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Original Article

The Domination Number and Bondage Number of Circular Arc Graphs with Different Types of Circular Arc Families of "A"

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Abstract *- Domination number and Bondage number are important parameters in graph theory that Characterize dominating structures and the effect of edge removal on domination respectively. Circular arc graphs are intersection graphs generated by families of arcs on a circle and exhibit structural properties that provide a useful setting for studying these parameters. In this paper, we investigate the domination number and bondage number of circular arc graphs by examining the relationships between the arrangements of arcs and the adjacency structure of the corresponding graphs. We establish results concerning dominating sets and analyze how the deletion of suitable edges can affect the domination number. Particular attention is given to connected and consecutive circular arc structures, together with structural conditions involving vertex degree. Based on this property, we derive formulas for the selected circular arc graphs. In this paper we have presented "The domination number and bondage number of circular arc graphs with different types of circular arc families of "A"*

Keywords *- Circular Arc Family, Circular Arc Graph, Overlap circular arc graph, Dominating Set, Domination Number, Bondage Number.*

1. Introduction

The history of graph theory may be specifically traced to 1735, when the Swiss mathematician Leonhard Euler solved the Konigsberg bridge problem. Graph theory in mathematics is the study of graphs. Graphs are one of the prime objects of study in discrete mathematics. In mathematics graph theory is the study of graphs, which are mathematical structures, used to model pair wise relations between objects. A graph structure is very suitable for representing relationships between objects in the abstract, and a large number of combinatorial problems can be modeled as problems on the graph structure. A survey on results and applications of dominating sets was presented by E. J. Cockade and S. T. Hedetniemi.

Let G be a simple graph. A subset D of V is said to be a dominating set of G if every vertex in V/D is adjacent to a vertex in D. The domination number $\gamma(G)$ of G is the minimum Cardinality of a dominating set [1], [2]. Let G be a graph and let A = $\{A_1, A_2 \ldots A_n\}$ be a circular – arc family on a circle, where each A_i is an arc. Then G is called a circular – arc graph if there is a one – to – one correspondence between V and A such that two vertices in V are adjacent if and only if their corresponding arcs in A intersect. We denote this circular - arc graph by G (A). If circular - arc family is proper, and then the corresponding graph is called a proper circular - arc graph. The bondage number B(G) of a non – empty graph G is The minimum cardinality among all sets of edges E_1, for which $\gamma(G - E_1) > \gamma(G)$ [3],[4].

A dominating set in a graph G is a subset of vertices $D \subseteq V$, such that every vertex not in D is adjacent to at least one vertex in D. The academic literature surrounding dominating sets spans foundational mathematics, computation complexity and modern network applications.

Systematic mathematical study began around 1960. The defective reference books in the field are authored by Teresaw. Haynes, Stephen Hedentniemi, and its comparison values, which codify standard terminology, bounds and variations.Defined as the minimum cardinality (size) of a dominating set in G. Extensive literature establishes upper and lower bounds relative to other graph invariant like maximum or minimum degree, order of the graph, and independence number.

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Testing whether domination number is less than or equal to any number is one of the classic non-polynomial-hard or NP-hard in general graphs, efficient linear-time or polynomial time exact algorithms exist for restricted graph families like interval family and circular-arc-families, including trees, split graphs, interval graphs, circular-arc-graphs, threshold graphs, and outer planar graphs.

Because finding the exact minimum dominating set is hard, extensive computer science literature focuses on greedy approximation algorithms and set covering reductions with algorithmic performance guarantees.The main major variations and extensions, the literature branches into structural variations of standard domination like connected dominating sets , total dominating sets , independent dominating sets , split dominating sets , non-split dominating sets and bondage number and also, cover dominating sets with degree constraints, such as perfect dominating set proper K-dominating sets, and hop domination.

2. Preliminaries

Existing studies on domination and bondage parameters have mainly considered general graph classes, circular arc graphs, interval graphs and selected special structures. However the domination number and bondage number associated with different structural configurations of circular arc graph corresponding to circular arc families require further investigation. In particular, there is a need to study how consecutive, overlap, regular and non-cyclic circular arc structures influence these parameters and how edge deletion changes the domination structure. This study addresses this gap by investigating domination and bondage numbers for selected classes of circular arc graphs and establishing corresponding results under specified structural conditions.

Let A = $\{A_1, A_2, \ldots\ldots, A_n\}$ is said to be a circular – arc family on a circle, where each A_i is an arc. Without loss of generality we may assume that the end points of all arcs are distinct and no arc covers the entire circle. Denote an arc I that begins At x and ends at the point y in the clockwise direction by (x, y). Define x to be the head and y to be the tail of the arc I and now, I is denoted by (x, y). Two arcs j and I are said to intersect each other if they have non – empty intersection.

Let G (V, E) be a graph. Let A = $\{A_1, A_2, \ldots A_n\}$ be a circular – arc family on a circle, where each A_i is an arc. Then G is called a circular – arc graph if there is a one – to – one correspondence between V and A such that two vertices in V are adjacent if and only if their corresponding arcs in A intersect. We denote this circular - arc graph by G (A). If circular - arc family is proper, and then the corresponding graph is called a proper circular - arc graph.

A subset D of V is said to be a dominating set of G if every vertex in V - D is adjacent to a vertex in D. A dominating set with minimum cardinality is said to be a minimum dominating set.

The cardinality of the dominating set of a graph G is called domination number of G. It is denoted by $\gamma(G)$.

The bondage number B(G) of a non – empty graph G is the minimum cardinality among all sets of edges E_1, for which $\gamma(G - E_1) > \gamma(G)$.

Let A = $\{A_1, A_2, \ldots.., A_n\}$ be a finite family of arcs on the circle. Two arcs A_i and A_j in A are said to overlap each other if they intersect each other but neither one of them contains the other. (or) We know that the overlap circular arc graph is a graph G in which each arc represents a vertex and two vertices are adjacent if and only if their corresponding circular arcs overlap and two arcs $A_i, A_j \in A$ are said to overlap if $A_i \cap A_j \neq \emptyset$ and $A_i \not\subseteq A_j$, $A_j \not\subseteq A_i$.

3. Main Theorem

Theorem-1: Let A = $\{A_1, A_2, \ldots.., A_n\}$ be a finite consecutive circular arc family and G be a regular circular arc graph with degree 2 corresponding to A, G must be connected. If n must be greater than or equal to 3, i.e. $\forall n \geq 3$. Then the domination number with floor function

$$\gamma(G) = \begin{cases} 1 \; if \; n = 3 \\ \left\lfloor \frac{n+k}{2k-1} \right\rfloor if \; n \geq 4 \end{cases}$$ Here k = degree of the vertex i.e. k = 2, n = 3, 4, 5…

Proof: Let A = $\{A_1, A_2, \ldots.., A_n\}$ be a finite consecutive circular arc family and G be a regular circular arc graph with degree 2 corresponding to A, and G must be connected.

Also the number of vertices must be greater than or equal to 3. Now we use the step function, in this we have two types 1) floor function 2) Ceiling function

Particularly in this theorem we can take floor function only, such that the floor function of a real number x, it is the greatest integer less than or equal to x. It is denoted by $\lfloor x \rfloor$.

So, Let us consider the parameter $\gamma(G) = \begin{cases} 1 \; if \; n = 3 \\ \left\lfloor \frac{n+k}{2k-1} \right\rfloor if \; n \geq 4 \end{cases}$, $\forall n \geq 3$ and k = 2 $\rightarrow$ (1)

Here n = number of vertices ($n \geq 3$)
k = degree of vertex (k = 2)

First we will show that the graph G is regular. By the definition of regular graph in which every vertex has same degree. As, if G is a k-regular circular-arc graph corresponding to circular- arc family A. In this we have , the minimum vertex degree = the maximum vertex degree i.e. $\delta = \Delta$.

