International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V72I2P102
Volume 72 Issue 2, 6-18, February 2026

Original Article

On Radio Number of Caterpillars

Alamgir Rahaman Basunia

Department of Applied Science, Haldia Institute of Technology, Haldia, India.

Corresponding Author : alam.basunia@gmail.com

Received: 07 December 2025 Revised: 15 January 2026 Accepted: 06 February 2026 Published: 21 February 2026

Abstract - *A radio labeling of a graph G is a function f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(u)-f(v)| \geq \operatorname{diam}(G)+1-d_G(u,v)$, where $\operatorname{diam}(G)$ and $d_G(u,v)$ are the diameter of G and the distance between u and v in G, respectively. The radio number $\operatorname{rn}(G)$ of G is the smallest number k such that G has radio labeling with $\max\{f(v): v \in V(G)\} = k$. A tree T is called a caterpillar if it has a path P of maximum length such that all the vertices other than the path P are at most distance 1 from the path P. In this paper, we determine the radio number of some special types of caterpillars.*

Keywords - *Caterpillar graph, Radio number, Span of a function.*

1. Introduction

By a graph $G=(V,E)$ we mean a finite, connected undirected graph with neither loops nor multiple edges. For graph theoretic terminology, we refer to the book [7]. Motivated by the Chhanel assignment problem given in Hale [15], Chartrand et al. [8] introduced the concept of radio k - labeling. For a positive integer k with $1 \leq k \leq diam(G)$, a radio k - labeling of G is an assignment f of non-negative integers to the vertices of G such that $|f(u)-f(v)| \geq k+1-d(u,v)$. If $k = diam(G)$, then radio k - labelling is simply called radio labeling. In other words, a radio labeling of a graph G is a function $f: V(G) \to \{0,1,2,\ldots\}$ such that $|\ f(u)-f(v)\ | \geq diam(G)+1-d(u,v)$. The span of f is defined as $span(f) = max\{|f(u)-f(v)|: u,v \in V(G)\}$. The minimum of spans of all possible radio labelings of G is called the radio number of G, denoted by $rn(G)$. The exact value of the radio number of several families of graphs, such as paths, square of cycles, and generalised prism, is given in [2,5,10]. Further results on radio number can be seen in [6, 9, 11, 16].

A tree T is called a caterpillar if it has a path $P_m = (v_0, v_1, \ldots, v_{m-1})$ of maximum length such that all the vertices other than the path P_m is atmost distance 1 from the path P_m. The path P_m is called the central path of T, and the set of caterpillars with the central path P_m is denoted by $\mathcal{C}(m)$. If further $deg(v_i) = 2+t$ for all v_i with $1 \leq i \leq m-2$ and for a fixed positive integer t, then the set of all such caterpillars is denoted by $\mathcal{C}(m;t)$.

The radio number of $\mathcal{C}(m;t)$ for $t=1$ is given in [12]. For $m=2p$, the radio number of $\mathcal{C}(m;t)$ has been given by Kola and Panigrahi [14]. They proved that $rn(\mathcal{C}(2p;t)) = 2[t(p-1)^2 + p(p-1)]+3$. In Theorem 2.5 of [14], they also proved that if $\mathcal{C}$ is a caterpillar with central path $P: v_0, v_1, \ldots, v_{2p-1}$ and $deg(v_i) = deg(v_{p+i-1}) = t_i + 2, t_i \geq 0, i = 1,2,3,\ldots,p-1$, then $rn(\mathcal{C}) = 2\left[(p-1)\sum_{i=1}^{p-1} t_i + p(p-1)\right]+3$. Later, Bantva et al.[1] found the radio number of $\mathcal{C}(m;t)$ for both even and odd values of m. Also, they observed that the radio number of $C(m;t)$ for even m, described in Theorem 2.3 of [14], is bigger than the actual $rn\mathcal{C}(m;t)$ by 1.

Let T be a caterpillar with a spline $P_m = (v_0, v_1, v_2, \ldots, v_{m-1})$.

