International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V72I10P107

Volume 72 Issue 10, 82-92, October 2026
© 2026 Seventh Sense Research Group®

Original Article

Reliability Analysis of Three Different Series System Model Using Combined Universal Generating Function (UGF) and Markov Process Technique

Raju Singh Gaur

Department of Mathematics, D.A-V College, Civil Lines, Kanpur, C.S.J.M. University, Kanpur, India.

Corresponding Author : bhuwan.rashi@gmail.com

Received: 24 August 2026 Revised: 30 September 2026 Accepted: 17 October 2026 Published: 29 October 2026

Abstract - *This paper examines the reliability of Mean Time Between Failure (MTBF) and performance distribution of three different models, which is built of two types of devices (i.e. unit or system). In each model, the devices are connected in series. The two types of devices are as follows:*

(1) First type device has three states. In normal mode, it operates satisfactorily and fails in open or closed (shorted) modes. The examples of such types of devices are a fluid flow valve and an electronic diode.

(2) Second type device has three states if it has successful, partial and catastrophic failure modes. The examples of such types of devices are instrumental panel, a system, etc.

The two types of devices are considering the three different models. The first model consists of both types of three-state devices, where the second type of three-state device controls the movement of first type. The example of this model is an instrumentation panel controlling the state of a fluid flow valve. The device is only repaired if the instrumentation panel fails in its partial mode. The second model consists of two first-type of three-state devices operating in a series configuration. When one of the three state devices fails in its shorted or closed mode, then repair is possible while the other one is still operating. The third model is composed of second-type of three-state devices or systems operating in a series configuration. The three-state devices are failed in its partial mode, then the repair is only possible, and the other device or system still operates successfully, or both elements function in their partial modes.

These three different models are considered for calculating reliability analysis, MTBF, performance distribution using combined universal generating function (UGF) and Markov process technique.

Keywords - *Mean time between failure, Markov process, Reliability, Repair, Universal generating function.*

1. Introduction

This paper presents reliability analysis of MTBF and performance distributions of three different models, which is built of two types of devices where each device contains three states. In each model, the devices are connected in series [10, 12]. We solve the models using combined UGF and Markov process technique [8]. The two types of device models are considered.

(1) First type of device has three states. In normal mode, it operates satisfactorily and fails in open or closed (shorted) modes. The typical examples of such a device are a fluid flow valve and an electronic diode, etc.

(2) The second type of device or system consists of three states if it has successful, partial and catastrophic failure modes. The typical examples of such a type of device are an instrument panel, a system, etc.

Shooman [1] and Gredeneco et al. [2] have applied a Markov modeling approach to analyze mathematical models. Shooman [1] described some simple active redundant networks, whereas Gredenco et. al. considered both active and non-active redundant network configurations. Srinivasan [3] developed two models to make a theoretical study of the effect of stand-by redundancy of an intermittently used repairable system. Levitin, Lisnianski, Ben-Haim and Elmakis analyzed redundancy optimization for series-parallel system expansion scheduling subject to availability constraints [4,5].

The three-state device models that are used in this paper are defined as follows:

This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/)

The first model consists of both types of three-state devices, where the second type of three-state device controls the movement of the first type [9, 11]. An example of this model is an instrumentation panel controlling the state of a fluid flow valve. The device is only repaired if the instrumentation panel fails in its partial mode.

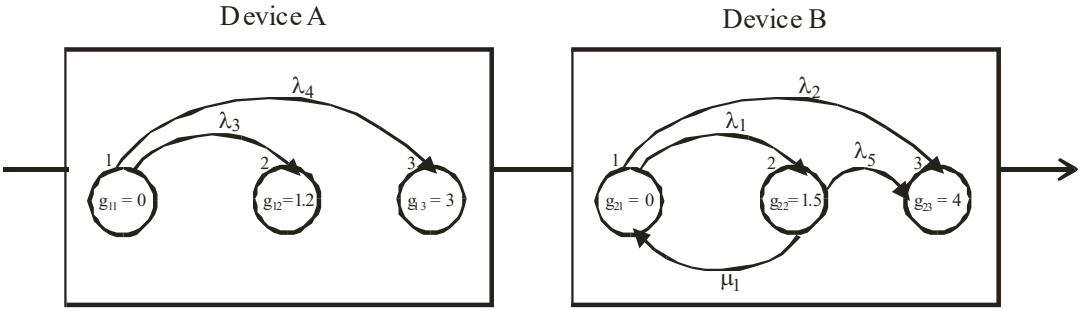
The second model consists of two first-type of three-state devices operating in a series configuration. When one of the three state devices fails in its shorted or closed mode, then repair is possible while the other one is still operating.

The third model is composed of the second type of three-state devices or systems operating in a series configuration. The three-state devices are failed in their partial mode, then the repair is only possible, and the other device or system still operates successfully, or both elements function in their partial modes.

The method is used in this paper for detail study, which is suggested by Levitin, Lisnianski and Tsokos et al. [6, 7]. The paper is organized as follows: Section 2 introduces the first model, which is containing both types of three-state devices. Section 3 is devoted to second model, which consists of two first-type of three-state devices. Section 4 describes the third model, which composed of second type of three-state devices, and Section 5 presents the conclusion.

2. Model 1

In this model, the two devices A and B are modeled in series. Each device has three state g_{11}, g_{12}, g_{13} and g_{21}, g_{22}, g_{23}, respectively, which connected in series. One device has normal, partial and catastrophic states, and the other has normal, open and closed mode states. Repair is possible when equipment fails in its partial mode. A typical practical example of such a system is a fluid flow value command from an instrumental control panel where the control panel represents the first type of device an instrumental control panel where the control panel represents the first type of device, and the fluid flow valve represents the second type.



$$u_1(z) = P_{11}(t)z^0 + P_{12}(t)z^{12} + P_{13}(t)z^3, \qquad u_2(z) = P_{21}(t)z^0 + P_{22}(t)z^{15} + P_{23}(t)z^4$$

Fig. 1 State-space diagrams and u-functions of the two types of three-state devices of fluid flow value

According to the Markov method, we build the following systems of differential equations for each device separately (using the state-space diagram presented in Figure 1.):

