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Original Article

Alpha Graceful Labeling in the Context of m-Shadow Graphs

J. H. Jani[1], V. J. Kaneria[2]

[1, 2]*Department of Mathematics, Saurashtra University, Rajkot, Gujarat, India.*

[1]*Corresponding Author : jaykeenhj@gmail.com*

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Abstract *- A graph $G = (V, E)$ is said to be α-graceful if there exists one-to-one function f from $V(G)$ to the set $\{0, 1, 2, \dots, |E(G)|\}$ such that each edge xy is assigned the label $|f(x) - f(y)|$, then the resulting edge labels are distinct, and there exists an integer k such that, $min\{f(x), f(y)\} \leq k < max\{f(x), f(y)\}$ for each edge xy of G. In this paper, we derive α-graceful labeling for the m-shadow graph of various standard graphs.*

Keywords *- Graceful labeling, **α**-graceful labeling, m-shadow graph.*

1. Introduction

Considered a simple, finite, connected, and undirected graph $G = (V, E)$, where V is the set of vertices, and E is the set of edges of the graph G. For all the terminology and notations related to graph theory, we follow Harary [7].

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. For various results and references related to graph labeling we follow a dynamic survey of graph labeling by Gallian [5].

In 1967, Rosa [9] introduced β-valuation and α-valuation of graph labeling. The β-valuation is later known as graceful labeling due to Golomb [6]. Also, α-valuation is a special type of graceful labeling, so it is known as α-graceful labeling or α- labeling. α-graceful labeling of graphs is useful in graph decomposition [4, 9]. For some existing results about α-labeling, refer [2, 3, 8, 10]. Vaidya and Shah [11] prove that the splitting graph and the shadow graph of a bistar are odd-graceful. Zeen El Deen and Elmahdy [12] showed δ graceful labeling for the shadow graphs of the paths, cycles, and fans.

Abdel-Aal [1] introduced the extension of the shadow graph, as m-shadow graph, and gave the notation for the m-shadow graph. In this paper, α-graceful labeling of m-shadow graph of path, complete bipartite graph, bistar graph, and cycle graph has been derived. Here are some definitions that are useful for this paper.

Definition 1.1 [5]**.** The graph obtained by joining the center vertices of two star graphs $K_{1,m}$ and $K_{1,n}$ with an edge is called *bistar* $B_{m,n}$.

Definition 1.2 [5]**.** The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies G_1 and G_2 of G and joining each vertex u of G_1 to the neighbors of the corresponding vertex v in G_2.

Definition 1.3 [1]**.** The *m-shadow graph* $D_m(G)$ of a connected graph G is constructed by taking m copies $G_1, G_2, \dots, G_m$ of G, and joining each vertex u in G_i to the neighbors of the corresponding vertex v in G_j for $1 \leq i, j \leq m$.

Definition 1.4 [9]**.** A function f is called a *graceful labeling* of a graph $G = (V, E)$ if $f: V \to \{0,1, \dots, q\}$ is injective and the induce function $f^*: E \to \{1, \dots, q\}$ defined as $f^*(uv) = |f(u) - f(v)|$ is bijective, $\forall\, uv \in E$. A graph G, which admits graceful labeling, is called a graceful graph.

Definition 1.5 [9]**.** The *α-graceful labeling* of a graph is a graceful labeling with the additional property that there exists an integer k such that for each edge uv of the graph, either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$. A graph which has α-graceful labeling is called α-graceful graph.

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Rosa [9] observed that an α-graceful graph G is necessarily bipartite. Also, from the definition, we can say that if f is a graceful labeling for the graph G and G is a bipartite graph with partitions A and B of $V(G)$ such that, for every edge uv of G, if $f(u) \leq k < f(v)$ for $u \in A$ and $v \in B$ for some positive integer k, then f is α-graceful labeling for the graph G.

2. Main Results

Theorem 2.1. The m-shadow graph of the path, $D_m(P_n)$ is α-graceful, for all $m, n \geq 2$.