We define a subclass $\mathcal{C}(m; r_1, r_2, \ldots, r_{p-2})$ of $\mathcal{C}(m)$ as following: If $m = 2p+1$, let

$$deg(v_i) = \begin{cases} 1, & i = 0, 2p \\ r_{i-1}+2, & 2 \leq i \leq p-1 \\ 2, & i = 1, p, 2p-1 \\ r_{i-p}+2, & p+1 \leq i \leq 2p-2, \end{cases}$$

If $m = 2p$, *let*

$$deg(v_i) = \begin{cases} 1, & i = 0, 2p-1 \\ r_{i-1}+2, & 2 \leq i \leq p-1 \\ 2, & i = 1, p \\ r_{i-p}+2, & p+1 \leq i \leq 2p-2 \end{cases}$$

That is, by $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$, we mean a caterpillar with a central path $P_m: v_0 v_1 v_2 \dots v_{m-1}$, where r_i vertices are connected to both v_{i+1} and v_{p+i} for each i with $1 \le i \le p-2$. Throughout the article, we take $m \ge 4$, because otherwise $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$ will be a path. Here in Figure 1, $m = 11, r_1 = 4, r_2 = 3$, and $r_3 = 5$. We shall find the radio number of $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$.

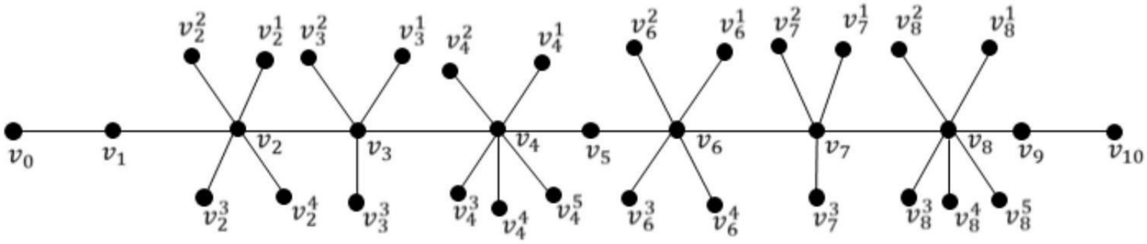

Fig. 1 $\mathcal{C}(\mathbf{11}: \mathbf{4}, \mathbf{3}, \mathbf{5})$

Definition 1.1. Let T be a tree rooted at x, then the weight of T at x is defined by

$$w_T(x) = \sum_{u \in V(T)} d(x, u).$$

The smallest weight among all possible roots of T is called the weight of T, and it is denoted by $w(T)$. That is,

$$w(T) = min\{w_T(x): w \in V(T)\}$$

A vertex x^* of a tree T is called a weight center or centroid of T if $w_T(x^*) = w(T)$.
Observation 1.1. For the Caterpillar $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$ the following are held:
(a) If $m = 2p + 1$, then $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$ has exactly one centroid, namely a_p.
(b) If $m = 2p$, then $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$ has two centroids a_{p-1} and a_p.
(c) $\left|V\left(\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})\right)\right| = m + 2\sum_{i=1}^{p-2} r_i$.
(d) $w\left(\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})\right) = \begin{cases} \frac{1}{4}\{n(m+2) - 2m\}, & \text{if } m = 2p, \\ \frac{1}{4}(n-1)(m+1), & \text{if } m = 2p+1. \end{cases}$

2. Lower bound for the radio number of $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$

This section deals with a lower bound for the radio number of $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$. In the next section, we show that this lower bound is the exact value of the radio number of $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$.

Theorem 2.1. For an n-vertex caterpillar $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$

$$rn\left(\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})\right) \ge \begin{cases} \frac{1}{2}n(m-2) + 1, & \text{if } m = 2p \\ \frac{1}{2}(n-1)(m-1) + 2, & \text{if } m = 2p+1 \end{cases}$$

Proof. Let f be a radio labelling of $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$ and $u_0, u_1, \dots, u_{n-1}$ be an arrangement of vertices such that $f(u_0) = 0 < f(u_1) < f(u_2) < \dots < f(u_{n-1}) = span(f)$ of $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$. Now, from the radio conditions, we have the following.

$$f(u_{i+1}) - f(u_i) \ge m - d(u_i, u_{i+1}), 0 \le i \le n-2. \quad (1)$$

Now let

$$f(u_{i+1}) - f(u_i) = m - d(u_i, u_{i+1}) + J_f(u_i, u_{i+1}) \quad (2)$$

where $J_f(u_i, u_{i+1})$ denotes the jump of f from u_i to u_{i+1}. The addition of these $n-1$ equations yields

$$
\begin{aligned}
f(u_{n-1}) &= \sum_{i=1}^{n-1} [f(u_i) - f(u_{i-1})] + f(u_0) \\
&= \sum_{i=0}^{n-2} \left[m - d(u_i, u_{i+1}) + J_f(u_i, u_{i+1})\right] + f(u_0) \\
&= (n-1)m - 2\sum_{i=0}^{n-1} L_s(u_i) + L_s(u_0) + L_s(u_{n-1}) + \sum_{i=0}^{n-2} \left[J_f(u_i, u_{i+1}) + 2\phi(u_i, u_{i+1})\right] + f(u_0) \\
&= (n-1)m - 2w\left(\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})\right) + f(u_0) + L_s(u_0) + L_s(u_{n-1}) + \sigma(f);
\end{aligned}
$$

