \{u_1u_2, u_1u_3, \ldots, u_1u_n\}$ be the vertex and edge set of the Star S_n, respectively.
Define the vertex labeling of the Star S_n by the 1-1 function as follows:
$f(u_i) = 2^{i-1}$, for all $1 \leq i \leq n$.

Now by the definition of induced edge function [3.1], for every edge $u_1u_i \in E$ and $1 < i \leq n$,

$$f^*(u_1u_i) = \begin{cases} 1 & if i \equiv 1(mod\ 2) \\ -1 & if i \equiv 0(mod\ 2) \end{cases}$$

Case (i): When n is odd
The number of labeled edges with the labels -1 and 1 is equal, and it is $\left\lfloor\frac{n}{2}\right\rfloor$ in each case, respectively.
i.e., $e_{f^*}(-1) = e_{f^*}(1) = \left\lfloor\frac{n}{2}\right\rfloor$
Case (ii): When n is even

In this case, the number of edges with the labels -1 and 1 are $\frac{n}{2}$ and $\frac{n-2}{2}$ respectively.
i.e.,$e_{f^*}(-1) = \frac{n}{2}$ and $e_{f^*}(1) = \frac{n-2}{2}$
In both the above-mentioned cases $|e_{f^*}(-1) - e_{f^*}(1)| \leq 1$.
Hence, Star S_n admits Liouville Product Cordial labeling for all $n \geq 2$.

Theorem 3.4: For all $n > 3$, there exists a Liouville Product Cordial labelling in Bistar $B_{n,n}$.
Proof: Let $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and $E = \{u_1u_i, v_1v_i, \ldots, u_1v_1; 1 < i \leq n\}$ be the vertex and edge set of the Bistar $B_{n,n}$, respectively.
Consider the vertex labeling of the Bistar by the 1-1 function as follows:

$$\begin{aligned} &f(u_1) = 2; \\ &f(v_1) = 1; \\ &f(u_i) = 2^{2k} and \\ &f(v_i) = 3^{2k}, for i = 2,3,\ldots,\left\lfloor\frac{n}{2}\right\rfloor + 1; k \in N \\ &f(u_i) = 2^{2k+1} and \\ &f(v_i) = 3^{2k+1}, for i = \left\lfloor\frac{n}{2}\right\rfloor + 2,\ldots,n; k \in N \end{aligned}$$

Then by the definition [3.1], for every edge in E,

$$f^*(u_1u_i) = \begin{cases} -1 & if i = 2,3,\ldots,\left\lfloor\frac{n}{2}\right\rfloor + 1 \\ 1 & if i = \left\lfloor\frac{n}{2}\right\rfloor + 2,\ldots,n \end{cases}$$

$$f^*(v_1v_i) = \begin{cases} 1 & if i = 2,3,\ldots,\left\lfloor\frac{n}{2}\right\rfloor + 1 \\ -1 & if i = \left\lfloor\frac{n}{2}\right\rfloor + 2,\ldots,n \end{cases}$$

$$f^*(u_1v_1) = -1$$

Among the $2n - 1$ edges, the number of edges with the labels -1 and 1 are n and $n - 1$ respectively.

i.e.,$e_{f^*}(-1) = n$ and $e_{f^*}(1) = n - 1$

$$\Rightarrow \left|e_{f^*}(-1) - e_{f^*}(1)\right| \leq 1$$

Hence, Bistar $B_{n,n}$ is a Liouville Product Cordial graph.

Theorem 3.5: For all $n \geq 3$, there exists a Liouville Product Cordial labelling in a Twig graph $TW(n)$.
Proof: Let $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-2}, w_1, w_2, \ldots, w_{n-2}\}$ and $E = E_1 \cup E_2 \cup E_3$, where $E_1 = \{u_i u_{i+1}, 1 \leq i \leq n-1\}$,

$$E_2 = \left\{u_{j+1} v_j, 1 \leq j \leq n-2\right\},$$
$$E_3 = \left\{u_{j+1} w_j, 1 \leq j \leq n-2\right\}$$

be the vertex and edge set of the graph $TW(n)$, respectively.

