International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V72I10P105

Volume 72 Issue 10, 53-62, October 2026
© 2026 Seventh Sense Research Group®

Original Article

Odd-Cycle Conditions in Enhanced Power Graphs: Integer Rounding and Normality

Luis Alfredo Dupont García[1], Daniel Gualtiero Mendoza Ramírez[2]

[1,2]*Facultad de Matemáticas, Universidad Veracruzana, Xalapa, Veracruz, México.*

Corresponding Author : ldupont@uv.mx

Received: 22 August 2026 Revised: 28 September 2026 Accepted: 15 October 2026 Published: 28 October 2026

Abstract *- This article studies the odd-cycle condition for the enhanced power graph of a finite group. The problem is approached through the subgroup structure of the group rather than through a general graph decomposition. For groups of coprime orders, the enhanced power graph of the direct product is the strong product of the corresponding enhanced power graphs. This observation gives a useful obstruction and shows, in particular, that a nilpotent group whose order has at least two distinct prime divisors satisfies the odd-cycle condition only in the cyclic case. For p-groups, the enhanced power graph agrees with the power graph, which allows the problem to be reduced to the arrangement of cyclic subgroups. The analysis yields the resulting classification for every prime. If $p \geq 5$, the group must be cyclic. For $p = 3$, the possibilities are cyclic groups and groups of exponent 3. For $p = 2$, besides cyclic groups, the condition holds when the Frattini subgroup has order at most 2 and for the dihedral, semi dihedral, and generalized quaternion families occurring in the classification. The p-group results also yield the complete abelian case. The present study further considers generalized dihedral groups and examples outside the nilpotent setting. Since the identity is a universal vertex, these graph-theoretic results apply directly to the associated incidence systems and give corresponding integer-rounding and normality statements for the relevant monomial algebras and edge polytopes.*

Keywords *- Cyclic subgroup, Edge polytope, Finite group, Integer rounding property, Normality.*

MSC 2020 *- Primary 05C25, Secondary 05C17, 13B22, 20D15, 90C10.*

1. Introduction

Let G be a finite group. Its enhanced power graph E(G) has vertex set G, and two different elements are adjacent when they belong to a common cyclic subgroup. This definition is close to that of the ordinary power graph, but the two graphs need not agree: in the power graph, one of the two elements must be a power of the other, whereas in E(G), it is enough that both occur in some cyclic subgroup. The enhanced power graph has been studied from several points of view, including its structure, maximal cliques, connectivity, direct products, perfectness, and forbidden subgraphs [1,2,5–7,10,13,14]. The central problem is to determine which finite groups have enhanced power graphs satisfying the Odd-Cycle Condition (OCC). Existing studies describe the structure, cliques, and forbidden subgraphs of enhanced power graphs [1,2,5–7,10,13,14], while graph-algebra results connect OCC to normality and integer rounding [3,9,11,12]. Structural results on enhanced power graphs and classical normality criteria address different questions: the former describe adjacency and subgroup geometry, whereas the latter characterize algebraic properties once OCC is known. Neither, by itself, identifies the finite-group families whose enhanced power graphs satisfy OCC. The present investigation addresses that gap through coprime products, Sylow subgroups, and cyclic-subgroup configurations. In particular, it identifies a cyclicity obstruction for mixed-prime nilpotent groups, distinguishes the cases $p \geq 5$, $p = 3$, and $p = 2$, and transfers the resulting classifications to the associated graph algebras. The algebraic equivalences used in this transfer are established results, rather than new normality criteria.

There are two simple facts that will be used repeatedly. The identity of G is adjacent to every other vertex of E(G), so deleting it does not affect an obstruction consisting of two disjoint anticomplete odd cycles. Also, if H is a subgroup of G, then E(H) is the subgraph of E(G) induced by H. Consequently, failure of the odd-cycle condition inside a subgroup is already enough to rule it out for the whole group.

The first substantial step concerns direct products of relatively prime order. If H and K have coprime orders, generation in $H \times K$ separates into the two coordinates. It follows that $E(H \times K)$ is the strong product of E(H) and E(K). This elementary

This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/)

formula is especially effective when one factor is cyclic. A non-edge in the other factor can then be used to produce two anticomplete odd cycles. From this, the analysis yields a cyclic-factor criterion and, as an immediate application, the classification of nilpotent groups whose orders involve more than one prime.

The prime-power case behaves differently. If P is a finite p-group, two elements that lie in a common cyclic subgroup generate comparable cyclic subgroups. Hence, E(P) is exactly the ordinary power graph of P. Finite power graphs are perfect [4], so in this setting, an obstruction to the odd-cycle condition may be reduced to two anticomplete triangles. This makes it possible to work directly with cyclic subgroups and their generators.

For odd primes, the outcome is relatively short. When $p \geq 5$, every noncyclic p-group contains enough disjoint cyclic structure to force a pair of anticomplete triangles. For $p = 3$, groups of exponent 3 behave differently because their nonidentity cyclic subgroups have order 3; the final result is that the odd-cycle condition holds precisely for cyclic 3-groups and for 3-groups of exponent 3. The case $p = 2$ needs a separate analysis. There, the relevant information is carried by cyclic subgroups of order at least 8 and by the involutions contained in cyclic subgroups of order at least 4. The intrinsic criterion obtained from these two features leads to the classification in terms of cyclic groups, groups with small Frattini subgroups, and the dihedral, semidihedral, and generalized quaternion families.