For device A:

$$\frac{dP_{11}(t)}{dt} = -(\lambda_3 + \lambda_4)P_{11}(t)$$

$$\frac{dP_{12}(t)}{dt} = \lambda_3 P_{11}(t)$$

$$\frac{dP_{13}(t)}{dt} = \lambda_4 P_{11}(t)$$

Initial conditions are:

$$P_{11}(0) = 1,\ P_{12}(0) = 0 = P_{13}(0)$$

For device B:

$$\frac{dP_{21}(t)}{dt} = -(\lambda_1 + \lambda_2)P_{21}(t) + \mu_1 P_{22}(t)$$

$$\frac{dP_{22}(t)}{dt} = -(\mu_1 + \lambda_5)P_{22}(t) + \lambda_1 P_{21}(t)$$

$$\frac{dP_{23}(t)}{dt} = \lambda_5 P_{22}(t) + \lambda_2 P_{21}(t)$$

Initial conditions are:

$$P_{21}(0) = 1,\ P_{22}(0) = 0 = P_{23}(0)$$

For devices A:

Using the Lapalace – Stieltjes transform, we obtain

$$L\{\dot{P}_{11}(t)\} = -(\lambda_3 + \lambda_4)L\{P_{11}(t)\} \Rightarrow sL\{P_{11}(t)\} - P_{11}(0) = -(\lambda_3 + \lambda_4)\,\widetilde{P}_{11}(s)$$

$$s\,\widetilde{P}_{11}(s) - 1 = -(\lambda_3+\lambda_4)\,\widetilde{P}_{11}(s)$$

$$\widetilde{P}_{11}(s) = \frac{1}{s+\lambda_3+\lambda_4}$$

$$L\{\dot{P}_{12}(t)\} = \lambda_3 L\{P_{11}(t)\} \Rightarrow sL\{P_{12}(t)\} - P_{12}(0) = \lambda_3\,\widetilde{P}_{11}(s)$$

$$s\,\widetilde{P}_{12}(s) - 0 = \lambda_3\,\widetilde{P}_{11}(s) \Rightarrow s\,\widetilde{P}_{12}(s) = \lambda_3\,\widetilde{P}_{11}(s)$$

$$s\,\widetilde{P}_{12}(s) = \frac{\lambda_3}{s+\lambda_3+\lambda_4} \Rightarrow \widetilde{P}_{12}(s) = \frac{\lambda_3}{s(s+\lambda_3+\lambda_4)}$$

$$L\{\dot{P}_{13}(t)\} = \lambda_4 L\{P_{11}(t)\} \Rightarrow sL\{P_{13}(t)\} - P_{13}(0) = \lambda_4\,\widetilde{P}_{11}(s)$$

$$s\,\widetilde{P}_{13}(s) - 0 = \lambda_4\,\widetilde{P}_{11}(s)$$

$$\widetilde{P}_{13}(s) = \frac{\lambda_4\,\widetilde{P}_{11(s)}}{s} \Rightarrow \widetilde{P}_{13}(s) = \frac{\lambda_4}{s(s+\lambda_3+\lambda_4)}$$

Using the inverse Lapalace – Stieltjes transform, we find the state probabilities as functions of time t:

$$P_{11}(t) = L^{-1}\{\frac{1}{\{s+(\lambda_3+\lambda_4)\}}\} \Rightarrow P_{11}(t) = e^{-(\lambda_3+\lambda_4)t}$$

$$P_{12}(t) = \lambda_3 L^{-1}\left\{\frac{1}{s(s+\lambda_3+\lambda_4)}\right\} = \lambda_3 L^{-1}\{\frac{1}{s}\left(\frac{1}{(s+(\lambda_3+\lambda_4))}\right)\}$$

$$= \lambda_3 \int_0^t e^{-(\lambda_3+\lambda_4)x}\,dx = \frac{\lambda_3(e^{-(\lambda_3+\lambda_4)x}{}_0^t}{-(\lambda_3+\lambda_4)}$$

$$P_{12}(t) = \frac{\lambda_3\{1-e^{-(\lambda_3+\lambda_4)t}\}}{(\lambda_3+\lambda_4)}$$

Similarly,

$$P_{13}(t) = \frac{\lambda_4\{1-e^{-(\lambda_3+\lambda_4)\,t}\}}{(\lambda_3+\lambda_4)}$$

The system reliability is then,

$$R(t) = 1 - \frac{\lambda_4\{1-e^{-(\lambda_3+\lambda_4)\,t}\}}{(\lambda_3+\lambda_4)}$$

For device B:

Using the Laplace – Stieltjes transform, we obtain:

$$L\{\dot{P}_{21}(t)\} = -(\lambda_1 + \lambda_2)L\{P_{21}(t)\} + \mu_1 L\{P_{22}(t)\} \Rightarrow s\,\widetilde{P}_{21}(s) - 1 = -(\lambda_1 + \lambda_2)\,\widetilde{P}_{21}(s) + \mu_1\,\widetilde{P}_{22}(s)$$

$$\widetilde{P}_{21}(s) = \frac{\mu_1\,\widetilde{P}_{22}(s)+1}{s+\lambda_1+\lambda_2}$$

$$\dot{P}_{22}(t) = -(\mu_1 + \lambda_5)\,P_{22}(t) + \lambda_1 P_{21}(t)$$

$$sL\{P_{22}(t)\} - P_{22}(0) = -(\mu_1 + \lambda_5)\,L\{P_{22}(t)\} + \lambda_1 L\{P_{21}(t)\}$$

$$\widetilde{P}_{22}(s) = \frac{\lambda_1\,\widetilde{P}_{21}(s)}{(s+\mu_1+\lambda_5)}$$

$$\widetilde{P}_{22}(s) = \frac{\lambda_1(\mu_1 \widetilde{P}_{22}(s)+1}{(s+\mu_1+\lambda_5)(s+\lambda_1+\lambda_2)}$$

$$\widetilde{P}_{22}(s) = \frac{\lambda_1}{(s+\mu_1+\lambda_5)(s+\lambda_1+\lambda_2)-\lambda_1\mu_1}$$

$$\widetilde{P}_{21}(s) = \frac{\mu_1\lambda_1}{(s+\mu_1+\lambda_5)(s+\lambda_1+\lambda_2)-\lambda_1\mu_1} + \frac{1}{(s+\lambda_1+\lambda_2)}$$

$$\widetilde{P}_{21}(s) = \frac{s+\mu_1+\lambda_5}{(s+\mu_1+\lambda_5)(s+\lambda_1+\lambda_2)-\lambda_1\mu_1}$$