Let the vertex labeling of the graph be defined by the 1-1 function $f: V \to N$ as follows:

$$f(u_i) = \begin{cases} 2, & if\ i = 1 \\ 2\, f(u_{i-1}), & if\ i \equiv 0\ (mod\ 2), \qquad 1 < i \leq n \\ 2^2\, f(u_{i-1}), & otherwise \end{cases}$$

and for $1 \leq j \leq n-2$,

$$f\left(v_j\right) = 3^{2j-1}$$
$$f\left(w_j\right) = 3^{2j}$$

Then by the definition [3.1] of induced edge function, for every edge in E, and for $1 < i \leq n$,

$$f^*(u_i u_{i-1}) = \begin{cases} -1 & if i \equiv 0(mod\ 2) \\ 1 & if i \equiv 1(mod\ 2) \end{cases}$$

Also for $1 \leq j \leq n-2$,

$$f^*\left(u_{j+1} v_j\right) = \begin{cases} -1 \rightleftarrows\ if\ \rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\ j \equiv 1 or 2(mod\ 4) \\ 1\quad if\ \rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\ j \equiv 3 or 0(mod\ 4) \end{cases}$$

$$f^*\left(u_{j+1} w_j\right) = \begin{cases} 1\ \rightleftarrows\ if\ \rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\ j \equiv 1 or 2(mod\ 4) \\ -1\quad if\ \rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\rightleftarrows\ j \equiv 3 or 0(mod\ 4) \end{cases}$$

Case (i): When n is odd
By the definition of induced edge function f^*, the number of edges with the labels -1 and 1 is equal, and it is $\frac{3n-5}{2}$ in each case, respectively.
Thus, $e_{f^*}(-1) = e_{f^*}(1) = \frac{3n-5}{2}$
Case (ii): When n is even
In this case, by the definition of induced edge function f^*, the number of edges with the labels -1 and 1 are $\left\lfloor\frac{3n-3}{2}\right\rfloor$ and $\left\lfloor\frac{3n-5}{2}\right\rfloor$ respectively.
i.e.,$e_{f^*}(-1) = \left\lfloor\frac{3n-3}{2}\right\rfloor$ and $e_{f^*}(1) = \left\lfloor\frac{3n-5}{2}\right\rfloor$
In both the above-mentioned cases, the labeled edges always satisfy the condition $\left|e_{f^*}(-1) - e_{f^*}(1)\right| \leq 1$.
Hence, for all $n \geq 3$, the Twig graph $TW(n)$ admits Liouville Product Cordial labeling.

Theorem 3.6: For all, $m, n \in N$, there exists a Liouville Product Cordial labelling for the Sparkler graph $P_{(m+n)}$.
Proof: Let $V = \{u_i; 1 \leq i \leq m\} \cup \left\{v_j; 1 \leq j \leq n\right\}$ and $E = \{u_i u_{i+!}; 1 \leq i < m\} \cup \left\{u_m v_j; 1 \leq j \leq n\right\}$ be the vertex and edge set of the graph, where $\{u_i; 1 \leq i \leq m\}$ are the vertices of the path P_m and $\left\{v_j; 1 \leq j \leq n\right\}$ are the vertices joined to the vertex u_m to form the sparkler graph $P_{(m+n)}$.
Let the vertex labeling be defined by the 1-1 function $f: V \to N$ as follows: for all $1 \leq i \leq m$,

$$f(u_i) = \begin{cases} 2^2 f(u_{i+1}) & if\ i \equiv 1(\mathrm{mod}\ 2);\ 1 \leq i \leq m-2 \\ 2 f(u_{i+1}) & if\ i \equiv 0(\mathrm{mod}\ 2);\ 1 \leq i \leq m-2 \\ 2 & if\ i = m-1 \\ 3 & if\ i = m \end{cases}$$

and for all $1 \leq i \leq n$,

$f(v_j) = 3^{j+1}$

Now, from the definition [3.1], for every edge, E, the following results hold good.