And also we know that $\Delta = \frac{2|E|}{|V|} \Rightarrow \Delta \times |V| = 2|E|$ and

$\delta = \frac{2|E|}{|V|} \Rightarrow \; \delta \times |V| = 2|E|$

$\therefore \; \Delta \times |V| = \delta \times |V|$

Finally we get $\delta = \Delta$, there must be exist for all regular graphs. Where $|E| = \frac{n \times k}{2}$ = total number of edges in a regular graph

Here n = number of vertices
k = vertex degree

Our aim to find the domination number through the formula of $\gamma(G)$.

Let us consider a circular arc family and k vertex degree from regular circular arc graph then $\gamma(G) = \left\lfloor \frac{n+k}{2k-1} \right\rfloor$ Where k = d (v)

In this it will arise two cases, since $n \geq 3$.

Case – 1: $\gamma(G) = 1$ *if* $n = 3$

Let n=3 then the dominating set DS = { v_1 } and the domination number $\gamma(G) = 1$.
Since $|DS| = 1$.

From equation (1) $\Longrightarrow \; \gamma(G) = \left\lfloor \frac{n+k}{2k-1} \right\rfloor$

Take the number of vertices n = 3 and k = 2 then we get $\gamma(G) = \left\lfloor \frac{3+2}{2(2)-1} \right\rfloor$

$$\gamma(G) = \left\lfloor \frac{5}{4-1} \right\rfloor$$
$$= \left\lfloor \frac{5}{3} \right\rfloor$$
$$= \lfloor 1.6 \rfloor$$
$$= 1$$

Therefore $\gamma(G) = 1$ if n = 3

In this case only the domination number is 1at n = 3.

Case – 2: $\gamma(G) = \left\lfloor \frac{n+k}{2k-1} \right\rfloor$ if n $\geq 4, k = 2$

Suppose n = 4 vertices then the dominating set DS = { v_1, v_3 } and the domination number $\gamma(G) = 2$ as follows from equation (1)

$\gamma(G) = \left\lfloor \frac{n+k}{2k-1} \right\rfloor$

Substitute n = 4 and k = 2 then we get $\gamma(G) = \left\lfloor \frac{4+2}{2(2)-1} \right\rfloor$

$$\gamma(G) = \left\lfloor \frac{6}{4-1} \right\rfloor$$

$$= \left\lfloor \frac{6}{3} \right\rfloor$$

$$= \lfloor 2 \rfloor$$

$$= 2$$

Therefore $\gamma(G) = 2$ if n = 4

Let n = 5 then the dominating set DS = { v_2, v_4 } and the domination number $\gamma(G) = 2$.

From equation (1) $\gamma(G) = \left\lfloor \frac{n+k}{2k-1} \right\rfloor$

Take n = 5 and k = 2 then we get $\gamma(G) = \left\lfloor \frac{5+2}{2(2)-1} \right\rfloor$

$$= \left\lfloor \frac{7}{3} \right\rfloor$$

$$= \lfloor 2.3 \rfloor$$

$$= 2$$

Therefore get $\gamma(G) = 2$ if n = 5

Similarly we will continue m number of vertices $\forall\, m \geq 4$ and k = 2 then we get the domination number $\gamma(G) = \left\lfloor \frac{m+2}{2(2)-1} \right\rfloor = \left\lfloor \frac{m+2}{3} \right\rfloor ; \forall\, m \geq 4$

Therefore the domination number for the consecutive regular circular arc graph with degree 2 is

$$\gamma(G) = \begin{cases} 1 \; if \; n = 3 \\ \left\lfloor \frac{n+k}{2k-1} \right\rfloor if \; n \geq 4 \end{cases}$$ and k = 2

Therefore the theorem is hold

Illustration – 1:

n = 2:

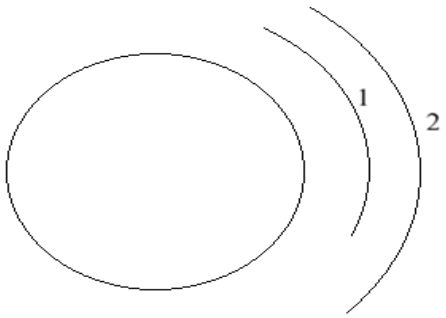


Fig. 1 Circular arc family A



Fig. 2 Circular arc graph G

Dominating set DS = {1} and domination number $\gamma(G) = 1$

Illustration – 2:
n = 16:

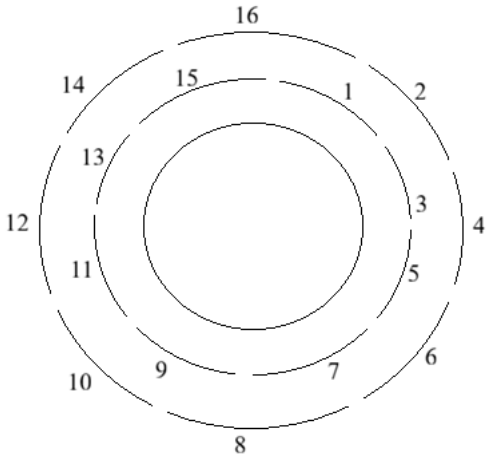


Fig. 3 Circular arc family A

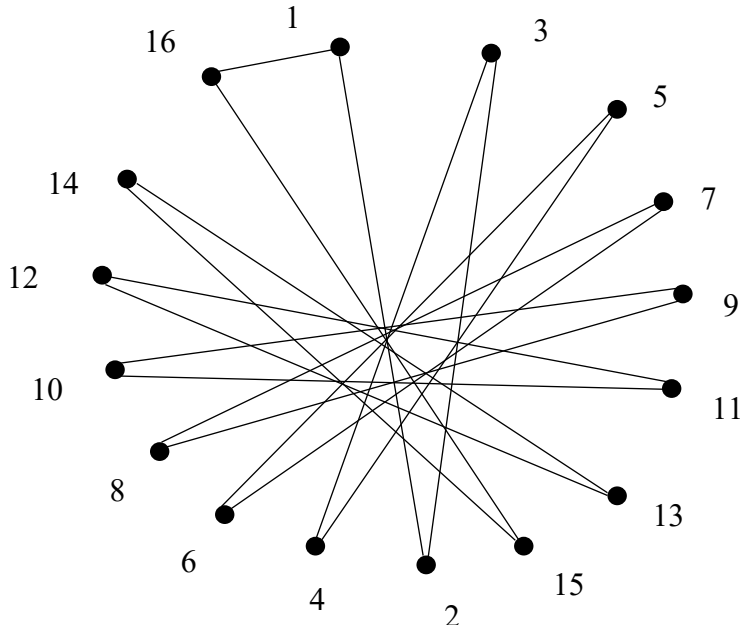


Fig. 4 Circular arc graph G

Dominating set DS = {1, 4, 7, 10, 13, 15} and
Domination number $\gamma(G) = 6$

Theorem-2: Let A = $\{A_1, A_2, \ldots.., A_n\}$ be a finite consecutive circular arc family without cycle and G be a circular arc graph corresponding to A, G must be connected. Where number of vertices must be greater than or equal to 2, i.e. $\forall n \geq 2$. Then the domination number with floor function $(G) = \left\lfloor \frac{n+2}{3} \right\rfloor$; $\forall\, n \geq 2$. Where n = 2, 3…

Proof: Let A = $\{A_1, A_2, \ldots, A_n\}$ be a finite consecutive circular arc family without cyclic and G be a circular arc graph corresponding to A, and G must be connected.

Also the number of vertices must be greater than or equal to 2. In this theorem we have to prove that the domination number with respect to the step function formula.

Now we use the step function and we have two types 1) floor function
2) Ceiling function

Here we are using only floor function , such that the floor function of a real number x, it is the greatest integer less than or equal to x. It is denoted by $\lfloor x \rfloor$.

Also, we know that A = $\{A_1, A_2, \ldots, A_n\}$be a finite consecutive circular arc family on a circle, then the resulting graph is called a consecutive circular arc graph without cyclic if $A_1 \cap A_2 \neq \emptyset$, $A_2 \cap A_3 \neq \emptyset$, and $A_3 \cap A_4 \neq \emptyset$, and so on but $A_n \cap A_1 = \emptyset$.