Now,

$$L\{\dot{P}_{23}(t)\} = \lambda_5 L\{P_{22}(t)\} + \lambda_2 L\{P_{21}(t)\}$$

$$s\widetilde{P}_{23}(s) = \lambda_5 \widetilde{P}_{22}(s) + \lambda_2 \widetilde{P}_{21}(s)$$

$$\widetilde{P}_{23}(s) = \frac{\lambda_1\lambda_5 + \lambda_2(s+\mu_1+\lambda_5)}{s\{(s+\mu_1+\lambda_5)(s+\lambda_1+\lambda_2)-\lambda_1\mu_1\}}$$

Solve the equation $(s+\mu_1+\lambda_5)(s+\lambda_1+\lambda_2)-\lambda_1\mu_1 = 0$ for as,
We obtain as follows:

$$s^2 + (\lambda_1+\lambda_2+\mu_1+\lambda_5)s + \mu_1\lambda_2 + \lambda_1\lambda_5 + \lambda_2\lambda_5$$

$$s_1 = \frac{-(\lambda_1+\lambda_2+\mu_1+\lambda_5)+\{(\lambda_1+\lambda_2+\mu_1+\lambda_5)^2-4(\mu_1\lambda_2+\lambda_1\lambda_5+\lambda_2\lambda_5)\}^{1/2}}{2}$$

$$s_2 = \frac{-(\lambda_1+\lambda_2+\mu_1+\lambda_5)-\{(\lambda_1+\lambda_2+\mu_1+\lambda_5)^2-4(\mu_1\lambda_2+\lambda_1\lambda_5+\lambda_2\lambda_5)\}^{1/2}}{2}$$

Hence, expanding by partial fractions, we see that

$$\widetilde{P}_{21}(s) = \frac{s_1+\mu_1+\lambda_5}{s_1-s_2}\left(\frac{1}{s-s_1}\right) - \frac{s_2+\mu_1+\lambda_5}{s_1-s_2}\left(\frac{1}{s-s_2}\right)$$

$$\widetilde{P}_{22}(s) = \frac{\lambda_1}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

$$\widetilde{P}_{23}(s) = \frac{\lambda_2}{(s_1-s_2)}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right) + (\lambda_5\lambda_1 + \lambda_2\mu_1 + \lambda_2\lambda_5)\left\{\frac{1}{s_1s_2}\left(\frac{1}{s}\right) + \frac{1}{s_1(s_1-s_2)}\left(\frac{1}{s-s_1}\right) - \frac{1}{s_2(s_1-s_2)}\left(\frac{1}{s-s_2}\right)\right\}$$

Using the inverse Laplace – Stieltjes transform, we find the state probabilities as functions of time t:

$$P_{21}(t) = \frac{s_1+\mu_1+\lambda_5}{s_1-s_2}e^{s1t} - \frac{s_2+\mu_1+\lambda_5}{s_1-s_2}e^{s2t}$$

$$P_{22}(t) = \frac{\lambda_1}{s_1-s_2}(e^{s1t} - e^{s2t})$$

$$P_{23}(t) = \frac{\lambda_2}{s_1-s_2}(e^{s1t} - e^{s2t}) + 1 + \frac{s_2e^{s1t} - s_1e^{s2t}}{s_1-s_2}$$

The system reliability is then,

$$R(t) = 1 - P_{23}(t)$$

$$R(t) = \frac{\lambda_2}{s_1-s_2}(e^{s2t} - e^{s1t}) + \frac{s_1e^{s2t} - s_2e^{s1t}}{s_1-s_2}$$

Similarly, the system MTBF is given by,

$$MTBF = \int_0^\infty R(t)\,dt = \int_0^\infty \left\{\frac{\lambda_2}{s_1-s_2}(e^{s2t} - e^{s1t}) + \frac{s_1e^{s2t} - s_2e^{s1t}}{s_1-s_2}\right.$$

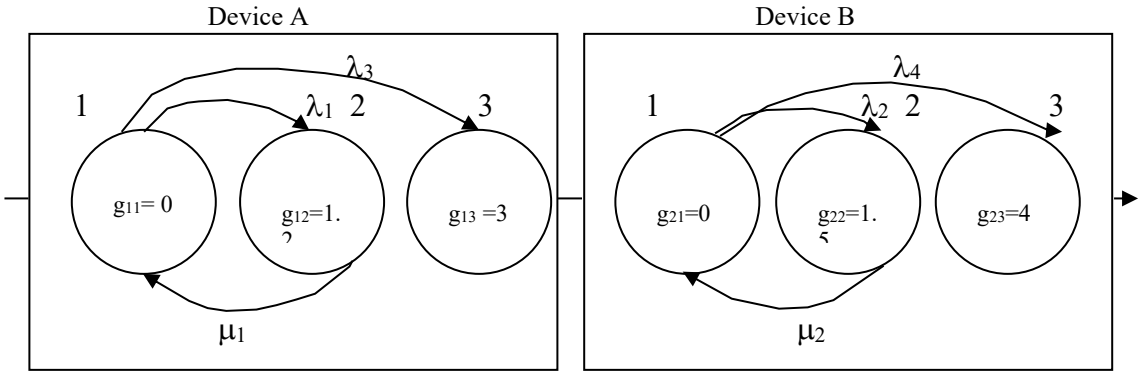
$$= -\frac{\lambda_2}{s_1s_2} - \frac{s_1+s_2}{s_1s_2}$$

$$= -\frac{(\lambda_2+s_1+s_2)}{s_1s_2}$$

$$= \frac{(\lambda_1+\mu_1+\lambda_5}{\mu_1\lambda_2+\lambda_1\lambda_5+\lambda_2\lambda_5}$$

3. Model 2

In this model, the two devices A and B are modeled in series. Each device has three state g_{11}, g_{12}, g_{13} and g_{21}, g_{22}, g_{23} respectively, which connected in series. One device is failed in its closed mode, then repair is possible, assuming the other one is still operating. A good, typical, practical example of this model is two fluid flow valves operating in series.