Once the p-groups are understood, the abelian classification follows without an additional case-by-case argument: a noncyclic abelian group satisfying the condition must be a p-group, and the preceding results leave only the families listed in Section 6. The article also examines generalized dihedral groups. Their enhanced power graphs have a particularly simple form: the elements outside the abelian subgroup are leaves at the identity. Thus, the odd-cycle question reduces exactly to the corresponding question for the abelian subgroup.

Relation to previous work. Structural studies of enhanced power graphs describe adjacency, clique structure and forbidden configurations [1,5–7,10,13,14], whereas the present problem asks whether two disjoint odd cycles can be anticomplete. The classification is therefore formulated in terms of cyclic-subgroup arrangements and coprime direct products. In contrast, the normality and integer-rounding implications are applications of existing graph-algebra criteria [3,9,11,12], not newly established general equivalences.

The algebraic consequences are obtained by applying established graph-theoretic normality and integer-rounding criteria to the enhanced power graphs classified here. The new contribution is the identification of the relevant finite-group families; the underlying graph-algebra equivalences are attributed to their original sources [3,9,11,12].

2. Preliminaries and Algebraic Transfer

Throughout, all groups and graphs are finite, and all graphs are simple and undirected. The identity element is denoted by e. Two disjoint vertex sets are anticomplete if there is no edge with one endpoint in each set. A universal vertex is adjacent to every other vertex. For graphs X, Y, their strong product $X \boxtimes Y$ has vertex set $V(X) \times V(Y)$, with distinct (x, y) and (x', y') adjacent precisely when, in each coordinate, the entries are equal or adjacent, and at least one coordinate is adjacent.

Definition 2.1. The enhanced power graph E(G) has vertex set G. Distinct $x, y \in G$ are adjacent if and only if they belong to a common cyclic subgroup; equivalently, $x \sim y \Longleftrightarrow \langle x, y\rangle$is cyclic.

The identity e is a universal vertex.

Lemma 2.2 (Clique containment). Every clique of E(G) is contained in a cyclic subgroup of G. In particular, every triangle is contained in a cyclic subgroup.

Proof. Let $S = \{x1,..., xr\}$ be a clique and put $B = \langle S\rangle$. Since $\langle xi, xj\rangle$is cyclic for every i, j, the elements of S commute pairwise, so B is finite abelian. For each prime p, take the p-parts of the xi. Any two lie in a common cyclic p-group, hence generate comparable cyclic subgroups. Therefore, the cyclic subgroups generated by the p-parts form a chain and together generate a cyclic Sylow p-subgroup of B. Since this holds for every p, B is cyclic. This agrees with the maximal-clique description in [1].

Lemma 2.3 (Subgroup inheritance). If $H \leq G$, then $E(H) = E(G)[H]$, the subgraph induced by H.

Proof. For $x, y \in H$, the subgroup $\langle x, y\rangle$is the same whether generated inside H or G.

Definition 2.4. A graph X satisfies the odd-cycle condition if every two vertex-disjoint odd cycles have an edge joining a vertex of one to a vertex of the other.

Proposition 2.5 (Deleting the identity and subgroup obstruction). Let G be finite. (a) E(G) satisfies OCC if and only if E(G)−{e} satisfies OCC. (b) If E(G) satisfies OCC and H ≤G, then E(H) satisfies OCC.

Proof. An OCC obstruction consists of two vertex-disjoint odd cycles with no edge between them. The universal vertex e cannot lie in either cycle of such an obstruction: if it belonged to one cycle, it would be adjacent to every vertex of the other. This proves (a). For (b), Lemma 2.3 identifies E(H) with an induced subgraph of E(G); any obstruction in E(H) would therefore also be an obstruction in E(G).

Remark 2.6. If two distinct connected components of E(G) −{e} each contain an odd cycle, then OCC fails. This is only a necessary component-level test: obstructions may also occur inside a single component.

Lemma 2.7 (Triangle reduction). Let X be a perfect graph. Then X fails OCC if and only if it contains two vertex-disjoint anticomplete triangles.

Proof. Two anticomplete triangles give an OCC obstruction. Conversely, suppose X fails OCC and let C1, C2 be vertex-disjoint anticomplete odd cycles. Put Xi = X[V (Ci)]. Each Xi is perfect. If Xi were triangle-free, then $\omega(Xi) \leq 2$ and perfectness would give $\chi(Xi) \leq 2$, contradicting the odd cycle Ci. Thus, each Xi contains a triangle, and these triangles remain anticomplete.

The perfectness assumption is essential, since enhanced power graphs need not be perfect in general. The argument invokes the preceding lemma only for p-groups, for which the enhanced power graph coincides with the ordinary power graph. Feng, Ma and Wang proved that every finite power graph admits a transitive orientation and is therefore perfect [4].

Let X be a connected graph on vertices x1,..., xn, with edge-incidence vectors v1,..., vq ∈ {0, 1}n. Put A = [v1 · · · vq], let K be a field, set R = K[x1,..., xn], and let I(X) denote the edge ideal. Throughout, x is a row vector, and 1 denotes an all-ones vector of the appropriate dimension.