Now let n = 2 here v_1, v_2 are the vertices and v_1 connected to v_2.Then the dominating set

DS = $\{v_1\}$ and the domination number$\gamma(G) = 1$.

Take n = 2 then we get $\gamma(G) = \left\lfloor \frac{2+2}{3} \right\rfloor = 1$

By the hypothesis $\gamma(G) = \left\lfloor \frac{n+2}{3} \right\rfloor$

Take n = 2 then we get $\gamma(G) = \left\lfloor \frac{2+2}{3} \right\rfloor$
$= \lfloor 1.3 \rfloor$
$= 1$

Therefore $\gamma(G) = 1$ if n = 2

put n = 3 at this place v_1,v_2, v_3 are the vertices and v_1 connected to the v_2, v_2 connected to v_3 but v_3 does not connected to v_1.Then the dominating set DS = $\{v_2\}$ and the domination number $\gamma(G) = 1$.

By the hypothesis $\gamma(G) = \left\lfloor \frac{n+2}{3} \right\rfloor$

Put n = 3 then we get $\gamma(G) = \left\lfloor \frac{3+2}{3} \right\rfloor$
$\gamma(G) = \left\lfloor \frac{5}{3} \right\rfloor$
$= \lfloor 1.6 \rfloor$
$= 1$

Therefore$\gamma(G) = 1$ if n = 3

Let n = 4 in this place v_1,v_2, v_3,v_4 are the vertices and v_1 connected to the v_2, v_2 connected to v_3 and v_3 connected to v_4 but v_4 does not connected to v_1.

Then the dominating set DS = $\{v_2, v_4\}$ and the domination number $\gamma(G) = 2$.

By the hypothesis $\gamma(G) = \left\lfloor \frac{n+2}{3} \right\rfloor$

Take n = 4 then we get $\gamma(G) = \left\lfloor \frac{4+2}{3} \right\rfloor$
$\gamma(G) = \left\lfloor \frac{6}{3} \right\rfloor$
$= \lfloor 2 \rfloor$
$= 2$

Therefore $\gamma(G) = 2$ if n = 4

Let n = 5 where v_1, v_2, v_3, v_4, v_5 are the vertices and v_1 connected to the v_2, v_2 connected to v_3 and v_3 connected to v_4, v_4 connected to v_5 but v_5 does not connected to v_1.

Then the dominating set DS = { v_2, v_4 } and the domination number $\gamma(G) = 2$.

By the hypothesis $\gamma(G) = \left\lfloor \frac{n+2}{3} \right\rfloor$

Take n = 5 then we get $\gamma(G) = \left\lfloor \frac{5+2}{3} \right\rfloor$

$$\gamma(G) = \left\lfloor \frac{7}{3} \right\rfloor$$

$$= \lfloor 2.3 \rfloor$$

$$= 2$$

Therefore $\gamma(G) = 2$ if n = 5

Similarly we take n number of vertices, $\forall\, n \geq 2$ then we get $\gamma(G) = \left\lfloor \frac{n+2}{3} \right\rfloor$

Illustration:

n = 14:

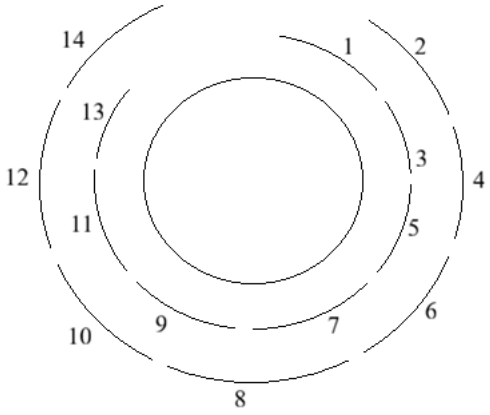


Fig. 5 Circular arc family A

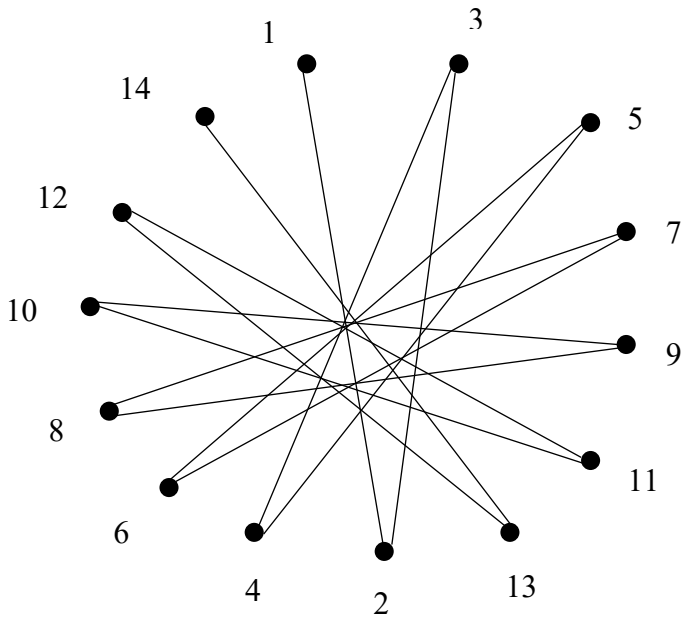


Fig. 6 Circular arc graph G

Dominating set DS = {2, 5, 8, 11, 13} and domination number $\gamma(G) = 5$

Theorem-3: Let A = $\{A_1, A_2, \ldots.., A_n\}$ be a finite consecutive circular arc family and G be a regular circular arc graph with degree 2 corresponding to A,G must be connected. Where n must be greater than or equal to 3, i.e. $\forall\, n \geq 3$. Then the bondage number

$$B(G) = \begin{cases} 2\ if\ n \geq 3\ , n \not\equiv 1(mod 3\ ) \\ 3\ if n \geq 4, n \equiv 1(mod\ 3) \end{cases}$$

Proof: Let A = $\{A_1, A_2, \ldots.., A_n\}$be a finite consecutive circular arc family and G be a regular circular arc graph with degree 2 corresponding to A, and G must be connected.

Also the number of vertices must be greater than or equal to 3. In this theorem we use congruent modulo 3. It can be written as a $\equiv b\ (mod\ 3)$ if and only if $3/\ (a-b)$ and a $\not\equiv b\ (mod\ 3)$ if and only if 3 ∤ (a-b).

We know that by the regular graph, in which every vertex has same degree. As if G is a K-regular graph from circular arc family A. Then the bondage number

$$B(G) = \begin{cases} 2\ if\ n \geq 3\ , n \not\equiv 1(mod 3\ ) \\ 3\ if n \geq 4, n \equiv 1(mod\ 3) \end{cases}$$

In this it will arise two cases

Case-1: B (G) = 2 if $n \geq 3, n \not\equiv 1(mod 3\ )$

Put n = 3 then the dominating set DS = $\{v_1\}$ and domination number $\gamma(G) = 1$.

Let us take e = $\{(v_1, v_2), (v_1, v_3)\}$ then the dominating set of G – e is DS (G - e) = $\{v_1, v_2\}$
And domination number of G - e is $\gamma(G - e) = 2$.
Therefore $\gamma(G - e) > \gamma(G)$, Then the bondage number B (G) = 2.

By the hypothesis if $n \not\equiv 1(mod 3\ )$ Then B (G) = 2
Put n = 3 then $3 \not\equiv 1(mod 3\ )$
⇒ 3 ∤ (3-1)
⇒ 3 ∤ 2
⇒ B (G) = 2
Our hypothesis is hold for n = 3.

Now Let n = 4 then the dominating set DS = $\{v_1, v_3\}$ and domination number$\gamma(G) = 2$.

Let us take e = $\{(v_1, v_2), (v_1, v_4), (v_2, v_3)\}$ then the dominating set of G – e is
DS (G - e) = $\{v_1, v_2, v_3\}$ and domination number of G - e is $\gamma(G - e) = 3$.
Therefore $\gamma(G - e) > \gamma(G)$, Then the bondage number B (G) = 3.