Proof. Consider m-copies of the graph P_n. Let $v_1^j, v_2^j, \dots, v_n^j$ be the vertices of j^{th} copy of P_n for $j = 1, 2, \dots, m$. Let G be a graph $D_m(P_n)$. Then, $|V(G)| = mn$ and $|E(G)| = m^2(n-1)$.
Now, consider the labeling function $f: V(G) \to \{0,1,2, \dots, m^2(n-1)\}$ defined as,

$$f(v_i^j) = \begin{cases} m(j-1)(n-1) + \dfrac{i-2}{2} & ;\text{if} \quad i - even \\ (m^2+1-j)(n-1) - \dfrac{i-1}{2} & ;\text{if} \quad i - odd \end{cases}$$

$\forall\, i = 1,2, \dots, n.\ \forall\, j = 1,2, \dots, m.$

Then, clearly, the function f defined here is injective and induces edge labeling $f^*: E(G) \to \{1,2, \dots, m^2(n-1)\}$ defined as $f^*(xy) = |f(x) - f(y)|$ for each edge $xy \in E(G)$ is bijective. So, given f is a graceful labeling for the graph G.
Now, note that the graph $G = D_m(P_n)$ is a bipartite graph with a partition $V_1 = \{v_i, v_i^j : i\ is\ even\}$ and $V_2 = \{v_i, v_i^j : i\ is\ odd\}$ of $V(G)$. Moreover, for

$$k = \begin{cases} m(m-1)(n-1) + \dfrac{n-2}{2} & ;\text{if} \quad n - even \\ m(m-1)(n-1) - \dfrac{n-3}{2} & ;\text{if} \quad n - odd \end{cases}$$

we have, $f(x) \leq k < f(y)$ for all edges $xy \in E(G)$, with $x \in V_1$ and $y \in V_2$. So, defined labeling f is α-graceful labeling for the graph $D_m(P_n)$, $\forall\, m, n \geq 2$.

□

Illustration 2.1. α-graceful labeling of a graph $D_4(P_5)$ is shown in Figure 1.

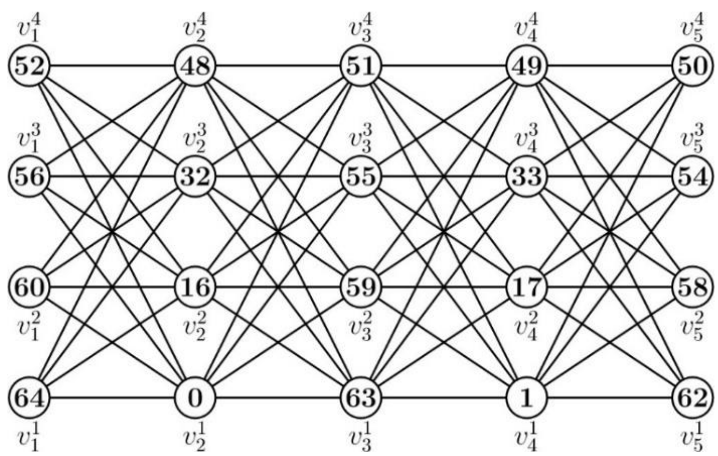

Fig. 1 The graph $D_4(P_5)$ with α-graceful labeling

Theorem 2.2. The m-shadow graph of a complete bipartite graph, $D_m(K_{r,t})$ is α-graceful, for all $m \geq 2$ and $r, t \geq 1$.

Proof. Consider m-copies of the complete bipartite graph $K_{r,t}$. Let $v_1^j, v_2^j, \dots, v_r^j, u_1^j, u_2^j, \dots, u_t^j$ be the vertices of j^{th} copy of $K_{r,t}$. Let G be the graph $D_m(K_{r,t})$. Then, $|V(G)| = m(r+t)$ and $|E(G)| = m^2rt$.
Now, define the labeling function $f: V(G) \to \{0,1,2, \dots, m^2rt\}$ as,

$$f(v_i^j) = (i-1)mt + (j-1)mrt \quad ;\text{for} \quad i = 1,2,\dots,r.$$
$$f(u_i^j) = m^2rt - (i-1) - (j-1)t \quad ;\text{for} \quad i = 1,2,\dots,t.$$

$\forall j = 1,2,\dots,m.$

Then, clearly, the function f defined here is injective and induces edge labeling $f^*: E(G) \to \{1,2,\dots,(m^2rt)\}$ defined as $f^*(xy) = |f(x) - f(y)|$ for all edge $xy \in E(G)$ is bijective. So, given f is a graceful labeling for graph G.

Now, note that the graph $G = D_m(K_{r,t})$ is a bipartite graph with a partition $V_1 = \{v_i, v_i^j : i = 1,2,\dots,r. \,\&\, j = 1,2,\dots,m.\}$ and $V_2 = \{u_i, u_i^j : i = 1,2,\dots,t. \,\&\, j = 1,2,\dots,m.\}$ of $V(G)$. Moreover, for $k = m^2rt - mt$, we have $f(x) \le k < f(y)$ for all edges $xy \in E(G)$ with $x \in V_1,\ y \in V_2$. So, defined labeling f is α-graceful labeling for the graph $D_m(K_{r,t})$, $\forall\, m \ge 2$ and $\forall\, r,t \ge 1$.