where $\sigma(f) = \sum_{i=0}^{n-2} \sigma_f(u_i, u_{i+1})$ and $\sigma_f(u_i, u_{i+1}) = J_f(u_i, u_{i+1}) + 2\phi(u_i, u_{i+1})$. Here total jump $J(f) = \sum_{i=0}^{n-2} J_f(u_i, u_{i+1})$. So the relation between $\sigma(f)$ and $J(f)$ is $\sigma(f) = J(f) + 2\sum_{i=0}^{n-2} \phi(u_i, u_{i+1})$. Under the notation of $\sigma(f)$ and using $f(u_0) = 0$, the Span of f is given by

$$span(f) = (n-1)m - 2w\left(\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})\right) + L_s(u_0) + L_s(u_{n-1}) + \sigma(f) \tag{3}$$

Case 1 : $m = 2p$.

Since $L_s(u_0) + L_s(u_{n-1}) \geq 1$ and $\sigma(f) \geq 0$, using Lemma 1.1(d), equation (3) gives us

$$
\begin{aligned}
span(f) &\geq (n-1)m - 2w\left(\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})\right) + 1 \\
&= (n-1)m - 2 \cdot \frac{1}{4}\{n(m+2) - 2m\} + 1 \\
&= \frac{2nm - 2m - nm - 2n + 2m}{2} + 1 \\
&= \frac{1}{2}n(m-2) + 1
\end{aligned}
$$

As the above is true for every radio labelling of $\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})$, so we obtain the result when $m = 2p$.

Case 2: $m = 2p + 1$.
Using Lemma 1.1(d), equation (3) gives us

$$
\begin{aligned}
span(f) &\geq (n-1)m - 2w\left(\mathcal{C}(m; r_1, r_2, \dots, r_{p-2})\right) + L_s(u_0) + L_s(u_{n-1}) + \sigma(f) \\
&= (n-1)m - 2 \cdot \frac{1}{4}(n-1)(m+1) + L_s(u_0) + L_s(u_{n-1}) + \sigma(f) \\
&= (n-1)\left[m - \frac{m+1}{2}\right] + L_s(u_0) + L_s(u_{n-1}) + \sigma(f) \\
&= \frac{1}{2}(n-1)(m-1) + L_s(u_0) + L_s(u_{n-1}) + \sigma(f)
\end{aligned}
$$

We claim that $L_s(u_0) + L_s(u_{n-1}) + \sigma(f) \geq 2$. If $s \notin \{u_0, u_{n-1}\}$ or $s \in \{u_0, u_{n-1}\}$ with $\{u_0, u_{n-1}\} \setminus \{s\}$ is not adjacent to s, then $L_s(u_0) + L_s(u_{n-1}) \geq 2$ and hence our claim is true. Again recall that $\sigma(f) = \sum_{i=0}^{n-2} \left[J_f(u_i, u_{i+1}) + 2\phi(u_i, u_{i+1})\right]$, and thus if $\phi(u_\ell, u_{\ell+1}) \geq 1$ for some ℓ, then $\sigma(f) \geq 2$ and in this case our claim is true. The only remaining case is the one when both of the following hold

(a) if one of u_0 and u_{n-1} is s and the other is adjacent to s, i.e., $L_s(u_0) + L_s(u_{n-1}) = 1$

(b) $\gamma: u_0, u_1, u_2, \dots, u_{n-1}$ is an alternating sequence.

Without loss of generality, we assume $u_0 = s$ and u_{n-1} is adjacent to s. Then either $u_\ell = a_0$ or $u_\ell = a_{m-1}$ for some ℓ with $2 \leq \ell \leq n-2$. We show that $J_f(u_{\ell-1}, u_\ell) + J_f(u_\ell, u_{\ell+1}) \geq 1$. From (2) with $i = \ell - 1, \ell$; and using $d(u_{\ell-1}, u_\ell) = L_s(u_{\ell-1}) + L_s(u_\ell)$ and $d(u_\ell, u_{\ell+1}) = L_s(u_\ell) + L_s(u_{\ell+1})$ because γ is an alternating sequence, we have

$$f(u_\ell)-f(u_{\ell-1})=m-L_s(u_{\ell-1})-L_s(u_\ell)+J_f(u_{\ell-1},u_\ell)$$
$$f(u_{\ell+1})-f(u_\ell)=m-L_s(u_\ell)-L_s(u_{\ell+1})+J_f(u_\ell,u_{\ell+1}).$$