For $m \equiv 0(mod\,2)$ and $1 \leq j \leq n$

$$f^*(u_i u_{i+1}) = \begin{cases} 1 & for i \equiv 1(mod\,2) \\ -1 & for i \equiv 0(mod\,2) \end{cases}$$

$$f^*(u_m v_j) = \begin{cases} -1 & for j \equiv 1(mod\,2) \\ 1 & for j \equiv 0(mod\,2) \end{cases}$$

and for $m \equiv 1(mod\,2)$ and $1 \leq j \leq n$

$$f^*(u_i u_{i+1}) = \begin{cases} -1 & for i \equiv 1(mod\,2) \\ 1 & for i \equiv 0(mod\,2) \end{cases}$$

$$f^*(u_m v_j) = \begin{cases} -1 & for j \equiv 1(mod\,2) \\ 1 & for j \equiv 0(mod\,2) \end{cases}$$

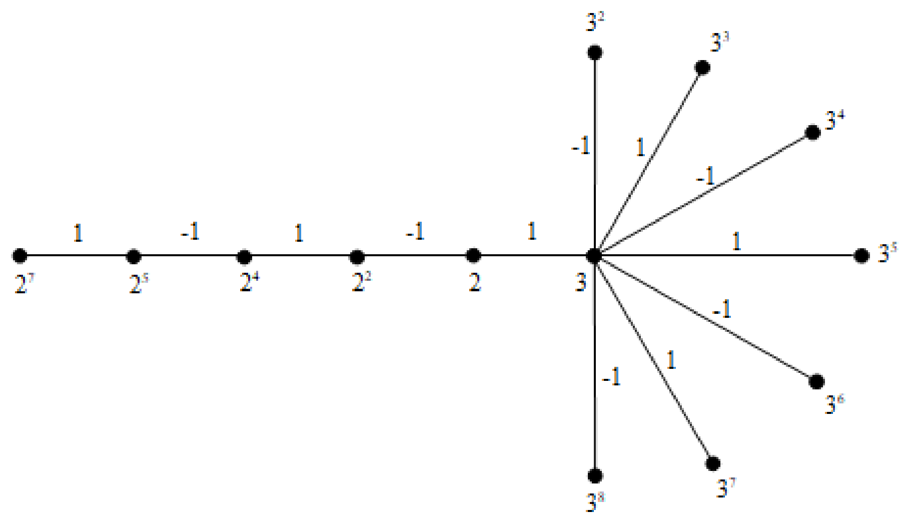

Fig. 2 LPCL of Sparkler Graph $P_{(6+7)}$

Case (i): When both m n are odd

By the definition of induced edge function f^*, the number of edges with the labels -1 and 1 are $\frac{m+n}{2}$ and $\frac{m+n-2}{2}$, respectively.

i.e.,$e_{f^*}(-1) = \frac{m+n}{2}$ and $e_{f^*}(1) = \frac{m+n-2}{2}$

Case (ii): When both m n are even

In this case, $e_{f^*}(-1) = \frac{m+n-2}{2}$ and $e_{f^*}(1) = \frac{m+n}{2}$

Case (iii): When either m n is odd

In this case, $e_{f^*}(-1) = e_{f^*}(1) = \left\lfloor \frac{m+n}{2} \right\rfloor$

In all the above-mentioned cases, the labeled edges satisfy the condition $|e_{f^*}(-1) - e_{f^*}(1)| \leq 1$.

Thus, Sparkler graph $P_{(m+n)}$ admits Liouville Product Cordial labeling.

Theorem 3.7: For all, $n, r \in N$, there exists a Liouville Product Cordial labelling in the Lobster graph $L_n(2, r)$.