□

Illustration 2.2. α-graceful labeling of a graph $D_3(K_{2,3})$ is shown in Figure 2.

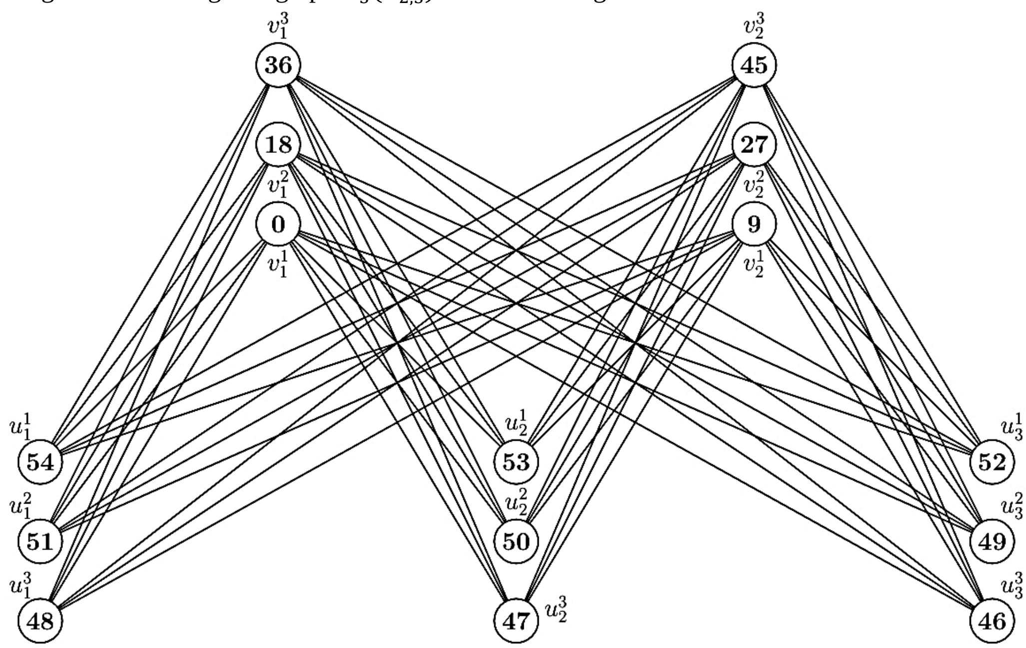

Fig. 2 The graph $D_3(K_{2,3})$ with α-graceful labeling

Theorem 2.3. The m-shadow graph of the bistar graph, $D_m(B_{r,t})$ is α-graceful, for all $m, r, t \ge 2$.

Proof. Consider the bistar graph $B_{r,t}$ with vertex set $\{u_0, u_i, v_0, v_j : 1 \le i \le r, 1 \le j \le t\}$ and edge set $\{u_0u_i, u_0v_0, v_0v_j : 1 \le i \le r, 1 \le j \le t\}$. Now consider m-copies of the graph $B_{r,t}$. Let $u_0^j, u_1^j, u_2^j, \dots, u_r^j,\ v_0^j, v_1^j, v_2^j, \dots, v_t^j$ be the vertices of j^{th} copy of $B_{r,t}$ for $j = 1, 2, \dots, m$. Let G be a graph $D_m(B_{r,t})$. Then, $|V(G)| = m(r+t+2)$ and $|E(G)| = m^2(r+t+1)$.

Now, define the labeling function $f: V(G) \to \{0, 1, 2, \dots, m^2(r+t+1)\}$ as,

$$f(u_0^j) = m(j-1)(r+t+1) \quad ;\text{for} \quad j = 1,2,\dots,m.$$
$$f(u_i^j) = (m^2 - j + 1)(r+t+1) - i + 1 \quad ;\text{for} \quad i = 1,2,\dots,r. \quad j = 1,2,\dots,m.$$
$$f(v_0^j) = (m^2 - j + 1)(r+t+1) - r \quad ;\text{for} \quad j = 1,2,\dots,m.$$
$$f(v_i^j) = m(j-1)(r+t+1) + i \quad ;\text{for} \quad i = 1,2,\dots,t. \quad j = 1,2,\dots,m.$$

Then, clearly, the function f defined here is injective and induces edge labeling $f^*: E(G) \to \{1,2,\dots,m^2(r+t+1)\}$ defined as $f^*(xy) = |f(x) - f(y)|$ for all edge $xy \in E(G)$ is bijective. So, given f is a graceful labeling for graph G.