Theorem 2.8 (Incidence IRP and normality theorem). For a connected graph X, the following are equivalent:
(i) x ≥0, xA ≤1 has the integer rounding property; (ii) x ≥0, xA ≥1 has the integer rounding property; (iii) X satisfies OCC;
(iv) the Rees algebra R[I(X)t] is normal;
(v) the edge subring K[xv1,..., xvq] is normal; (vi) the augmented monomial algebra K[t, x1t,..., xnt, xv1t,..., xvqt] is normal. Moreover, these conditions are equivalent to normality of the edge polytope PX = conv(v1,..., vq) with respect to the edge lattice LX = Zv1 + · · · + Zvq in its affine span.

Proof. Theorem 3.2 of Dupont, Rentería-Márquez and Villarreal [3] characterizes the integer rounding property of x ≥ 0, xA ≤ 1 for connected graphs by the odd-cycle condition. Their Theorem 3.3 identifies the equivalent integer rounding property of x ≥ 0, xA ≥ 1 and the normality of the Rees algebra, the homogenized edge subring, and the augmented monomial algebra. The ordinary edge subring and its homogenized form are isomorphic as abstract affine semigroup rings because all edge incidence vectors have coordinate sum two; consequently, their normality properties agree. The normality criterion for edge rings and the associated edge polytopes is attributed to Simis, Vasconcelos and Villarreal [9] and Ohsugi and Hibi [11]. The polytope is considered relative to its edge lattice. These are previously established graph-algebra criteria, not new theorems of the present work.

3. Coprime Direct Products

Lemma 3.1 (Coprime subgroup splitting). Let H, K be finite groups with gcd(|H|, |K|) = 1. For x = (h1, k1) and y = (h2, k2) in H × K, ⟨x, y⟩= ⟨h1, h2⟩× ⟨k1, k2⟩.

Proof. Put S = ⟨x, y⟩, U = ⟨h1, h2⟩, and V = ⟨k1, k2⟩. Clearly S ≤U × V, and both coordinate projections are surjective. If (u, v) ∈S, then o(u) and o(v) are coprime. By the Chinese remainder theorem, there exists r with r ≡1 (mod o(u)) and r ≡0 (mod o(v)), so (u, v)r = (u, e) ∈S. Similarly (e, v) ∈S. Hence, S = U × V.

Theorem 3.2 (Coprime strong-product formula). If H, K are finite groups with gcd(|H|, |K|) = 1, then the following identity holds:

$$E(H \times K) = E(H) \boxtimes E(K)$$

Proof. By the preceding lemma, the subgroup generated by two pairs is the direct product of the corresponding generated subgroups. Because the two factors have coprime orders, their direct product is cyclic if and only if each factor is cyclic. This is exactly the adjacency criterion for the strong product.

Lemma 3.3. E(G) is complete if and only if G is cyclic.

Proof. The cyclic case is immediate. Conversely, choose g ∈G of maximum order. If G⊭ ⟨g⟩, take x /∈⟨g⟩. Completeness gives a cyclic subgroup containing g and x, properly containing ⟨g⟩, contradicting maximality of o(g).

Theorem 3.4 (Cyclic-factor amplification). Let H be a finite group and let m ≥ 2 satisfy gcd(|H|,m)=1. Then E(H × C_m) satisfies OCC if and only if H is cyclic.

$$E(H \times C_m) \; satisfies \; OCC \Leftrightarrow H \; is \; cyclic$$

Proof. If H is cyclic, then H × C_m is cyclic, so its enhanced power graph is complete. Conversely, let H be noncyclic and choose nonadjacent elements a,b ∈ H. If m ≥ 3, the two fibers {a} × C_m and {b} × C_m are anticomplete cliques of size m; each contains a triangle. Suppose m = 2. Coprimality forces |H| to be odd, so a and b have odd orders at least three. Choose two distinct generators a_1,a_2 of ⟨a⟩ and two distinct generators b_1,b_2 of ⟨b⟩. Every cross-pair a_i,b_j is nonadjacent: adjacency would place a and b in a common cyclic subgroup. The two edges a_1a_2 and b_1b_2 are therefore anticomplete. Under the strong-product identity of Theorem 3.2, each edge together with the C_2 factor induces K_4, giving two anticomplete triangles. Hence, OCC fails.

Corollary 3.5 (Centralizing coprime obstruction). Let G contain a noncyclic subgroup H and an element x of prime order q such that q ∤|H| and x centralizes H. Then E(G) does not satisfy OCC.

Proof. Since q ∤|H|, H ∩⟨x⟩= 1, and centralization gives H⟨x⟩≅ H × Cq. Theorem 3.4 shows that this subgroup fails OCC, and Proposition 2.5(b) finishes the proof.

4. Nilpotent Groups

Theorem 4.1. Let G be a finite nilpotent group whose order is divisible by at least two distinct primes. Then E(G) satisfies OCC if and only if G is cyclic.

Proof. Write G = P1 × · · · × Pr, r ≥ 2, as the direct product of its Sylow subgroups. If all Pi are cyclic, then G is cyclic. Otherwise, choose a noncyclic Pi, choose j⊭ i, and choose Cq ≤Pj of prime order. By subgroup inheritance, E(Pi × Cq) is induced in E(G), and Theorem 3.4 shows that it fails OCC. Conversely, if G is cyclic, E(G) is complete, and hence satisfies OCC. The use of the induced-subgraph property is legitimate because two elements of a subgroup generate the same subgroup in the ambient group; hence, every obstructing pair of odd cycles in Pi × Cq remains an obstruction in G.

5. Prime-Power Groups

Proposition 5.1. Let P be a finite p-group. Then E(P) = Pow(P), the ordinary undirected power graph.