By the hypothesis if $n \not\equiv 1(mod 3\ )$ Then B (G) = 2
Take n = 4 then $4 \not\equiv 1(\ mod 3\ )$
⇒ 3 ∤ (4-1)
⇒ 3 ∤ 3
⇒ B (G) ≠ 2

This is contradicting to the hypothesis. This case only true for all n ≥ 3, $n \not\equiv 1(mod 3\ )$.

Case-2: B (G) = 3 if $n \geq 4, n \equiv 1(mod 3\ )$
In this case we start at n = 4.

Let n = 4 then the dominating set DS = $\{v_1, v_3\}$ and domination number $\gamma(G) = 2$.

Let us take e = $\{(v_1, v_2), (v_1, v_4), (v_2, v_3)\}$ then the dominating set of G – e is

DS (G - e) = {v_1, v_2, v_3} And domination number of G - e is $\gamma(G-e)=3$.

Therefore $\gamma(G-e)>\gamma(G)$, Then the bondage number B (G) = 3.

By the hypothesis if $n \equiv 1(mod3)$ Then B (G) = 3

Take n = 4 then $4 \equiv 1(mod3)$
$\Rightarrow 3 / (4-1)$
$\Rightarrow 3 / 3$
$\Rightarrow$ B (G) $= 3$

Put n = 7 then the dominating set DS = {v_2, v_5, v_7} and domination number $\gamma(G)=3$.

Let us take e = {(v_1, v_2), (v_1, v_7), (v_2, v_3)} then the dominating set of G – e is

DS (G - e) = {v_1, v_2, v_4, v_6} and domination number of G - e is $\gamma(G-e)=4$.

Therefore$\gamma(G-e) > \gamma(G)$, Then the bondage number B (G) = 3.

Take n = 7 then $7 \equiv 1(mod3)$
$\Rightarrow 3 / (7-1)$
$\Rightarrow 3 / 6$
$\Rightarrow$ B (G) $= 3$

Similarly this is true for all if n ≥ 4, $n \equiv 1(mod3)$

Hence the proof.

Illustration – 1:
n = 18:

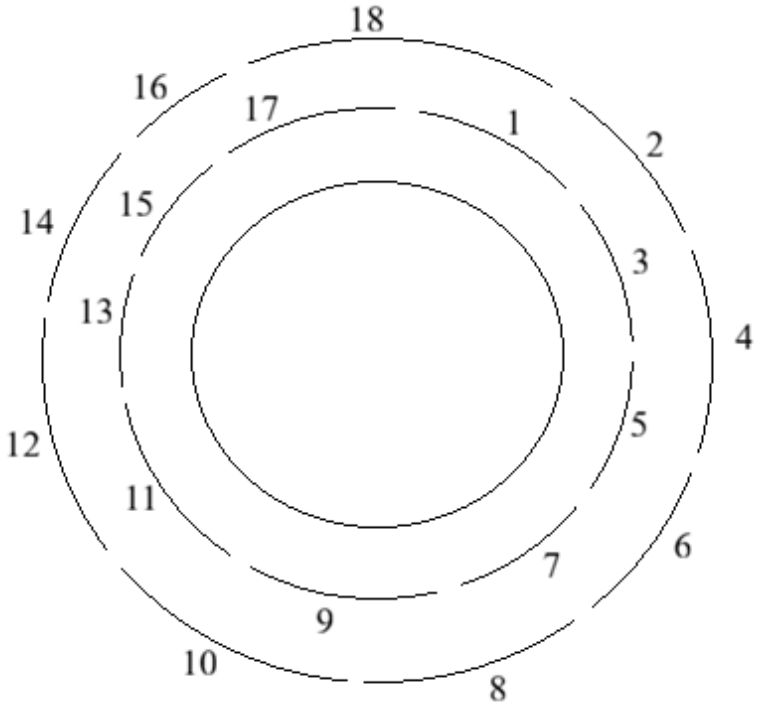


Fig. 7 Circular arc family A

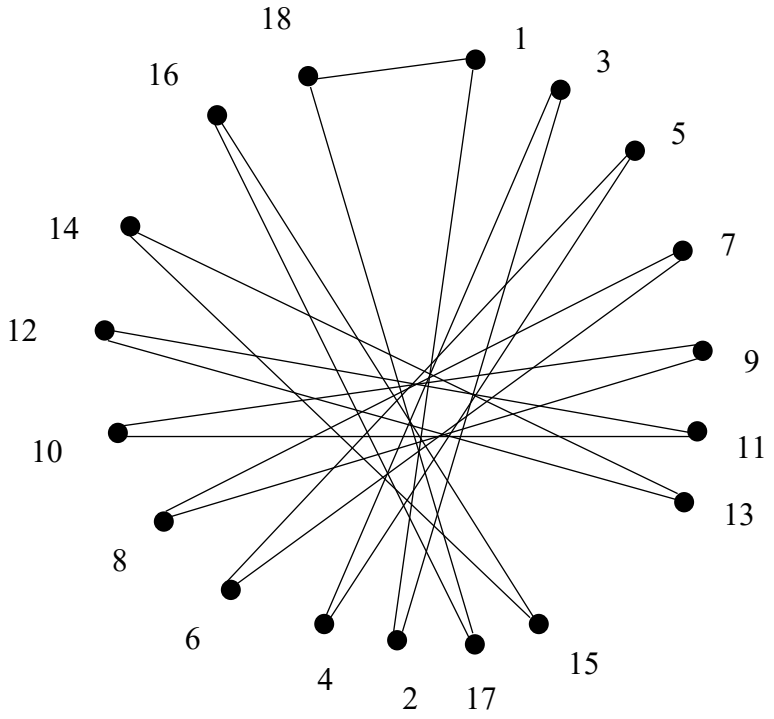


Fig. 8 Circular arc graph G

Domination set DS = {1, 4, 7, 10, 13, 16} and domination number $\gamma(G) = 6$

e = {(1, 2), (1, 18)}

G – e:

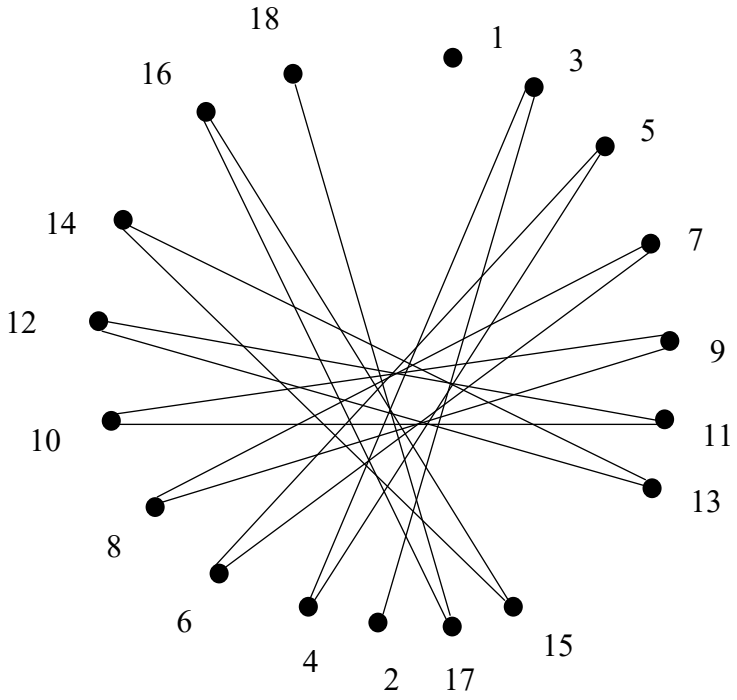


Fig. 9 Circular arc graph G - e

Dominating set DS (G - e) = {1, 3, 6, 9, 12, 15, 17} and
Domination number $\gamma(G-e)=7$ therefore $\gamma(G-e) > \gamma(G)$

$$\Rightarrow B(G) = 2$$

Illustration – 2:
n = 19:

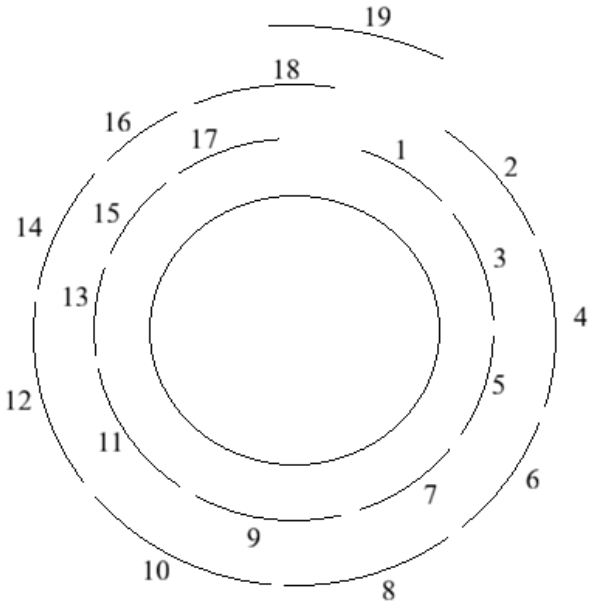


Fig. 10 Circular arc family A

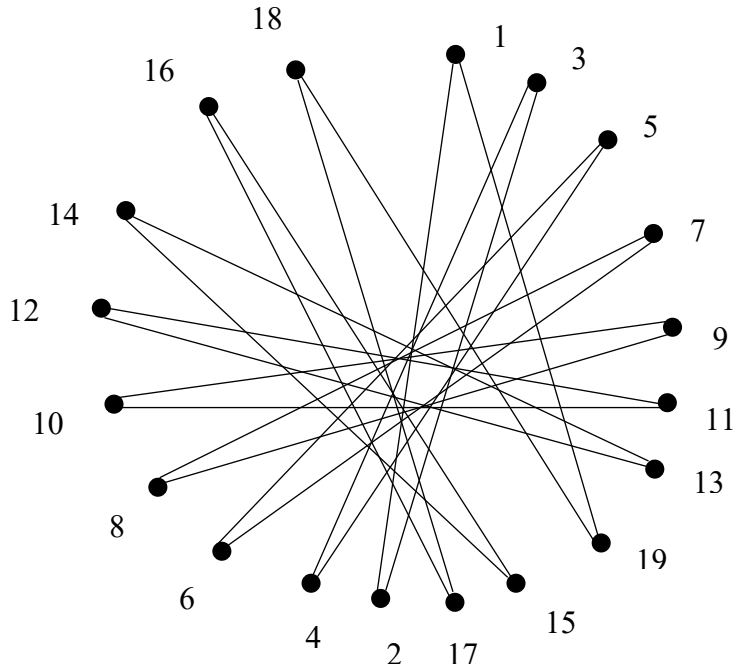


Fig. 11 Circular arc graph G

Dominating set DS = {1, 4, 7, 10, 13, 16, 18} and domination number $\gamma(G)=7$
e = {(1, 2), (2, 3), (1, 19)}

G – e :

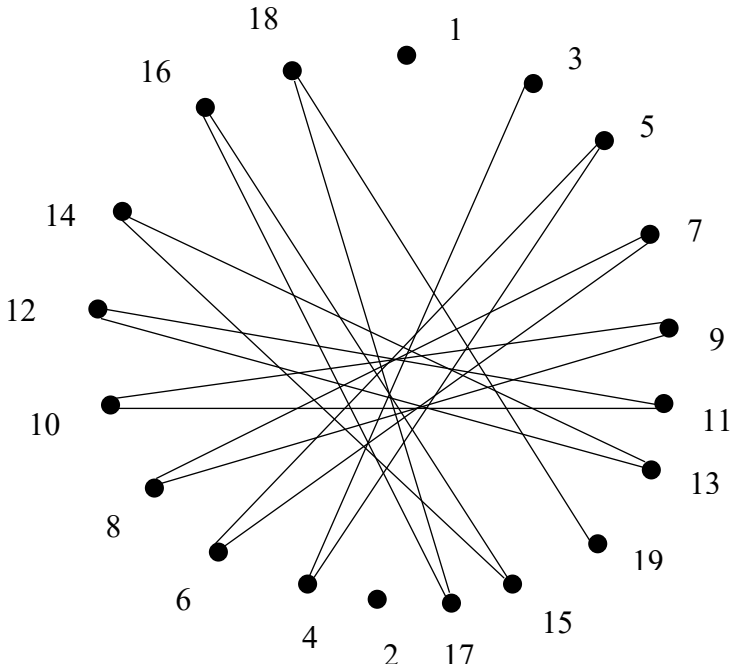


Fig. 12 Circular arc graph G - e

Dominating set DS (G - e) = {1, 2, 3, 6, 9, 12, 15, 17} and domination number

$$\text{Therefore } \gamma(G-e) > \gamma(G) \qquad \begin{matrix} \gamma(G-e) = 8 \\ \Longrightarrow \text{B (G)} = 3 \end{matrix}$$

Theorem-4: Let A = $\{A_1, A_2, \ldots, A_n\}$be a finite overlap circular arc family and G be an overlap circular arc graph corresponding to A, G must be connected. Where numbers of vertices are must be greater than or equal to 3, i.e. $\forall\, n \geq 3, and$ such that for all i, j ∈ $V(G)$, the vertices v_i and v_j are adjacent if j = i + 3 when i is odd an if j ∈ $i+1, i+2$ when i is even also, if $v_i \cap v_j = \emptyset$ if $v_i \subseteq v_j$ or $v_j \subseteq v_i$, then the bondage number $B(G) = \begin{cases} 2\ if\ n = 4 \\ 1\ if\ n \neq 4 \end{cases}$; $\forall\, n \geq 3$

Proof: Let A = $\{A_1, A_2, \ldots, A_n\}$be a finite circular arc family and G be an overlap circular arc graph corresponding to A, G must be connected. The number of vertices must be greater than or equal to 3.
We know that the overlap circular arc graph is a graph G in which each arc represents a vertex and two vertices are adjacent if and only if their corresponding circular arcs overlap and two arcs A_i, $A_j \in A$ are said to overlap if $A_i \cap A_j \neq \emptyset$ and $A_i \nsubseteq A_j$, $A_j \nsubseteq A_i$

In this theorem particularly dealing with the following kind of overlap circular arc family, such that for i, j ∈ $V(G)$, the vertices v_i and v_j are adjacent if j = i + 3 when i is odd and

if j ∈ $i+1, i+2$ when i is even

also, if $v_i \cap v_j = \emptyset$ if $v_i \subseteq v_j$ or $v_j \subseteq v_i$

Let us consider a finite overlap circular arc family and $\forall n \geq 3$ then the bondage number

$$\text{B (G)} = \begin{cases} 2\ if\ n = 4 \\ 1\ if\ n \neq 4 \end{cases}$$

In this it will arise two cases.

Case-1: B (G) = 2 if n = 4
Put n = 4 then the dominating set DS = {v_1, v_3} and domination number $\gamma(G) = 2$.

Let us take e = {(v_1, v_4), (v_2, v_4)} then

The dominating set of G – e is DS (G - e) = {v_1, v_3, v_4} and
Domination number of G - e is $\gamma(G - e) = 3$.
Therefore $\gamma(G - e) > \gamma(G)$, Then the bondage number B (G) = 2.

Case-2: B (G) = 1 if n ≠ 4
Suppose n = 3 then the dominating set DS = {v_3} and domination number $\gamma(G) = 1$.

Let us consider e = {(v_1, v_3)} then

The dominating set of G – e is DS (G - e) = {v_1, v_3} then the Domination number of G - e is $\gamma(G - e) = 2$.

Therefore $\gamma(G - e) > \gamma(G)$, then the bondage number B (G) = 1.

Le t n = 5 then the dominating set DS = {v_3, v_4} and domination number $\gamma(G) = 2$.