$u_1(z) = P_{11}(t)z^0 + P_{12}(t)z^{12} + P_{13}(t)z^3$ $\quad$ $u_2(z) = P_{21}(t)z^0 + P_{22}(t)z^{15} + P_{23}(t)z^4$

Fig. 2 State-space diagrams and u-functions of the two types of three-state devices of fluid flow value

According to the Markov method, we build the following systems of differential equations for each device separately (using the state-space diagram presented in Figure 2):

For device A:

$$\frac{dP_{11}(t)}{dt} = -(\lambda_1 + \lambda_3) P_{11}(t) + \mu_1 P_{12}(t)$$
$$\frac{dP_{12}(t)}{dt} = \lambda_1 P_{11}(t) - \mu_1 P_{12}(t)$$
$$\frac{dP_{13}(t)}{dt} = \lambda_3 P_{11}(t)$$

Initial conditions are:

$$P_{11}(0) = 1,\ P_{12}(0) = 0 = P_{13}(0)$$

For device B:

$$\frac{dP_{21}(t)}{dt} = -(\lambda_2 + \lambda_4) P_{21}(t) + \mu_2 P_{22}(t)$$
$$\frac{dP_{22}(t)}{dt} = \lambda_2 P_{21}(t) - \mu_2 P_{22}(t)$$
$$\frac{dP_{23}(t)}{dt} = \lambda_4 P_{21}(t)$$

Initial conditions are:

$$P_{21}(0) = 1,\ P_{22}(0) = P_{23}(0)$$

For device A:

Using the Laplace – Stieltjes transform, we obtain

$$L\{\acute{P}_{11}(t)\} = -(\lambda_1 + \lambda_3)L\{P_{11}(t)\} + \mu_1 L\{P_{12}(t)\}$$
$$s\widetilde{P}_{11}(s) - 1 = -(\lambda_1 + \lambda_3)\widetilde{P}_{11}(s) + \mu_1 \widetilde{P}_{12}(s)$$
$$\widetilde{P}_{11}(s) = \frac{\mu_1 \widetilde{P}_{12}(s)+1}{s+\lambda_1+\lambda_3}$$
$$L\{\acute{P}_{12}(t)\} = \lambda_1 L\{P_{11}(t)\} - \mu_1 P_{12}(t)\}$$
$$s\widetilde{P}_{12}(s) - 0 = \lambda_1 \widetilde{P}_{11}(s) - \mu_1 \widetilde{P}_{12}(s)$$
$$\widetilde{P}_{12}(s) = \frac{\lambda_1 \widetilde{P}_{11}(s)}{s+\mu_1}$$
$$L\{\acute{P}_{13}(t)\} = \lambda_3 L\{P_{11}(t)$$
$$s\widetilde{P}_{13}(s) = \lambda_3 \widetilde{P}_{11}(s)$$
$$\widetilde{P}_{13}(s) = \frac{\lambda_3 \widetilde{P}_{11}(s)}{s}$$

Now,

$$\widetilde{P}_{11}(s) = \frac{(s+\mu_1)}{s^2+(\lambda_1+\lambda_3+\mu_1)s+\mu_1\lambda_3}$$

$$\widetilde{P}_{12}(s) = \frac{\lambda_1}{s^2+(\lambda_1+\lambda_3+\mu_1)s+\mu_1\lambda_3}$$

$$\widetilde{P}_{13}(s) = \frac{\lambda_3(s+\mu_1)}{s(s^2+(\lambda_1+\lambda_3+\mu_1)s+\mu_1\lambda_3)}$$

Solving the equation $s^2 + (\lambda_1 + \lambda_3 + \mu_1)s + \mu_1\lambda_3 = 0$ for s,
We obtain as follows:

$$s_1 = \frac{-(\lambda_1+\lambda_3+\mu_1)+\{(\lambda_1+\lambda_3+\mu_1)^2-4\mu_1\lambda_3\}^{1/2}}{2}$$

$$s_2 = \frac{-(\lambda_1+\lambda_3+\mu_1)-\{(\lambda_1+\lambda_3+\mu_1)^2-4\mu_1\lambda_3\}^{1/2}}{2}$$

Hence, expanding by partial fractions, we see that

$$\widetilde{P}_{11}(s) = \frac{s_1+\mu_1}{s_1-s_2}\left(\frac{1}{s-s_1}\right) - \frac{s_2+\mu_1}{s_1-s_2}\left(\frac{1}{s-s_2}\right)$$

$$\widetilde{P}_{12}(s) = \frac{\lambda_1}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

$$\widetilde{P}_{13}(s) = \frac{\lambda_3}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right) + \lambda_3\mu_1\left\{\frac{1}{s_1s_2}\left(\frac{1}{s}\right) + \frac{1}{s_1(s_1-s_2)}\left(\frac{1}{s-s_1}\right) - \frac{1}{s_2(s_1-s_2)}\left(\frac{1}{s-s_2}\right)\right\}$$

Using the inverse Laplace-Stieltjes transform, we find the state probabilities as functions of time t:

$$P_{11}(t) = \frac{s_1+\mu_1}{s_1-s_2}e^{s1t} - \frac{s_2+\mu_1}{s_1-s_2}e^{s2t}$$

$$P_{12}(t) = \frac{\lambda_1}{s_1-s_2}\left(e^{s1t} - e^{s2t}\right)$$

$$P_{13}(t) = \frac{\lambda_3}{s_1-s_2}\left(e^{s1t} - e^{s2t}\right) + 1 + \frac{s_2e^{s1t}-s_1e^{s2t}}{s_1-s_2}$$

The system reliability is then

$$R(t) = 1 - P_{13}(t)$$

$$R(t) = \frac{\lambda_3}{s_1-s_2}\left(e^{s2t} - e^{s1t}\right) + \frac{s_1e^{s2t}-s_2e^{s1t}}{s_1-s_2}$$