Proof. Every power-graph edge is an enhanced-power-graph edge. Conversely, if x ∼y in E(P), then ⟨x, y⟩is a cyclic p-group, whose subgroups form a chain. Thus ⟨x⟩≤⟨y⟩or conversely.

Finite power graphs are perfect [4]; hence, Lemma 2.7 applies.

Theorem 5.2 (Odd primes p ≥ 5). Let P be a finite p-group with p ≥ 5. Then E(P) satisfies OCC if and only if P is cyclic.

Proof. Suppose P is noncyclic. For odd p, a finite p-group with a unique subgroup of order p is cyclic [8]; hence, P contains distinct subgroups A, B of order p. Each of A \ {e} and B \ {e} induces Kp−1, and they are anticomplete because a cyclic p-group has a unique subgroup of order p. Since p −1 ≥ 4, they contain anticomplete triangles. Conversely, if P is cyclic, E(P) is complete, and OCC holds.

Theorem 5.3 (The case p = 3). Let P be a finite 3-group. Then the following classification holds:

$$E(P)\ satisfies\ OCC\ \Leftrightarrow\ P\ is\ cyclic\ or\ exp(P)\ =\ 3$$

Proof. By Proposition 5.1, the perfectness of finite power graphs [4], and Lemmas 2.2 and 2.7, E(P) satisfies OCC if and only if the cyclic subgroups of order at least 9 form a chain: incomparable such subgroups supply anticomplete generator triangles. Indeed, each cyclic subgroup of order at least nine has at least six generators, so three distinct generators can be selected from each. An edge between generators of the two subgroups would make their generated cyclic subgroups comparable, contradicting incomparability.

Conversely, each triangle avoiding the identity is contained in a cyclic subgroup by Lemma 2.2. In a cyclic 3-group of order 3, there are only two nonidentity vertices, so any such triangle requires a cyclic subgroup of order at least 9. If the cyclic subgroups of order at least 9 form a chain, two triangles in these subgroups cannot be anticomplete: every vertex in the smaller cyclic subgroup is adjacent to every vertex in the larger one. If P is cyclic, OCC is immediate. If exp(P) = 3, there are no cyclic subgroups of order at least 9.

Conversely, assume the chain condition and exp(P) = 3^a with a ≥ 2. Let M = ⟨g⟩ be the largest such cyclic subgroup of order 3^a. Every element of order at least 9 lies in M. Every conjugate of M has the same order and hence equals M, so M ⊴ P. If x ∈ P \ M, then o(x) = 3; also xg /∈ M, hence o(xg) = 3. Write $x^{-1}gx = g^r$. Since $x^3 = e$, $r^3 \equiv 1 \pmod{3^a}$, and reducing modulo 3 gives $r \equiv 1 \pmod 3$, because the multiplicative group of the field with three elements has order two and contains no nontrivial element of order three.

$$r^2 + r + 1 = 3(1 + 3s + 3s^2)$$

Set r=1+3s. Direct expansion yields $r^2+r+1=3(1+3s+3s^2)$. Since the factor in parentheses is congruent to 1 modulo 3, $v_3(r^2+r+1)=1$. Moreover, $x^3=e$ and $x^{-1}gx=g^r$ imply $(xg)^3=g^{(r^2+r+1)}$, an element of order 3^(a−1)>1 because g has order 3^a and a≥2.

$$r^2 + r + 1 = 3(1 + 3s + 3s^2)$$
$$(xg)^3 = g^(r^2 + r + 1) \neq e$$

This contradicts the previously established equality o(xg)=3, which would require $(xg)^3=e$. Consequently, P=M and P is cyclic.

Theorem 5.4 (Intrinsic criterion for 2-groups). Let P be a nontrivial finite 2-group. Then E(P) satisfies OCC if and only if either P has exponent 2, or P has exponent at least 4 and conditions (a) and (b) below both hold:
(a) The cyclic subgroups of order at least 8 are linearly ordered by inclusion. (b) All cyclic subgroups of order at least 4 contain the same involution z. In the exponent-two case, neither condition is required.

Proof. If exp(P)=2, then distinct nonidentity elements cannot belong to a common cyclic subgroup. Hence, E(P) is a star with center e and satisfies OCC. The common-involution condition is not asserted in this case, since no cyclic subgroup of order at least four exists. Suppose henceforth that exp(P) ≥ 4.

Suppose first that OCC holds. Two incomparable cyclic subgroups of order at least eight have disjoint sets of generators, each containing a triangle; no generator of one can be adjacent to a generator of the other, since the cyclic subgroups generated by adjacent elements in a 2-group are comparable. This proves (a). If two cyclic subgroups of order at least four have different involutions, the sets of nonidentity elements of their respective order-four subgroups form two disjoint triangles. A cross-edge would force their involutions to agree, which is impossible. Thus (b) holds.

Conversely, suppose that exp(P) ≥ 4 and that (a) and (b) hold. If OCC failed, Proposition 5.1 and Lemma 2.7 would provide two vertex-disjoint anticomplete triangles T and U. Lemma 2.2 places each triangle in a cyclic subgroup of order at least 4. If both containing subgroups have order at least 8, condition (a) makes them comparable, and every vertex of T is adjacent to every vertex of U, a contradiction. If one containing subgroup has order 4, its triangle must consist of its three nonidentity elements and hence contains z. The other triangle either contains z, contradicting disjointness, or has all its vertices adjacent to z, contradicting anticompleteness. Thus, OCC holds.