Adding the above two equations and using $f(u_{\ell+1})-f(u_{\ell-1})\geq m-L_s(u_{\ell-1})-L_s(u_{\ell+1})$, we get

$$J_f(u_{\ell-1},u_\ell)+J_f(u_\ell,u_{\ell+1})\geq 2L_s(u_\ell)+2\phi(u_{\ell-1},u_{\ell+1})-m. \tag{4}$$

Since $L_s(a_0))=p=L_s(a_{m-1})$), it follows that $L_s(u_\ell)=p$. Also $\phi(u_{\ell-1},u_{\ell+1})\geq 1$ because $s=u_0$ and hence $s\notin\{u_{\ell-1},u_{\ell+1}\}$. Therefore, (4) with $m=2p+1$ gives us $J_f(u_{\ell-1},u_\ell)+J_f(u_\ell,u_{\ell+1})\geq 1$. Thus $L_s(u_0)+L_s(u_{n-1})+\sigma(f)\geq 2$ is hold. On account of all of the above facts, we have $rn\left(\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})\right)\geq\frac{1}{2}(n-1)(m-1)+2$, when $m=2p+1$.

3. An ordering of the vertices of $\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})$.

We define an ordering of the vertices of $\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})$, which will be used in the proof of the main theorem. Let us rename all the vertices of $\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})$. The diameter of $\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})$ is $m-1$. Let $N_1(v_i)$ denote the set of all pendant vertices adjacent to v_i and $N[v]=N(v)\cup\{v\}$. Thus

$$N_1(v_i)=\left\{v_i^t: 1\leq t\leq r_i, 1\leq t\leq r_j\right\}$$

Case-1: Let $m=2p+1$.
The following steps give an ordering $u_0,u_1,\ldots u_{n-1}$ of vertices for $\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})$.
Step-1: Partition the vertex set $V\left(\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})\right)$ into $(p-1)$ sets $S_0,S_1,S_2,\ldots S_{p-2}$; where,

$$S_0=\{v_p,v_{2p},v_0,v_{p+1},v_1\}$$
$$S_j=N_1[v_{j+1}]\cup N_1(v_{p+j})\cup\{v_{p+j+1}\}, 1\leq j\leq p-2.$$

Clearly, $|S_j|=2r_j+2, 1\leq j\leq p-2$. For $1\leq j\leq p-2$, let us define $n_j=\sum_{t=2}^{j}|S_{t-1}|$ and $n_1=0$.
Step-2: The ordering of the vertices of S_0 is given below
$u_0=v_p, u_1=v_{2p}, u_2=v_0, u_3=v_{p+1}$ and $u_4=v_1$.

The remaining vertices are ordered by using the following rule.
For $1\leq j\leq p-2, 1\leq i\leq r_j+2$.

$$u_{4+n_j+i}=\begin{cases}\dfrac{i+1}{2}, & i \text{ is odd and } 1\leq i\leq r_j\\ v_{p+j}^2, & i \text{ is even and } 1\leq i\leq r_j\\ v_{j+1}^{\frac{i}{2}}, & i=2r_j+1\\ v_{p+j+1}, & i=2r_j+2\end{cases}$$

Thus, we get $\gamma=u_0,u_1,u_2,\ldots,u_{n-1}$ as an ordering of the vertices of $\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})$.
Case-2: Let $m=2p$.
The following steps give an ordering $u_0,u_1,\ldots u_{n-1}$ of vertices for $\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})$.
Step-1: Partitioned the vertex set $V\left(\mathcal{C}(m;r_1,r_2,\ldots,r_{p-2})\right)$ into $(p-1)$ sets $S_0,S_1,S_2,\ldots,S_{p-2}$, where

$$S_0=\{v_p,v_0,v_{p+1},v_1\}$$
$$S_j=N_1[v_{j+1}]\cup N_1(v_{p+j})\cup\{v_{p+j+1}\}, 1\leq j\leq p-2.$$

Recall, $n_j=\sum_{t=2}^{j}|S_{t-1}|$ and $n_1=0$.
Step-2: The ordering of the vertices of S_0 is given below
$u_0=v_p, u_1=v_0, u_2=v_{p+1}$, and $u_3=v_1$.