Let us take e = {(v_1, v_4)} then the dominating set of G – e is DS (G - e) = {v_1, v_2, v_5} and

Domination number of G - e is $\gamma(G - e) = 3$.

Therefore $\gamma(G - e) > \gamma(G)$, Then the bondage number B (G) = 1

Set n = 6 then the dominating set DS = {v_3, v_4} and domination number $\gamma(G) = 2$.

Let us consider e = {(v_1, v_4)} then
The dominating set of G – e is DS (G - e) = {v_1, v_3, v_4} and
Domination number of G - e is $\gamma(G - e) = 3$.

Therefore $\gamma(G - e) > \gamma(G)$, Then the bondage number B (G) = 1

Similarly take n = k, $\forall\, n \neq 4$ then we get the bondage number is 1, i.e., B (G) = 1;
$\forall\, n \neq 4$.
. Hence the theorem is hold

Illustration – 1:
n = 4:

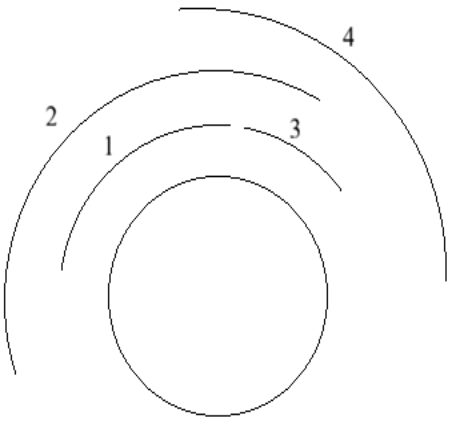


Fig. 13 Circular arc family A

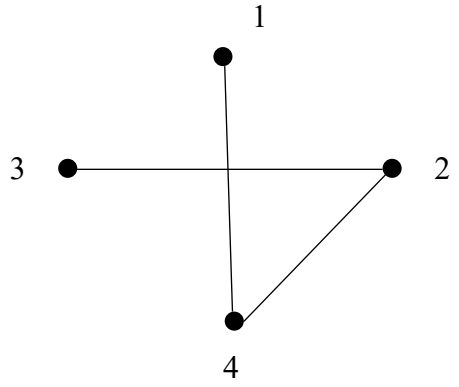


Fig. 14 Circular arc graph G

Dominating set DS = {1, 3} and domination number $\gamma(G) = 2$
e = {(2, 4), (4, 1)}
G – e:

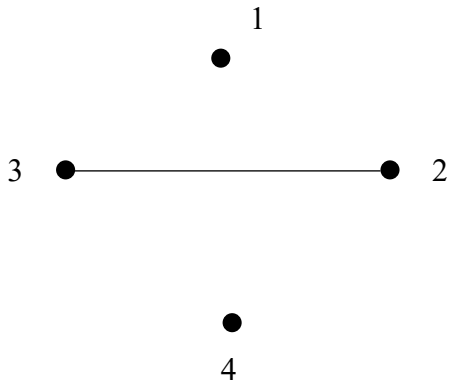


Fig. 15 Circular arc graph G – e

Dominating set DS (G - e) = {1, 3, 4} and domination number $\gamma(G - e) = 3$

Therefore $\gamma(G - e) > \gamma(G) \Longrightarrow \mathrm{B}\,(\mathrm{G}) = 2$

Illustration - 2:
n = 15:

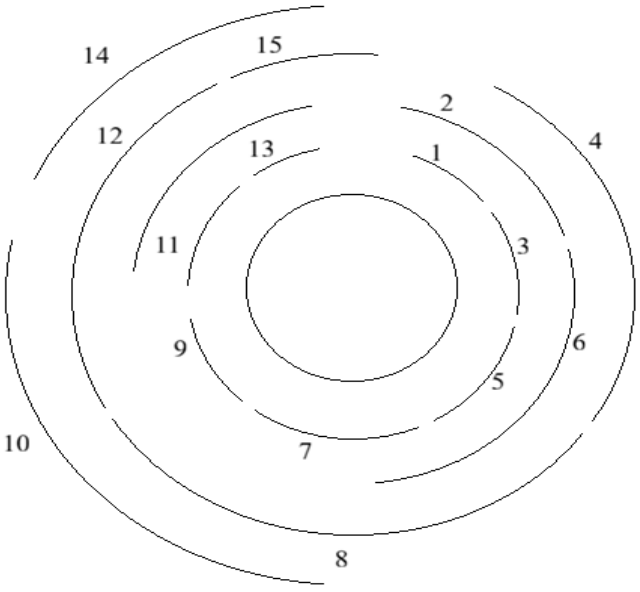


Fig. 16 Circular arc family A

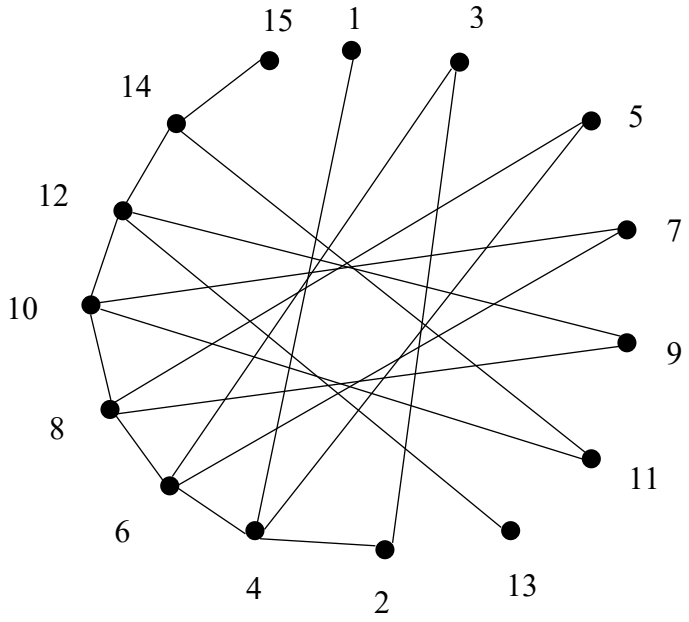


Fig. 17 Circular arc graph G

Dominating set DS = {4, 6, 12, 14} and domination number $\gamma(G) = 4$
e = {(1, 4)}

G – e:

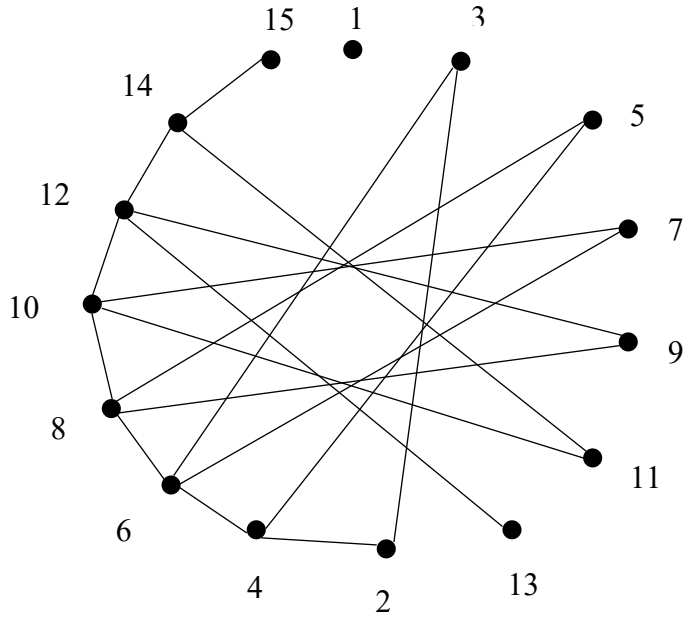


Fig. 18 Circular arc graph G – e

Dominating set DS (G - e) = {1, 4, 6, 12, 14} and domination number $\gamma(G - e) = 5$

Therefore $\gamma(G - e) > \gamma(G)$

$\Longrightarrow \mathrm{B}(\mathrm{G}) = 1$

Theorem-5: Let A = $\{A_1, A_2, \ldots, A_n\}$be a finite consecutive circular arc family without cycle and G be a circular arc graph related to A, G must be connected. Where n must be greater than or equal to 2, i.e. $\forall n \geq 2$. Then the bondage number

$$B(G) = \begin{cases} 1 \; if \; n \not\equiv 1(mod\ 3) \\ 2 \; if \; n \equiv 1(\ mod 3) \end{cases} ; \forall\, n \geq 2$$

Proof: Let A = $\{A_1, A_2, \ldots, A_n\}$ be a finite consecutive circular arc family (without cycle) and G be a circular arc graph with corresponding to A, and G must be connected.