Similarly, the system MTBF is given by

$$MTBF = \int_0^\infty R(t)dt$$

$$= \int_0^\infty \left\{\frac{\lambda_3}{s_1-s_2}\left(e^{s2t} - e^{s1t}\right) + \frac{s_1e^{s2t}-s_2e^{s1t}}{s_1-s_2}\right\}dt$$

$$= -\frac{\lambda_3}{s_1s_2} - \frac{s_1+s_2}{s_1s_2}$$

$$= -\frac{(\lambda_3+s_1+s_2)}{s_1s_2}$$

$$= \frac{(\lambda_1+\mu_1)}{\mu_1\lambda_3}$$

For device B:

Similarly, using the Laplace-Stieltjes transform, we obtain

$$\widetilde{P}_{21}(s) = \frac{(s+\mu_2)}{s^2+(\lambda_2+\lambda_4+\mu_2)s+\mu_2\lambda_4}$$

$$\widetilde{P}_{22}(s) = \frac{\lambda_2}{s^2+(\lambda_2+\lambda_4+\mu_2)s+\mu_2\lambda_4}$$

$$\widetilde{P}_{23}(s) = \frac{\lambda_4(s+\mu_2)}{s(s^2+(\lambda_2+\lambda_4+\mu_2)s+\mu_2\lambda_4}$$

Solving the equation $s^2 + (\lambda_2 + \lambda_4 + \mu_2)\ s + \mu_2\lambda_4 = 0$ for s,

We obtain as follows:

$$s_1 = \frac{-(\lambda_2+\lambda_4+\mu_2)+\{(\lambda_2+\lambda_4+\mu_2)^2-4\mu_2\lambda_4\}^{1/2}}{2}$$

$$s_2 = \frac{-(\lambda_2+\lambda_4+\mu_2)-\{(\lambda_2+\lambda_4+\mu_2)^2-4\mu_2\lambda_4\}^{1/2}}{2}$$

Hence, expanding by partial fractions, we see that

$$\widetilde{P}_{21}(s) = \frac{s_1+\mu_2}{s_1-s_2}\left(\frac{1}{s-s_1}\right) - \frac{s_2+\mu_2}{s_1-s_2}\left(\frac{1}{s-s_2}\right)$$

$$\widetilde{P}_{22}(s) = \frac{\lambda_2}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

$$\widetilde{P}_{23}(s) = \frac{\lambda_4}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right) + \lambda_4\mu_2\left\{\frac{1}{s_1s_2}\left(\frac{1}{s}\right) + \frac{1}{s_1(s_1-s_2)}\left(\frac{1}{s-s_1}\right) - \frac{1}{s_2(s_1-s_2}\left(\frac{1}{s-s_2}\right)\right\}$$

Using the inverse Laplace-Stieltjes transform, we find the state probabilities as functions of time t:

$$P_{21}(t) = \frac{s_1+\mu_2}{s_1-s_2}\, e^{s1t} - \frac{s_2+\mu_2}{s_1-s_2}\, e^{s2t}$$

$$P_{22}(t) = \frac{\lambda_2}{s_1-s_2}\left(e^{s1t} - e^{s2t}\right)$$

$$P_{23}(t) = \frac{\lambda_4}{s_1-s_2}\left(e^{s1t} - e^{s2t}\right) + 1 + \frac{s_2e^{s1t} - s_1e^{s2t}}{s_1-s_2}$$

The system reliability is then

R (t) = 1- $P_{23}(t)$

$$R(t) = \frac{\lambda_4}{s_1-s_2}\left(e^{s2t} - e^{s1t}\right) + \frac{s_1e^{s2t} - s_2e^{s1t}}{s_1-s_2}$$

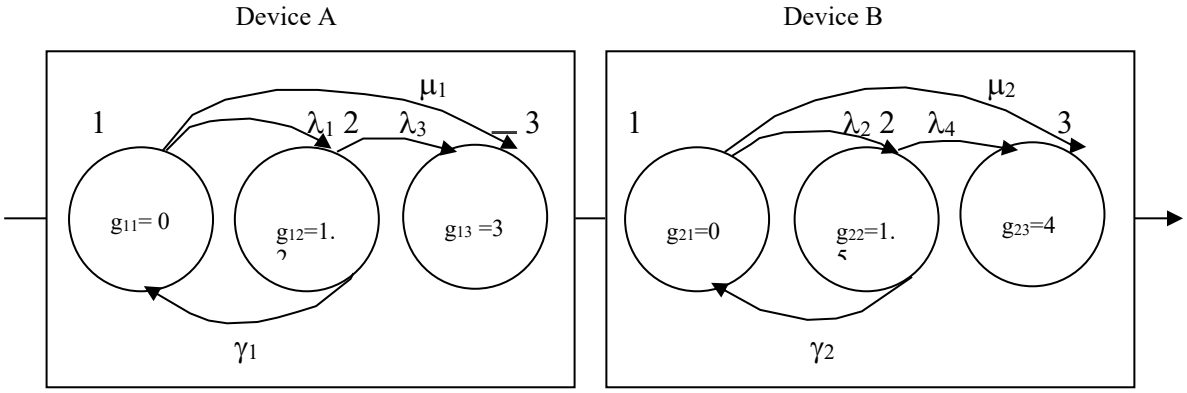
Similarly, the system MTBF is given by

$$MTBF = \int_0^\infty R(t)\, dt = \int_0^\infty \left\{\frac{\lambda_4}{s_1-s_2}\left(e^{s2t} - e^{s1t}\right) + \frac{s_1e^{s2t} - s_2e^{s1t}}{s_1-s_2}\right\} dt$$

$$= -\frac{\lambda_4}{s_1s_2} - \frac{s_1+s_2}{s_1s_2} = \frac{-(\lambda_4+s_1+s_2)}{s_1s_2} = \frac{(\lambda_2+\mu_2}{\mu_2\lambda_4}$$

4. Model 3

In this model, the two devices A and B are modeled in series. Each device has three state g_{11}, g_{12}, g_{13} and g_{21}, g_{22}, g_{23}, respectively, which are connected in series. The device is failed in its partial failure mode then repair is possible, the other three-state device is operating successfully, or both devices are operating in their partial failure mode. The typical practical example of this model is two automatic machines performing some operations on the assembly line.