Now, note that the graph $G = D_m(B_{r,t})$ is a bipartite graph with a partition $V_1 = \{u_0^j,\ v_i^j : 1 \le i \le n,\ 1 \le j \le m.\}$ and $V_2 = \{v_0^j, u_i^j : 1 \le i \le r, 1 \le j \le t\}$ of $V(G)$. Moreover, for $k = (m^2 - m)(r+t+1) + t$, we have $f(x) \le k < f(y)$ for all edges $xy \in E(G)$ with $x \in V_1,\ y \in V_2$. So, defined labeling f is α-graceful labeling for the graph $D_m(B_{r,t})$, $\forall\, m, r, t \ge 2$.

□

Illustration 2.3. α-graceful labeling of a graph $D_3(B_{4,3})$ is shown in Figure 3.

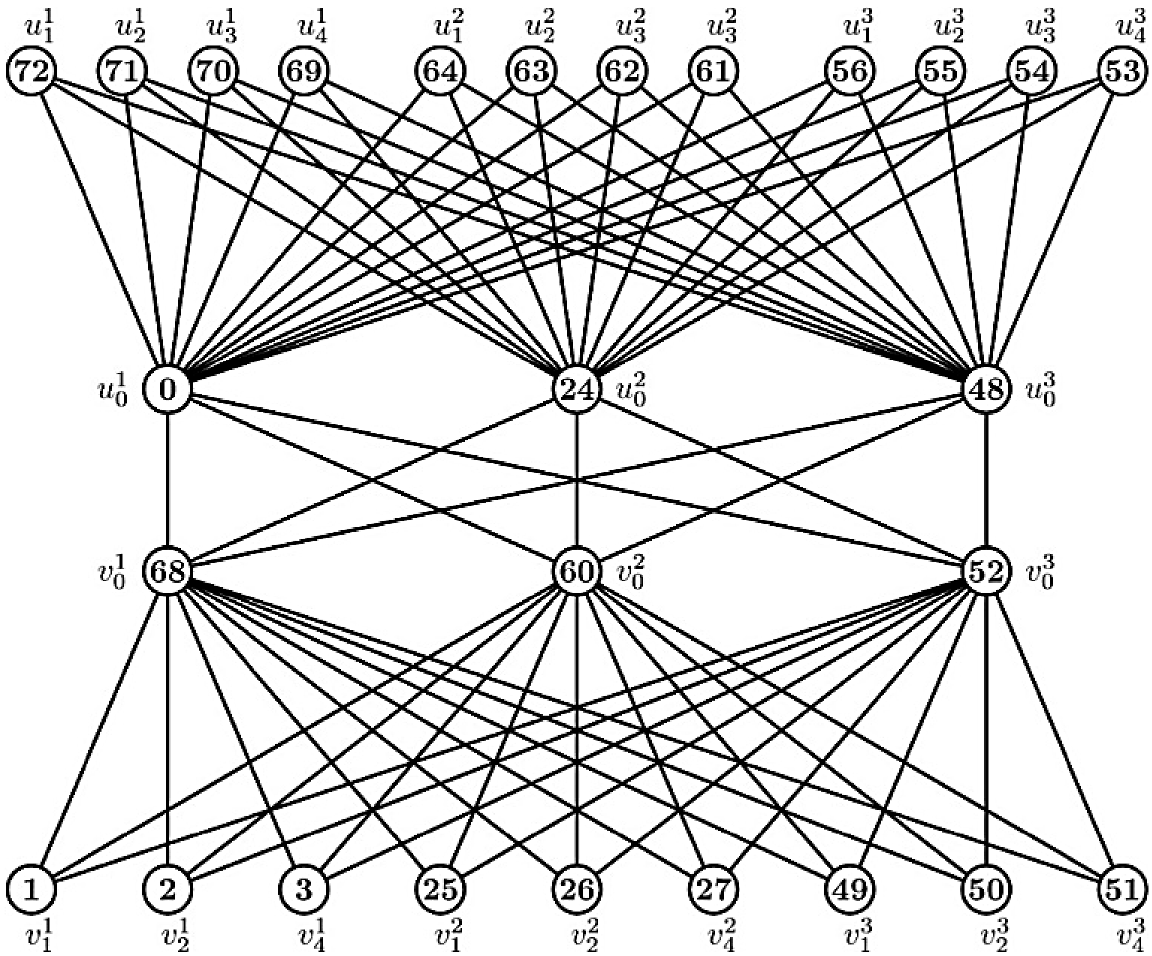

Fig. 3 The graph $D_3(B_{4,3})$ with α-graceful labeling

Theorem 2.4. The m-shadow graph of the cycle, $D_m(C_n)$ is α-graceful, for all $m \geq 2$ and $n \geq 4$ with $n \equiv 0 (\text{mod } 4)$.