Theorem 5.5 (Classification of finite 2-groups satisfying OCC). Let P be a nontrivial finite 2-group. Then E(P) satisfies OCC if and only if one of the following holds:
(i) P is cyclic; (ii) $|\Phi(P)| \leq 2$; (iii) P is a dihedral, semidihedral, or generalized quaternion 2-group of exponent at least 8.

Proof. The intrinsic criterion of Theorem 5.4 is applied separately to the two exponent ranges. First, assume that $\exp(P) \leq 4$. Condition (a) then imposes no restriction. If P has no element of order four, then $\exp(P)=2$ and $\Phi(P)=1$. Otherwise, by Theorem 5.4(b), there is an involution z such that every element of order four has a square z. Thus, every square belongs to {e, z}. Let $P^2 = \langle x^2 : x \in P \rangle$. The quotient P/P^2 has exponent 2 and is therefore abelian, so $[P, P] \leq P^2$. Using $\Phi(P) = P^2[P,P]$, the analysis yields $\Phi(P) = P^2 \leq \langle z \rangle$ and hence $|\Phi(P)| \leq 2$.

For the converse in this range, suppose that P is noncyclic and $|\Phi(P)| \leq 2$. Since $P/\Phi(P)$ is elementary abelian, $x^2 \in \Phi(P)$ for every $x \in P$. If $\Phi(P)$ is trivial, P has exponent 2, and OCC follows directly from Theorem 5.4. If $\Phi(P) = \{e,z\}$, every element of order 4 has square z, while no element can have order 8 or more, because its square would have order at least 4 inside $\Phi(P)$. Hence, $\exp(P) \leq 4$, condition (a) is empty, and condition (b) is satisfied. Theorem 5.4, therefore, gives OCC.

Now suppose $\exp(P) = 2^a \geq 8$. If P is cyclic, OCC is immediate. Hence, assume that P is noncyclic. By Theorem 5.4(a), the cyclic subgroups of order at least 8 form a chain. Choose its largest member $M = \langle g \rangle$, with $|M| = 2^a$. Every element of order at least 8 lies in M, and conjugates of M have the same order; hence $M \lhd P$. Elements outside M, in particular elements xg^k with $x \notin M$, have order at most 4. If x outside M centralized M, then xg would also lie outside M and $(xg)^4=x^4g^4=g^4\neq e$, since x has order at most four and $a\geq 3$. This contradicts the bound on orders outside M. Since M is cyclic, M is contained in C_P(M); the preceding contradiction excludes every element outside M. Consequently, C_P(M)=M.

$$(yg)^4 = g\text{^}(2(r^2+1)) = e$$

It remains to establish $x^2 \in M$. Suppose otherwise and put $y=x^2$. Since every element outside M has order at most four, y is an involution. Write $x^{-1}gx=g^r$ with r odd. Then $y^{-1}gy=g\text{^}(r^2)$, and $(yg)^4=g\text{^}(2(r^2+1))$. Since yg lies outside M, its order is at most four. Consequently $2\text{^}(a-1)$ divides r^2+1. But $r^2+1\equiv 2 \pmod 8$, which is impossible when $a\geq 3$. Therefore $x^2 \in M$.

$$(yg)^4 = g^{2(r2+1)} = e$$
$$(xg^k)^2 = x^2 g^{k(r+1)}$$

Let $z = g\text{^}(2^{a-1})$. For x outside M, x^2 belongs to M, and x has order at most four. If x has order four, the common-involution condition forces $x^2=z$; if x has order two, $x^2=e$. Hence $x^2 \in \{e,z\}$. Applying the same observation to xg^k gives $(xg^k)^2=x^2g\text{^}\{k(r+1)\} \in \{e,z\}$ for every integer k. Comparing k=0 and k=1 shows that $g\text{^}(r+1)$ belongs to $\langle z \rangle$, and hence $r+1\equiv 0 \pmod{2^{a-1}}$. Thus $r\equiv -1$ or $2^{a-1}-1 \pmod{2^a}$. Since C_P(M)=M, the conjugation homomorphism $P/M \to Aut(M)$ is injective. Both possible nonidentity actions cannot occur simultaneously: their product is $2^{a-1}+1$ modulo 2^a, which is congruent to 1 (rather than −1) modulo 2^{a-1} and therefore fails the required congruence $r\equiv -1 \pmod{2^{a-1}}$. Therefore |P: M|=2.

$$(r-1)t \equiv 0 \ (mod\ 2^a)$$

The two permitted actions are inversion and the semidihedral action. The modular action $r \equiv 1 + 2^{a-1} \pmod{2^a}$ is excluded by $r \equiv -1 \pmod{2^{a-1}}$. The remaining possibilities are identified directly from the presentations in the next paragraph.

Since |P: M|=2, the group is generated by g and any x outside M. Its relations may be written as $g\text{^}(2^a)=1$, $x^{-1}gx=g^r$, and $x^2=g^t$, with r and t restricted as above. These relations describe the actual group because M has order 2^a and the quotient has order two. Since x commutes with x^2, $(r-1)t \equiv 0 \pmod{2^a}$. The common-involution condition restricts t to 0 or 2^{a-1} modulo 2^a. For $r=-1$, the two values give the dihedral and generalized quaternion presentations. For $r=2^{a-1}-1$, replacing x by xg^k changes its square by $g\text{^}\{k(r+1)\}$; here $r+1=2^{a-1}$, so a suitable parity of k reduces either value of t to zero. This yields the semidihedral presentation.