We know that A = $\{A_1, A_2, \ldots, A_n\}$ be a finite circular arc family on a circle, then the resulting graph is called a consecutive circular arc graph (without cycle) if $A_1 \cap A_2 \neq \emptyset$, $A_2 \cap A_3 \neq \emptyset$, $A_3 \cap A_4 \neq \emptyset$ and so on but $A_n \cap A_1 = \emptyset$.

Also the number of vertices must be greater than or equal to 2. In this theorem we use congruent modulo 3. It can be written as a $\equiv b\ (mod\ 3)$ if and only if $3/\ (a-b)$ and a $\not\equiv b\ (mod\ 3)$ if and only if 3 ∤ (a-b).

Let us consider a finite consecutive circular arc family (without cycle) and $\forall\, n \geq 2$ then the bondage number

$$B\ (G) = \begin{cases} 1 \;\; if\ n \not\equiv 1\ (mod\ 3) \\ 2 \;\; if \;\; n \;\equiv 1\ (\ mod\ 3) \end{cases}$$

In this it will arise two cases

Case-1: B (G) = 2 if $n \equiv 1(mod\ 3)$

Suppose n = 4 then the dominating set DS = $\{v_2, v_4\}$ and domination number $\gamma(G) = 2$.
Since e = $\{(v_1, v_2)\}$ is not possible by theorem such that the bondage number is not increase in G.

Let us consider e = $\{(v_1, v_2), (v_2, v_3)\}$ then the dominating set of G – e is
DS (G - e) = $\{v_1, v_2, v_3\}$ and domination number of G - e is $\gamma(G-e) = 3$.

Therefore $\gamma(G-e) > \gamma(G)$, Then the bondage number B (G) = 2.

By the hypothesis if $n \equiv 1(mod\ 3)$ Then B (G) = 2
Take n = 4 then $4 \equiv 1(\ mod 3\ )$

$\Rightarrow 3\ /\ (4\text{-}1)$
$\Rightarrow 3\ /\ 3$
$\Rightarrow B\ (G) = 2$

Put n = 7 then the dominating set DS = $\{v_2, v_5, v_7\}$ and domination number $\gamma(G) = 3$.

Let us take e = $\{(v_1, v_2), (v_2, v_3)\}$ then the dominating set of G – e is
DS (G - e) = $\{v_1, v_2, v_4, v_6\}$ and domination number of G - e is $\gamma(G-e) = 4$.
Therefore $\gamma(G-e) > \gamma(G)$, Then the bondage number B (G) = 2.
By the hypothesis

Take n = 7 then $7 \equiv 1\ (mod\ 3\ )$

$\Rightarrow 3\ /\ (7\text{-}1)$
$\Rightarrow 3\ /\ 6$
$\Rightarrow B\ (G) = 2$

Similarly $B(G) = 2\ if\ n \equiv 1\ (mod\ 3\ )$

Case-2: B (G) = 1 if $n \not\equiv 1(mod\ 3)$

Let n = 2 then the dominating set DS = $\{v_1\}$ and domination number $\gamma(G) = 1$.

Let us take e = $\{(v_1, v_2)\}$ then the dominating set of G – e is DS (G - e) = $\{v_1, v_2\}$

And domination number of G - e is $\gamma(G-e) = 2$.

Therefore $\gamma(G-e) > \gamma(G)$, Then the bondage number B (G) = 1.

By the hypothesis if $n \not\equiv 1\ (mod\ 3\ )$ Then B (G) = 1

Put n = 2 then $2 \not\equiv 1\ (mod\ 3\ )$
⇒ 3 ∤ (2-1)
⇒ 3 ∤ 1
⇒ B (G) = 1
Our hypothesis is hold for n = 2.

Now Let n = 3 then the dominating set DS = $\{v_2\}$ and domination number $\gamma(G) = 1$

Let us consider e = $\{(v_1, v_2)\}$ then the dominating set of G – e is
DS (G - e) = $\{v_1, v_2\}$ and domination number of G - e is $\gamma(G-e) = 2$.

Therefore $\gamma(G-e) > \gamma(G)$, Then the bondage number B (G) = 1.

By the hypothesis if $n \not\equiv 1\ (mod 3\ )$ Then B (G) = 1

Take n = 3 then $3 \not\equiv 1\ (mod\ 3\ )$
⇒ 3 ∤ (3-1)
⇒ 3 ∤ 2
⇒ B (G) = 1

Similarly our hypothesis is true for all $n \not\equiv 1\ (mod 3\ )$, i.e, B (G) = 1 if $n \not\equiv 1\ (mod\ 3\ )$.

Hence the theorem is hold

Illustration - 1:
n = 14:

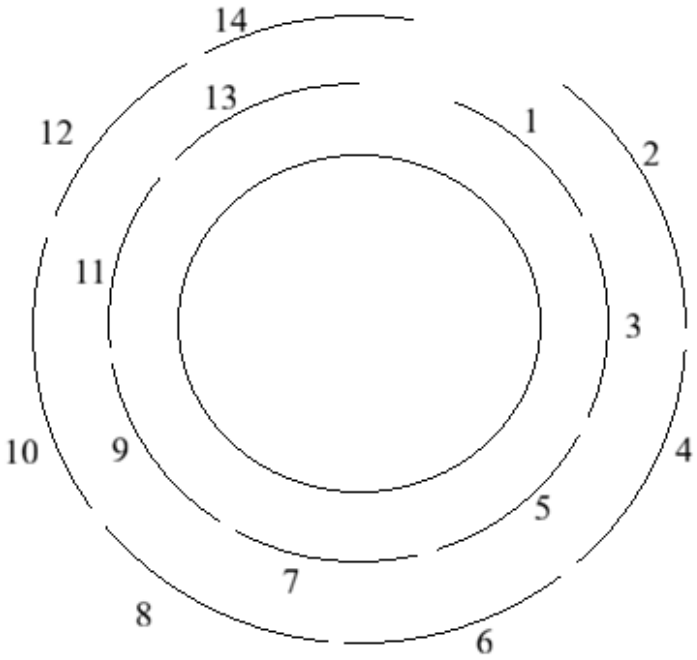


Fig. 19 Circular arc family A

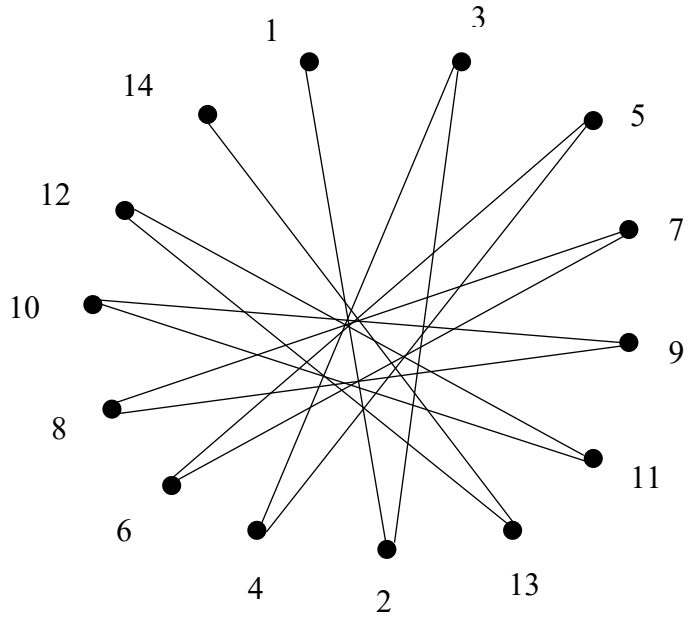


Fig. 20 Circular arc graph G

Dominating set DS = {2, 5, 8, 11, 13} and domination number $\gamma(G) = 5$

e = {(1, 2)}

G – e:

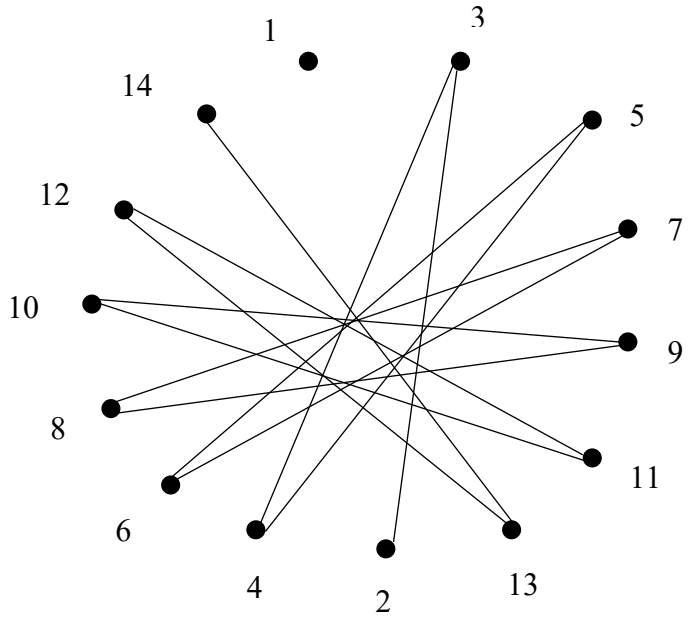


Fig. 21 Circular arc graph G – e

Dominating set DS (G - e) = {1, 3, 6, 9, 11} and domination number $\gamma\ (G - e) = 5$

Therefore $\gamma\ (G - e) >\ \gamma\ (G) \Longrightarrow$ B (G) $= 1$

Illustration – 2:
n = 13:

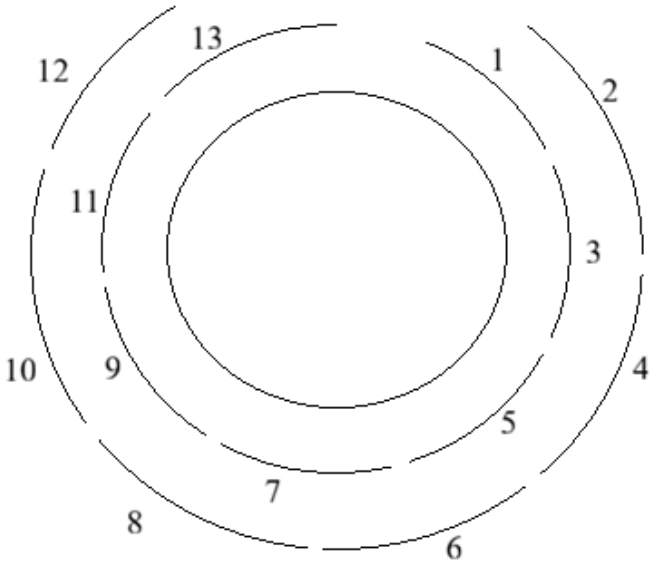


Fig. 22 Circular arc family A

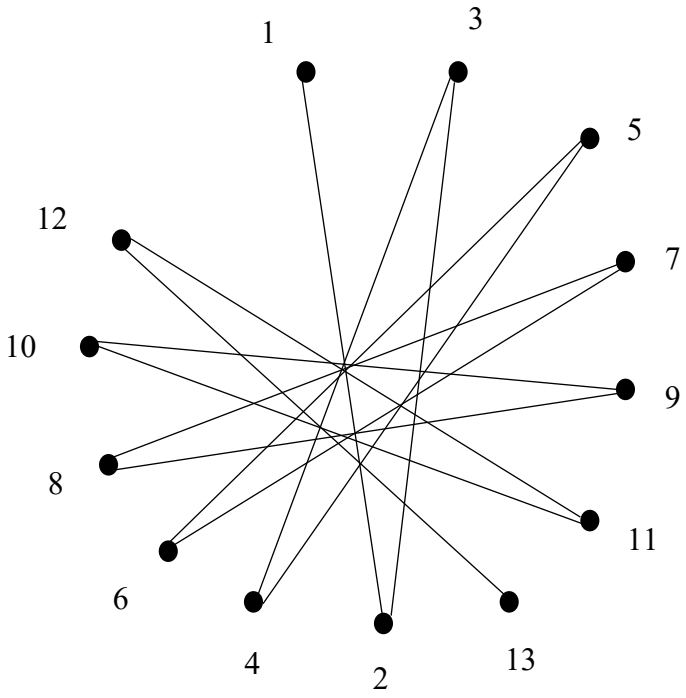


Fig. 23 Circular arc graph G

Dominating set DS = {2, 5, 8, 11, 13} and domination number $\gamma\ (G) = 5$
e = {(1, 2), (2, 3)}

G – e:

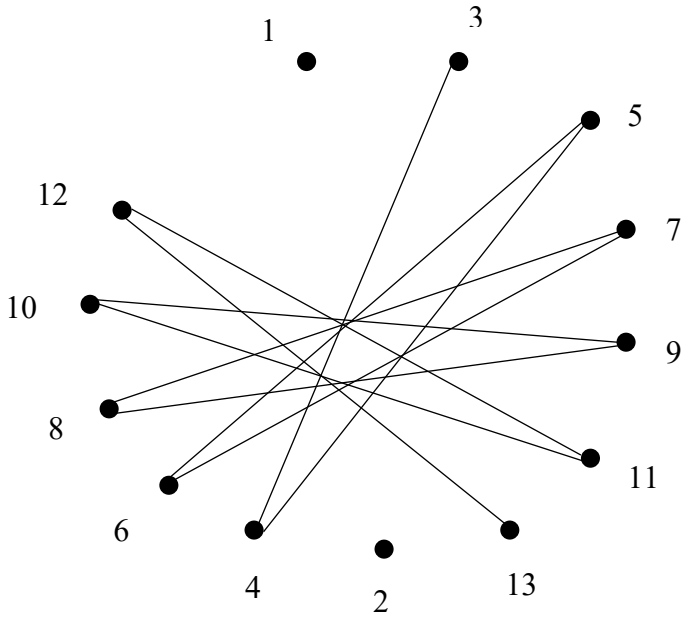


Fig. 25 Circular arc graph G – e

Dominating set DS (G - e) = {1, 2, 4, 7, 10, 12} and domination number $\gamma\ (G - e) = 6$
Therefore $\gamma\ (G - e) > \gamma\ (G) \implies$ B (G) $= 2$

4. Conclusion

In this paper, we investigated the domination number and bondage number of circular-arc graphs corresponding to different types of circular-arc families. In particular, we considered consecutive circular-arc families, regular circular-arc graphs of degree 2, overlap circular-arc families, and consecutive circular-arc families without cycles. For the considered graph structures, we established results for the domination number and bondage number by analyzing the structural relationships among the corresponding arcs and vertices.

The results demonstrate that the domination structure of a circular-arc graph is closely related to the arrangement and intersection properties of its arcs. Furthermore, the bondage number provides information about the effect of edge deletion on the domination number and hence gives a measure of the stability of the dominating structure under edge failures.

The present study is restricted to the specific circular-arc configurations considered in this paper. Therefore, several directions remain open for further investigation. Future work may consider general circular-arc graphs, k-regular circular-arc graphs, proper circular-arc graphs, and other classes of circular-arc families, as well as other domination parameters such as total, connected, independent, efficient, and certified domination. The study of vertex-bondage and algorithmic approaches for determining domination and bondage numbers also provides promising directions for future research.

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