Proof. Consider m-copies of a cycle C_n with $n \equiv 0(mod\ 4)$. Let $v_1^j, v_2^j, \dots, v_n^j$ be the vertices of j^{th} copy of C_n for $j = 1, 2, \dots, m$. Let G be a graph $D_m(C_n)$. Then, $|V(G)| = mn$ and $|E(G)| = m^2n$.

Now, consider the labeling function $f: V(G) \to \{0,1,2, \dots, m^2n\}$ defined as,

$$f(v_i^j) = \begin{cases} \dfrac{i-2}{2} + (j-1)mn & ;\text{if} \quad i - even, \quad i \leq \dfrac{n}{2} \\ \dfrac{i}{2} + (j-1)mn & ;\text{if} \quad i - even, \quad i > \dfrac{n}{2} \\ m^2n - \dfrac{i-1}{2} - (j-1)n & ;\text{if} \qquad i - odd \end{cases}$$

$\forall\ i = 1,2, \dots, n.\ \ \forall\ j = 1,2, \dots, m.$

Then, clearly, the function f defined here is injective and induces edge labeling $f^*: E(G) \to \{1,2, \dots, m^2n\}$ defined as $f^*(xy) = |f(x) - f(y)|$ for each edge $xy \in E(G)$ is bijective. So, given f is a graceful labeling for graph G.

Now, note that here $n \equiv 0(mod\ 4)$, so the graph $G = D_m(C_n)$ is a bipartite graph with a partition $V_1 = \{v_i, v_i^j : i\ is\ even\}$ and $V_2 = \{v_i, v_i^j : i\ is\ odd\}$ of $V(G)$. Moreover, for $k = \frac{n}{2} + mn(m-1)$, we have, $f(x) \leq k < f(y)$ for all edges $xy \in E(G)$, with $x \in V_1$ and $y \in V_2$. So, defined labeling f is α-graceful labeling for the graph $D_m(C_n)$, $\forall\ m \geq 2$ and $n \geq 4$ with $n \equiv 0(mod\ 4)$. □

Illustration 2.4. α-graceful labeling of a graph $D_3(C_8)$ is shown in Figure 4.

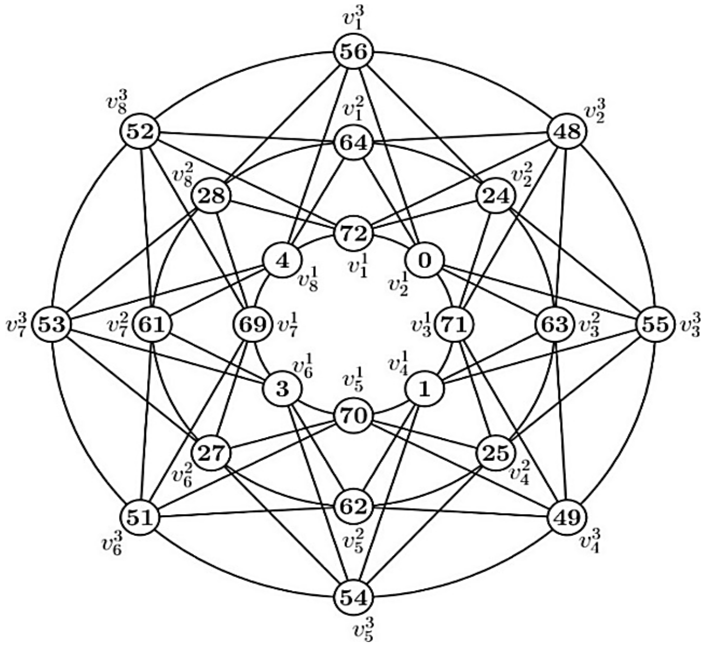

Fig. 4 The graph $D_3(C_8)$ with α-graceful labeling

3. Conclusion

We discussed here the alpha graceful labeling of various m-shadow graphs. Illustrations improve the understanding of the labeling pattern obtained in the results. To obtain similar results for different graph families using various graph operations is an open area of research.

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