$$u_1(z) = P_{11}(t)z^0 + P_{12}(t)z^{12} + P_{13}(t)z^3 \qquad u_2(z) = P_{21}(t)z^0 + P_{22}(t)z^{15} + P_{23}(t)z^4$$

Fig. 3 State-space diagrams and u-functions of the two types of three-state devices of fluid flow value

According to the Markov method, we build the following systems of differential equations for each device separately (using the state-space diagram presented in Figure. 3):

For device A:

$$\frac{dP_{11}(t)}{dt} = -(\lambda_1+\mu_1)P_{11}(t) + \gamma_1P_{12}(t)$$

$$\frac{dP_{12}(t)}{dt} = -(\lambda_3+\gamma_1)P_{12}(t) + \lambda_1P_{11}(t)$$

$\frac{dP_{13}(t)}{dt} = (\mu_1 + \lambda_3)P_{13}(t)$

Initial conditions are:
$P_{11}(0) = 1,\ P_{12}(0) = 0 = P_{13}(0)$

For device B:
$\frac{dP_{21}(t)}{dt} = - (\lambda_2 + \mu_2)P_{21}(t) + \gamma_2 P_{22}(t)$
$\frac{dP_{22}(t)}{dt} = - (\lambda_4 + \gamma_2)P_{22}(t) + \lambda_2 P_{21}(t)$
$\frac{dP_{23}(t)}{dt} = (\mu_2 + \lambda_4)P_{23}(t)$

Initial conditions are :
$P_{21}(0) = 1,\ P_{22}(0) = 0 = P_{23}(0)$

For devices A:

Using the Laplace–Stieltjes transform, we obtain

$\widetilde{P}_{11}(s) = \frac{(s+\gamma_1+\lambda_3)}{(s+\gamma_1+\lambda_3)(s+\lambda_1+\mu_1)-\lambda_1\gamma_1}$

$\widetilde{P}_{12}(s) = \frac{\lambda_1}{(s+\gamma_1+\lambda_3)(s+\lambda_1+\mu_1)-\lambda_1\gamma_1}$

$\widetilde{P}_{13}(s) = \frac{\lambda_1\lambda_3+\mu_1(s+\gamma_1+\lambda_3)}{s\{(s+\gamma_1+\lambda_3)(s+\lambda_1+\mu_1)-\lambda_1\gamma_1\}}$

Solve the equation $(s + \gamma_1 + \lambda_3)(s + \lambda_1 + \mu_1) - \lambda_1\gamma_1 = 0$ for s,
We obtain as follows:

$s^2 + (\lambda_1 + \mu_1 + \gamma_1 + \lambda_3)\, s + \mu_1\gamma_1 + \lambda_1\lambda_3 + \mu_1\lambda_3$

$s_1 = \frac{-(\lambda_1+\mu_1+\gamma_1+\lambda_3)+\{(\lambda_1+\mu_1+\gamma_1+\lambda_3)^2-4(\mu_1\gamma_1+\lambda_1\lambda_3+\mu_1\lambda_3)\}^{1/2}}{2}$

$s_2 = \frac{-(\lambda_1+\mu_1+\gamma_1+\lambda_3)-\{(\lambda_1+\mu_1+\gamma_1+\lambda_3)^2-4(\mu_1\gamma_1+\lambda_1\lambda_3+\mu_1\lambda_3)\}^{1/2}}{2}$

Hence, expanding by partial fractions, we see that

$\widetilde{P}_{11}(s) = \frac{s_1+\gamma_1+\lambda_3}{s_1-s_2}\left(\frac{1}{s-s_1}\right) - \frac{s_2+\gamma_1+\lambda_3}{s_1-s_2}\left(\frac{1}{s-s_2}\right)$

$\widetilde{P}_{12}(s) = \frac{\lambda_1}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$

$\widetilde{P}_{13}(s) = \frac{\mu_1}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right) + \mu_1\gamma_1 + \lambda_1\lambda_3 + \mu_1\lambda_3\left\{\frac{1}{s_1s_2}\left(\frac{1}{s}\right) + \frac{1}{s_1(s_1-s_2)}\left(\frac{1}{s-s_1}\right) - \frac{1}{s_2(s_1-s_2)}\left(\frac{1}{s-s_2}\right)\right\}$

Using the inverse Laplace-Stieltjes transform, we find the state probabilities as functions of time t:

$P_{11}(t) = \frac{s_1+\gamma_1+\lambda_3}{s_1-s_2}e^{s1t} - \frac{s_2+\gamma_2+\lambda_3}{s_1-s_2}e^{s2t}$

$P_{12}(t) = \frac{\lambda_1}{s_1-s_2}(e^{s1t} - e^{s2t})$

$P_{13}(t) = \frac{\mu_1}{s_1-s_2}(e^{s1t} - e^{s2t}) + 1 + \frac{s_2e^{s1t}-s_1e^{s2t}}{s_1-s_2}$

The system reliability is then
R (t) = 1 - $P_{13}(t)$

R (t) = $\frac{\mu_1}{s_1-s_2}(e^{s2t} - e^{s1t}) + \frac{s_1e^{s2t}-s_2e^{s1t}}{s_1-s_2}$

Similarly, the system MTBF is given by

MTBF = $\int_0^\infty R\,(t)\,dt = \int_0^\infty \{\frac{\mu_1}{s_1-s_2}(e^{s2t} - e^{s1t}) + \frac{s_1e^{s2t}-s_2e^{s1t}}{s_1-s_2}\}\,dt$

$= \frac{-(\mu_1+s_1+s_2)}{s_1s_2} = \frac{(\lambda_1+\gamma_1+\lambda_3)}{\mu_1\gamma_1+\lambda_1\lambda_3+\mu_1\lambda_3}$

For device B:
Similarly, using the Laplace-Stieltjes transform, we obtain

$$\widetilde{P}_{21}(s) = \frac{(s+\gamma_2+\lambda_4)}{(s+\gamma_2+\lambda_4)(s+\lambda_2+\mu_2)-\lambda_2\gamma_2}$$

$$\widetilde{P}_{22}(s) = \frac{\lambda_2}{(s+\gamma_2+\lambda_4)(s+\lambda_2+\mu_2)-\lambda_2\gamma_2}$$

$$\widetilde{P}_{23}(s) = \frac{\lambda_2\lambda_4+\mu_2(s+\gamma_2+\lambda_4)}{s\{(s+\gamma_2+\lambda_4)(s+\lambda_2+\mu_2)-\lambda_2\gamma_2}$$