Conversely, in each of these three presentations, every element outside $\langle g \rangle$ has order at most four and every such element of order four has square $z=g\text{^}(2^{a-1})$; elements of order four inside $\langle g \rangle$ also have square z; all cyclic subgroups of order at least eight lie in $\langle g \rangle$ and hence form a chain. Theorem 5.4, therefore applies. This proves the classification.

Remark 5.6. The modular maximal-cyclic case is not omitted from the index-two argument: it is eliminated by the congruence forced by OCC. Notice also that Theorem 5.4 does not depend on the classification of groups with a cyclic subgroup of index two; its two intrinsic conditions may therefore be used on their own.

Corollary 5.7 (Generalized quaternion groups). If Q_{2^n} is a generalized quaternion 2-group with $n \geq 3$, then $E(Q_{2^n})$ satisfies OCC.

Proof. For $n = 3$, the group Q_8 has a common involution in all cyclic subgroups of order four and no cyclic subgroup of order eight. For $n \geq 4$, the result follows from the generalized quaternion case of Theorem 5.5.

6. Finite Abelian Groups

Corollary 6.1 (Abelian 3-groups). Let A be a finite abelian 3-group. Then E(A) satisfies OCC exactly in the following cases: A is cyclic, or $A \cong (C_3)^d$ for some $d \geq 2$.

Proof. By Theorem 5.3, a 3-group satisfies OCC precisely when it is cyclic or has exponent 3. A finite abelian group of exponent 3 is an elementary abelian 3-group, hence isomorphic to $(C_3)^d$. In the noncyclic case $d \geq 2$.

Proposition 6.2 (Abelian 2-groups). Let A be a finite abelian 2-group. Then E(A) satisfies OCC if and only if A is cyclic, $A \cong (C_2)^d$ for some $d \geq 2$, or $A \cong C_4 \times (C_2)^d$ for some $d \geq 1$.

Proof. Cyclic groups clearly satisfy OCC. If A has exponent 2, there are no cyclic subgroups of order at least 4, so the exponent-two alternative of Theorem 5.4 applies; this gives the groups $(C_2)^d$. Now take $A = C_4 \times (C_2)^d$. Every element of order 4 has the same square, namely the nonidentity element in the square of the C_4-factor. Thus, all cyclic subgroups of order 4 contain one common involution, and A has no cyclic subgroup of order at least 8. Theorem 5.4 again gives OCC. It remains to show that no other noncyclic abelian 2-group occurs. Suppose first that $\exp(A) = 4$ and write $A \cong (C_4)^r \times (C_2)^s$. If $r \geq 2$, two coordinate C_4-subgroups have different involutions, contrary to Theorem 5.4(b). Hence $r = 1$. Finally, assume $\exp(A) = 2^a$ with $a \geq 3$. Write $A \cong C_{2^a} \times B$ with $B \neq 1$, and choose an involution $b \in B$. If g generates C_{2^a}, the cyclic subgroups $\langle(g,e)\rangle$ and $\langle(g,b)\rangle$ are distinct, incomparable, and both have order 2^a. This contradicts Theorem 5.4(a). The listed groups are therefore exactly the possibilities.

Theorem 6.3 (Complete abelian classification). Let A be a nontrivial finite abelian group. Then E(A) satisfies OCC if and only if one of the following mutually exclusive alternatives holds:
(i) A is cyclic; (ii) $A \cong (C_2)^d$ for some $d \geq 2$; (iii) $A \cong (C_3)^d$ for some $d \geq 2$;
(iv) $A \cong C_4 \times (C_2)^d$ for some $d \geq 1$.

Proof. If |A| is divisible by at least two primes, Theorem 4.1 gives cyclicity. Thus, every noncyclic example is a p-group. Apply Theorem 5.2, Theorem 5.3, and Proposition 6.2.

Corollary 6.4 (Abelian-subgroup obstruction). If G is a finite group and E(G) satisfies OCC, then every nontrivial abelian subgroup $A \leq G$ belongs to one of the four classes in Theorem 6.3.

Proof. Apply Proposition 2.5(b) and Theorem 6.3.

7. Generalized dihedral groups

Let A be a finite abelian group and define $\mathrm{Dih}(A) = A \rtimes \langle\tau\rangle$, where $\tau^2 = e$ and $\tau a \tau = a^{-1}$ for all $a \in A$.

Theorem 7.1. The graph E(Dih(A)) is obtained from E(A) by attaching |A| leaves at the identity. Consequently,

$$E(Dih(A)) \text{ satisfies } OCC \iff E(A) \text{ satisfies } OCC$$

Proof. Every element outside A has the form $a\tau$ and satisfies $(a\tau)^2 = e$. If $a\tau$ were adjacent to $x \in A \setminus \{e\}$, the two would lie in a cyclic subgroup and commute. But $(a\tau)x = ax^{-1}\tau$ and $x(a\tau) = xa\tau$, so $x = x^{-1}$; x and $a\tau$ are distinct involutions, and a cyclic group cannot contain two distinct involutions. Two distinct elements outside A are likewise distinct involutions and cannot lie in a common cyclic subgroup. Thus, all elements outside A are leaves at e, and leaves lie on no cycle.