Proof: Let $V = \{u_i\} \cup \{u_{i,j}\} \cup \{u_{i,j,k}\}$, where $1 \leq i \leq n; j = 1,2; 1 \leq k \leq r$ and $E = E_1 \cup E_2 \cup E_3$, where $E_1 = \{u_i u_{i+1}; 1 \leq i < n\}$,

$$E_2 = \{u_i u_{i,j}; 1 \leq i \leq n and j = 1,2\},$$
$$E_3 = \{u_{i,j} u_{i,j,k}; 1 \leq i \leq n; j = 1,2 and 1 \leq k \leq r\}$$

be the vertex and edge set of the Lobster graph $L_n(2, r)$, respectively.

Consider the vertex labeling of the graph by the 1-1 function $f: V \rightarrow N$ as follows:

$$f(u_i) = \begin{cases} 2, & if\ i = 1 \\ 2\,f(u_{i-1}), & if\ i \equiv 0\ (mod\ 2), \\ 2^2\,f(u_{i-1}), & otherwise \end{cases} \quad 1 < i \leq n$$

Moreover, for the remaining vertices, the following two cases hold good:
Case (i): When $1 \leq i \leq n$ and n or $i \equiv 0(mod\ 4)$

$$f(u_{i,j}) = \begin{cases} 3^{2i} & if j = 1 \\ 3^{2i+1} & if j = 2 \end{cases}$$

and for $1 \leq k \leq r$,

$$f(u_{i,j,k}) = \begin{cases} p_i^{2k-1} & if j = 1 \\ p_i^{2k+5} & if j = 2 \end{cases}$$

Case (ii): When $1 \leq i \leq n$ and $i \equiv 2(mod\ 4)$ or $i \equiv 3(mod\ 4)$

$$f(u_{i,j}) = \begin{cases} 3^{2i+1} & if j = 1 \\ 3^{2i} & if j = 2 \end{cases}$$

and for $1 \leq k \leq r$,

$$f(u_{i,j,k}) = \begin{cases} p_i^{2k} & if j = 1 \\ p_i^{2k+6} & if j = 2 \end{cases}$$

where p p_i is any prime other than 2 and 3.
By the definition of the induced edge function [3.1], for every edge in E,

$$f^*(u_i u_{i+1}) = \begin{cases} 1 & if i \equiv 0(mod\ 2) \\ -1 & if i \equiv 1(mod\ 2); 1 \leq i < n \end{cases}$$

$$f^*(u_i u_{i,j}) = \begin{cases} -1 & if j = 1 \\ 1 & if j = 2 \end{cases}$$

$$f^*(u_{i,j} u_{i,j,k}) = \begin{cases} -1 & if j = 1 \\ 1 & if j = 2 \end{cases}$$

where $1 \leq i \leq n and 1 \leq k \leq r.$
Case (i): When n is odd
The number of labeled edges with the labels -1 and 1 is equal, and it is $\left\lfloor \frac{n}{2} \right\rfloor + n(1 + r)$ in each case, respectively.
i.e., $e_{f^*}(-1) = e_{f^*}(1) = \left\lfloor \frac{n}{2} \right\rfloor + n(1 + r)$
Case (ii): When n is even
In this case, the number of edges with the labels -1 and 1 are $n\left(\frac{3}{2} + r\right)$ and $n\left(\frac{3}{2} + r\right) - 1$ respectively.
i.e.,$e_{f^*}(-1) = n\left(\frac{3}{2} + r\right)$ and $e_{f^*}(1) = n\left(\frac{3}{2} + r\right) - 1$
In both cases, it is clear that the labeled edges always satisfy the condition $|e_{f^*}(-1) - e_{f^*}(1)| \leq 1$.
Hence, the Lobster graph $L_n(2, r)$, for all, $n, r \in N$ is a Liouville Product Cordial graph.

4. Conclusion

In this paper, a new labeling technique called Liouville Product Cordial labeling has been introduced and studied for certain graph classes such as W_n,T_n,$K_{(1,n)}$, $B_{(n,n)}$, $TW(n)$, $P_{(m+n)}$ and $L_n(2, r)$. In future work, this labeling technique for broader classes, such as bipartite graphs and complete bipartite graphs, can be investigated, which may reveal additional structural properties of this labeling scheme.

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