Solving the equation$(s+\gamma_2+\lambda_4)$ (s+$\lambda_2+\mu_2$) $-\lambda_2\gamma_2$ = 0 for s,
We obtain as follows:

$$s^2 + (\lambda_2+\mu_2+\gamma_2+\lambda_4)s + \mu_2\gamma_2 + \lambda_2\lambda_4 + \mu_2\lambda_4$$

$$s_1 = \frac{-(\lambda_2+\mu_2+\gamma_2+\lambda_4)+\{(\lambda_2+\mu_2+\gamma_2+\lambda_4)^2-4(\mu_2\gamma_2+\lambda_2\lambda_4+\mu_2\lambda_4)\}^{1/2}}{2}$$

$$s_2 = \frac{-(\lambda_2+\mu_2+\gamma_2+\lambda_4)-\{(\lambda_2+\mu_2+\gamma_2+\lambda_4)^2-4(\mu_2\gamma_2+\lambda_2\lambda_4+\mu_2\lambda_4)\}^{1/2}}{2}$$

Hence, expanding by partial fractions, we see that

$$\widetilde{P}_{21}(s) = \frac{s_1+\gamma_2+\lambda_4}{s_1-s_2}\left(\frac{1}{s-s_1}\right) - \frac{s_2+\gamma_2+\lambda_4}{s_1-s_2}\left(\frac{1}{s-s_2}\right)$$

$$\widetilde{P}_{22}(s) = \frac{\lambda_2}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

$$\widetilde{P}_{23}(s) = \frac{\mu_2}{s_1-s_2}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right) + \mu_2\gamma_2 + \lambda_2\lambda_4 + \mu_2\lambda_4\left\{\frac{1}{s_1s_2}\left(\frac{1}{s}\right) + \frac{1}{s_1(s_1-s_2)}\left(\frac{1}{s-s_1}\right) - \frac{1}{s_2(s_1-s_2)}\left(\frac{1}{s-s_2}\right)\right\}$$

Using the inverse Laplace-Stieltjes transform, we find the state probabilities as functions of time t:

$$P_{21}(t) = \frac{s_1+\gamma_2+\lambda_4}{s_1-s_2}e^{s1t} - \frac{s_2+\gamma_2+\lambda_4}{s_1-s_2}e^{s2t}$$

$$P_{22}(t) = \frac{\lambda_2}{s_1-s_2}(e^{s1t} - e^{s2t})$$

$$P_{23}(t) = \frac{\mu_2}{s_1-s_2}(e^{s1t} - e^{s2t}) + 1 + \frac{s_2e^{s1t} - s_1e^{s2t}}{s_1-s_2}$$

The system reliability is then
R (t) = 1- $P_{23}(t)$

$$R(t) = \frac{\mu_2}{s_1-s_2}(e^{s2t} - e^{s1t}) + \frac{s_1e^{s2t} - s_2e^{s1t}}{s_1-s_2}$$

Similarly, the system MTBF is given by

$$MTBF = \int_0^\infty R(t)\,dt = \int_0^\infty \left\{\frac{\mu_2}{s_1-s_2}(e^{s2t} - e^{s1t}) + \frac{s_1e^{s2t} - s_2e^{s1t}}{s_1-s_2}\right\} dt$$

$$= \frac{-(\mu_2+s_1+s_2)}{s_1s_2} = \frac{(\lambda_2+\gamma_2+\lambda_4)}{\mu_2\gamma_2+\lambda_2\lambda_4+\mu_2\lambda_4}$$

After determining the state probabilities for each device of three different type models, we obtain the following performance distributions:
For device A

$$g_1 = \{g_{11}, g_{12}, g_{13}\} = \{0, 1\ 2, 3\}, P_1(t) = \{P_{11}(t), P_{12}(t), P_{13}(t)\}$$

For device B

$$g_2 = \{g_{21}, g_{22}, g_{23}\} = \{0, 1\ 5, 4\}, P_2(t) = \{P_{21}(t), P_{22}(t), P_{23}(t)\}$$

Having the sets $g_j, P_j(t)$ for j = 1, 2 obtained in the first step, we can define the u-functions of the individual devices as:

$$u_1(z) = P_{11}(t)Z^{g11} + P_{12}(t)Z^{g12} + P_{13}(t)Z^{g13}$$
$$= P_{11}(t)Z^0 + P_{12}(t)Z^{12} + P_{13}(t)Z^3$$

$$u_2(z) = P_{21}(t)Z^{g21} + P_{22}(t)Z^{g22} + P_{23}(t)Z^{g23}$$
$$= P_{21}(t)Z^0 + P_{22}(t)Z^{15} + P_{23}(t)Z^4$$

Using the composition operator $\Omega_{\phi s}$ for fluid flow value with flow dispersion, we obtain the resulting u-function for the entire series MSS

$U(Z) = \Omega_{øs}(u_1(z), u_2(z))$ by the following recursive procedure.

$$\Omega_{øs}\big(u_1(z), u_2(z)\big) = \Omega_{øs}(P_{11}(t)Z^0 + P_{12}(t)Z^{12} + P_{13}(t)Z^3, P_{21}(t)Z^0 + P_{22}(t)Z^{15} + P_{23}(t)Z^4)$$

$$= (P_{11}(t)P_{21}(t)Z^{\min\{0,0\}} + P_{11}(t)P_{22}(t)Z^{\min\{0,15\}} + P_{11}(t)P_{23}(t)Z^{\min\{0,4\}} + P_{12}(t)P_{21}(t)Z^{\min\{12,0\}} + P_{12}(t)P_{22}(t)Z^{\min\{12,15\}} + P_{12}(t)P_{23}(t)Z^{\min\{12,4\}} + P_{13}(t)P_{21}(t)Z^{\min\{3,0\}} + P_{13}(t)P_{22}(t)Z^{\min\{3,15\}} + P_{13}(t)P_{23}(t)Z^{\min\{3,4\}}$$

$$= (P_{11}(t)P_{21}(t)Z^0 + P_{11}(t)P_{22}(t)Z^0 + P_{11}(t)P_{23}(t)Z^0 + P_{12}(t)P_{21}(t)Z^0 + P_{12}(t)P_{22}(t)Z^{12} + P_{12}(t)P_{23}(t)Z^{12} + P_{13}(t)P_{21}(t)Z^0 + P_{13}(t)P_{22}(t)Z^{15} + P_{13}(t)P_{23}(t)Z^3)$$