Corollary 7.2. For finite abelian A, E(Dih(A)) satisfies OCC exactly when A is one of the four types in Theorem 6.3.

Corollary 7.3. Let D_{2n} = Dih(C_n) denote the dihedral group of order 2n. Then E(D_{2n}) satisfies OCC for every n ≥ 1. In particular, $D_2 \cong C_2$ and $D_4 \cong C_2 \times C_2$.

8. Algebraic Consequences

The algebraic conclusions in this section apply previously established criteria for connected graphs. The odd-cycle condition is verified here by classifying the relevant finite groups; normality and integer-rounding properties are then consequences of the cited graph-algebra results, not independent classification hypotheses.

Theorem 8.1. Let G be a nontrivial finite group, and let A_G denote the vertex-edge incidence matrix of E(G). Then the following properties are equivalent:

(i) E(G) satisfies OCC; (ii) x ≥ 0, xA_G ≤ 1 has the integer rounding property; (iii) x ≥ 0, xA_G ≥ 1 has the integer rounding property;

(iv) the Rees algebra of the edge ideal of E(G) is normal;

(v) the edge subring (equivalently, its standard homogenized form) is normal; (vi) the corresponding augmented monomial algebra is normal. Equivalently, the edge polytope of E(G) is normal with respect to the edge lattice generated by its incidence vectors.

Proof. The identity element is adjacent to every other vertex of E(G). Since G is nontrivial, E(G) is connected and has at least one edge. The connected-graph equivalences of [3, Theorems 3.2–3.3], summarized in Theorem 2.8, therefore apply to its vertex-edge incidence matrix A_G. The remaining group-specific conclusion follow from the classifications established above.

Corollary 8.2. The equivalent algebraic properties in Theorem 8.1 hold: (a) for a finite nilpotent group of non-prime-power order exactly when it is cyclic; (b) for every finite p-group exactly in the cases of Theorems 5.2, 5.3, and 5.5; (c) for a nontrivial finite abelian group exactly in the four cases of Theorem 6.3; and (d) for every generalized quaternion group.

9. Examples and Scope

Example 9.1. For every n ≥1, E(Cn) = Kn, so every cyclic group satisfies OCC.

Example 9.2. For (C3)d, all nontrivial cyclic subgroups have order 3; after deleting the identity, the enhanced power graph is a disjoint union of copies of K2. In contrast, C5 × C5 fails OCC because two distinct order-5 subgroups give anticomplete copies of K4. Thus, the obstruction for C5 × C5 is already visible in two distinct order-five subgroups, whereas elementary abelian 3-groups have no triangles after removal of the identity.

Example 9.3. The groups $C_4 \times (C_2)^d$ illustrate the common-involution alternative in Theorem 5.4: all their cyclic subgroups of order 4 contain the same involution, and there are no cyclic subgroups of larger 2-power order. Hence, OCC holds. The situation changes for $C_4 \times C_4$. Its two coordinate C_4-subgroups have different involutions; after removing the identity, each contributes a triangle, and the two triangles are anticomplete. Thus, OCC fails.

Example 9.4. Consider $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Its three cyclic subgroups of order 4 all contain −1 as their unique involution. Since Q_8 has no cyclic subgroup of order at least 8, the chain condition is vacuous, and the common-involution condition holds with z = −1. Therefore, E(Q_8) satisfies OCC. This is an instance of the common-involution alternative, not an application of the exponent-two exception.

Example 9.5. Let H be a noncyclic group of odd order. The factor C_2 has order coprime to |H|, so Theorem 3.4 applies with m = 2. It follows that E(H × C_2) does not satisfy OCC.

Example 9.6. Take D_{2n} = Dih(C_n). The rotation subgroup C_n has a complete enhanced power graph. By Theorem 7.1, the remaining n elements are leaves incident only with the identity. Leaves cannot belong to a cycle, so they create no OCC obstruction. Consequently, E(D_{2n}) satisfies OCC for every n ≥ 1.

Example 9.7 (Nonnilpotent behavior). Nonnilpotent groups already show that the preceding subgroup restrictions are not sufficient for a general classification. For A_4, remove the identity from E(A_4). The three involutions then become isolated vertices, while the four subgroups of order 3 give four separate edges. No odd cycle remains, and therefore A_4 satisfies OCC. A different phenomenon occurs in $S_3 \times C_3$. Let c generate C_3 and choose distinct transpositions s,t ∈ S_3. The vertex sets

$$\{(s,e),(s,c),(s,c^2)\} \quad and \quad \{(t,e),(t,c),(t,c^2)\}$$

Each induces a triangle. There is no edge between the two triangles: such an edge would place two elements whose S_3-components are noncommuting transpositions in one cyclic subgroup, which is impossible. Hence $E(S_3 \times C_3)$ fails OCC. This example also shows why Corollary 6.4 has no converse. Every abelian subgroup of $S_3 \times C_3$ belongs to one of the types permitted by Theorem 6.3, but the whole group still fails OCC. Thus, the nonnilpotent problem cannot be decided solely from the abelian subgroups.

10. Conclusion

The classification also identifies the boundary of the subgroup method. For nilpotent groups, the Sylow decomposition and the coprime-factor obstruction give a complete answer. For general nonnilpotent groups, the examples in Section 9 show that checking abelian subgroups alone is insufficient. A natural next problem is to characterize the interaction of noncommuting elements and cyclic subgroups that produces two anticomplete odd cycles.

The classification separates the odd-cycle condition for enhanced power graphs into distinct group-theoretic regimes. A noncyclic nilpotent group with at least two prime divisors cannot satisfy OCC; for p-groups, the answer depends on p and, in the 2-group case, on the arrangement of cyclic subgroups and their involutions. These conclusions also determine the corresponding cases of the established incidence-rounding and graph-algebra normality criteria. Generalized dihedral groups provide a further reduction to the abelian case. The examples outside the nilpotent setting show that restrictions on abelian subgroups alone do not characterize OCC. A natural next question is to identify additional structural obstructions for nonnilpotent finite groups without assuming a direct-product decomposition.

The classifications obtained here concern the odd-cycle condition, rather than the general structure of enhanced power graphs. Previous structural descriptions of these graphs [1, 5–7, 10, 13, 14] and established graph-algebra normality criteria [3, 9, 11, 12] serve distinct purposes: the former motivate the group-theoretic setting, while the latter translate the new group classifications into algebraic consequences. A classification for arbitrary nonnilpotent groups remains outside the scope of the present results.

Author Contributions

LADG conceived the study, developed the mathematical framework and principal results, and prepared the manuscript. DGMR contributed to the verification of the mathematical arguments and to critical revision of the manuscript. Both authors read and approved the final version.

Funding

No funds, grants, or other financial support were received for this work.

Conflict of Interest

The authors declare that they have no conflict of interest.

Generative AI and AI-Assisted Technologies

OpenAI ChatGPT was used during manuscript preparation for drafting assistance, language editing, editorial revision, and bibliographic organization. The authors reviewed the mathematical arguments and references and remain fully responsible for the accuracy, originality, integrity, and scientific content of the manuscript.

Data Availability

No research datasets were generated or analysed in this theoretical study.

References

[1] Ghodratollah Aalipour et al., "On the Structure of the Power Graph and the Enhanced Power Graph of a Group," *The Electronic Journal of Combinatorics*, vol. 24, no. 3, pp. 1-18, 2017. [CrossRef] [Google Scholar] [Publisher Link]

[2] David G. Costanzo et al., "The Cyclic Graph (Deleted Enhanced Power Graph) of a Direct Product," *Involve, A Journal of Mathematics*, vol. 14, no. 1, pp. 167-179, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[3] Luis A. Dupont, Carlos Rentería-Márquez, and Rafael H. Villarreal, "Systems with the Integer Rounding Property in Normal Monomial Subrings," *Annals of the Brazilian Academy of Sciences*, vol. 82, no. 4, pp. 801-811, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[4] Min Feng, Xuanlong Ma, and Kaishun Wang, "The Structure and Metric Dimension of the Power Graph of a Finite Group," *European Journal of Combinatorics*, vol. 43, pp. 82-97, 2015. [CrossRef] [Google Scholar] [Publisher Link]

[5] Xuanlong Ma et al., "A Survey on Enhanced Power Graphs of Finite Groups," *Electronic Journal of Graph Theory and Applications*, vol. 10, no. 1, pp. 89-111, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[6] Xuanlong Ma et al., "Forbidden Subgraphs in Enhanced Power Graphs of Finite Groups," *Journal of the Royal Academy of Exact, Physical and Natural Sciences. Series A. Mathematics*, vol. 118, no. 3, pp. 1-18, 2024. [CrossRef] [Google Scholar] [Publisher Link]

[7] Ramesh Prasad Panda, Sandeep Dalal, and Jitender Kumar, "On the Enhanced Power Graph of a Finite Group," *Communications in Algebra*, vol. 49, no. 4, pp. 1697-1716, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[8] Derek John Scott Robinson, *A Course in the Theory of Groups*, 2nd ed., GTM 80, Springer, New York, 1996. [CrossRef] [Google Scholar] [Publisher Link]

[9] Aron Simis, Wolmer V. Vasconcelos, and Rafael H. Villarreal, "The Integral Closure of Subrings Associated to Graphs," *Journal of Algebra*, vol. 199, no. 1, pp. 281-289, 1998. [CrossRef] [Google Scholar] [Publisher Link]

[10] Samir Zahirović, Ivica Bošnjak, and Rozália Madarász, "A Study of Enhanced Power Graphs of Finite Groups," *Journal of Algebra and its Applications*, vol. 19, no. 4, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[11] Hidefumi Ohsugi, and Takayuki Hibi, "Normal Polytopes Arising from Finite Graphs," *Journal of Algebra*, vol. 207, no. 2, pp. 409-426, 1998. [CrossRef] [Google Scholar] [Publisher Link]

[12] Rafael H. Villarreal, *Monomial Algebras*, 2nd ed., Monographs and Research Notes in Mathematics, CRC Press, 2015. [Google Scholar]

[13] Ivica Bošnjak, Rozália Madarász, and Samir Zahirović, "Some New Results Concerning Power Graphs and Enhanced Power Graphs of Groups," *Discrete Applied Mathematics*, vol. 357, pp. 86-93, 2024. [CrossRef] [Google Scholar] [Publisher Link]

[14] Mark L. Lewis, and Carmine Monetta, "On Neighbourhoods in the Enhanced Power Graph Associated with a Finite Group," *Bulletin of the Australian Mathematical Society*, vol. 112, no. 1, pp. 149-154, 2025. [CrossRef] [Google Scholar] [Publisher Link]