Taking into account that

$$P_{11}(t) + P_{12}(t) + P_{13}(t) = 1, \quad P_{21}(t) + P_{22}(t) + P_{23}(t) = 1$$

We obtain the u-function that determines the output performance distribution g, P (t) of the entire MSS in the following form:

$$U(Z) = \sum_{i=1}^{4} P_i(t) Z^{gi}$$

Where

g_1=0, $P_1(t) = P_{11}(t)\big(P_{21}(t) + P_{22}(t) + P_{23}(t)\big) + P_{21}(t)\big(P_{12}(t) + P_{13}(t)\big)$

$= P_{11}(t)(1) + P_{21}(t)\big(P_{12}(t) + P_{13}(t)\big)$

$P_1(t) = P_{11}(t) + P_{21}(t)\big(1 - P_{11}(t)\big)$

$g_2 = 1.2\ tons/min, \quad P_2(t) = P_{12}(t)(P_{22}(t) + P_{23}(t))$

$g_3 = 1.5\ tons/min, \quad P_3(t) = P_{13}(t)P_{22}(t)$

$g_4 = 3\ tons/min, \quad P_4(t) = P_{13}(t)P_{23}(t)$

Based on the entire MSS u–function U (Z) we obtain the MSS reliability indices:

The instantaneous MSS availability for the constant demand level W = 1.5 tons/min {according to the UGF technique}

$A(t) = \delta_A(U(Z), w = \delta_A(\sum_{i=1}^{4} P_i(t) Z^{gi}, 1.5)$

$= \sum_{i=1}^{4} P_i(t) 1(F(g_i, 1.5) \geq 0) = P_3(t) + P_4(t) = P_{13}(t)(P_{22}(t) + P_{23}(t))$

The instantaneous mean output performance at any instant t > 0

$E_t = \delta_E(U(Z) = \sum_{i=1}^{4} P_i(t) g_i = 0P_1(t) + 1.2P_2(t) + 1.5P_3(t) + 3P_4(t))$

The instantaneous performance deficiency D_t at any time t for the constant demand w = 1.5 tons/min:

$D_t = \delta_D(U(Z), w) = \sum_{i=1}^{4} P_i(t) \max(1.5 - g_i, 0)$

$= P_1(t)(1.5 - 0) + P_2(t)(1.5 - 1.2) = 1.5P_1(t) + 3P_2(t)$

5. Conclusion

Equal failure and repair rates were assumed for second and third models to simplify reliability analysis and MTBF. However, the reliability analysis and MTBF can be solved when failure and repair rates are not equal.

All in all, it can be concluded that, the three different models offer to analyze three-state devices systems using UGF and Markov process technique for solving reliability analysis, MTBF and performance distributions.

References

[1] Martin L. Shooman, *Probabilistic Reliability: An Engineering Approach*, Mc Graw-Hill, New York, 1968. [Google Scholar]

[2] Boris Vladimirovich Gnedenko et al., *Mathematical Models of Reliability Theory*, Academic Press, New York, 1969. [Publisher Link]

[3] V.S. Srinivasan, "The Effect of Standby Redundancy in System's Failure with Repair Maintenance," *Operations Research*, vol. 14, no. 6, pp. 1024-1036, 1966. [CrossRef] [Google Scholar] [Publisher Link]

[4] Gregory Levitin et al., "Redundancy Optimization for Series–Parallel Multi-State Systems," *IEEE Transactions on Reliability*, vol. 47, no. 2, pp. 165-172, 1998. [CrossRef] [Google Scholar] [Publisher Link]

[5] Gregory Levitin, "Multistate Series-Parallel System Expansion Scheduling Subject to Availability Constraints," *IEEE Transactions on Reliability*, vol. 49, no. 1, pp. 71-79, 2000. [CrossRef] [Google Scholar] [Publisher Link]

[6] Anatoly Lisnianski, and Gregory Levitin, *Multi-State System Reliability*, Word Scientific, New Jersey, 2003. [CrossRef] [Google Scholar] [Publisher Link]

[7] Chris P. Tsokos, and I.N. Shimi, *The Theory and Applications of Reliability with Emphasis on Bayesian and Nonparametric Methods*, Academic Press, New York, 1977. [CrossRef] [Google Scholar] [Publisher Link]

[8] Mohsen Babaei, and Amin Rashidi-Baqhi, "Universal Generating Function-based Narrow Reliability Bounds to Evaluate Reliability of Project Completion Time," *Reliability Engineering & System Safety*, vol. 218, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[9] Pedram Pourkarim Guilani et al., "Reliability Evaluation of Non-Repairable Three-State Systems using Markov Model and its Comparison with the UGF and the Recursive Methods," *Reliability Engineer and System Safety*, vol. 129, pp. 29-35, 2014. [CrossRef] [Google Scholar] [Publisher Link]

[10] Siqi Qiu, and Xinguo Ming, "A Fuzzy Universal Generating Function-based Method for the Reliability Evaluation of Series Systems with Performance Sharing Between Adjacent Units Under Parametric Uncertainty," *Fuzzy Sets and Systems*, vol. 424, pp. 155-169, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[11] Fatma Beria Oktay, and Özge Elmastaş Gültekin, "Combined Markov and UGF Methods for Multi-State Repairable Phased Mission Systems," *International Journal of Reliability, Risk and Safety: Theory and Application*, vol. 6, no. 1, pp. 1-9, 2023. [CrossRef] [Google Scholar] [Publisher Link]

[12] Hamisu Ismail Ayagi et al., "Structural Impact between Linear and Circular Consecutive k-out-of-n: G System Using Signature Reliability Combined with Universal Generating Function Approach," *Dutse Journal of Pure and Applied Sciences*, vol. 12, no. 3b, pp. 305-314, 2026. [Google Scholar] [Publisher Link]