The remaining vertices are ordered by using the following rule.
For $1 \leq j \leq p-2, 1 \leq i \leq r_j + 2$.

$$u_{3+n_j+i} = \begin{cases} \dfrac{i+1}{2}, & i \text{ is odd and } 1 \leq i \leq r_j \\ v_{p+j}^{2}, & i \text{ is even and } 1 \leq i \leq r_j \\ v_{j+1}^{\frac{i}{2}}, & i = 2r_j + 1 \\ v_{p+j+1}, & i = 2r_j + 2 \end{cases}$$

Thus, we obtain $\gamma = u_0, u_1, u_2, \ldots, u_{n-1}$ as an ordering of the vertices of $\mathcal{C}(m; r_1, r_2, \ldots, r_{p-2})$.
The ordering of the vertices of $\mathfrak{C}(11; 4,3,5)$ is given in Figure 2.
Let us define k as

$$k = \begin{cases} 4 & \text{if } m = 2p+1 \\ 3 & \text{if } m = 2p \end{cases}$$

Lemma 3.1. For any three consecutive vertices u_t, u_{t+1} and u_{t+2} in γ,
(a) $L(u_t) + L(u_{t+1}) \leq \left\lfloor \frac{m}{2} \right\rfloor + 1$ and $L(u_{t+1}) + \phi(u_t, u_{t+2}) \leq \left\lfloor \frac{m}{2} \right\rfloor$, for all t, except $t = 1$.
(b) $L(u_t) + L(u_{t+1}) \leq m - 1$ and $L(u_{t+1}) + \phi(u_t, u_{t+2}) \leq \left\lfloor \frac{m}{2} \right\rfloor + 1$, for $t = 1$.

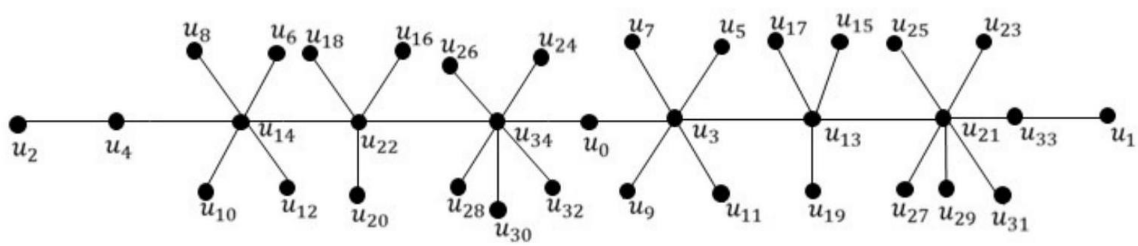

Fig. 2 Ordering of vertices of $\mathfrak{C}(11; 4, 3, 5)$

Proof. (a) We partition $V\left(\mathcal{C}(m; r_1, r_2, \ldots, r_{p-2})\right) \setminus \{u_0, u_1, \ldots, u_k\}$ into ($p-2$) sets given below.

$$S_j = N_1[v_{j+1}] \cup N_1(v_{p+j}) \cup \{v_{p+j+1}\}, 1 \leq j \leq p-2.$$

For $1 \leq j \leq p-2$, let us denote $n_j = \sum_{t=2}^{j} |S_{t-1}|$. For $1 \leq j \leq p-1$, we define

$$\begin{aligned} S_j' &= \left\{u_{k+n_j+i} : 1 \leq i \leq |S_j|\right\} \\ &= \left\{u_{k+n_j+i} : 1 \leq i \leq 2r_j + 2\right\} \end{aligned}$$

It is clear that

$$L(v_\ell) = \begin{cases} p - \ell, & \ell \leq p \\ \ell - p, & \ell > p, \end{cases}$$

$$L(v_\ell^t) = \begin{cases} p + 1 - i, & \ell \leq p - 1 \\ \ell + 1 - p, & \ell > p. \end{cases}$$

Case-1: $t \leq k$ and $t \neq 1$. We take the following to subcases according as $m = 2p+1$ or $m = 2p$.
Subcase 1: Let $m = 2p+1$. In this case $k = 4$. For $t = 0, u_t = u_0, u_{t+1} = u_1 = v_{2p}$ and $u_{t+2} = u_2 = v_0$. Thus $L(u_0) + L(u_1) = L(v_p) + L(v_{2p}) = p$ and $L(u_1) + L(u_2) = L(v_{2p}) + L(v_0) = 2p$. Also, for $t = 0$, we have $L(u_{t+1}) + \phi(u_t, u_{t+2}) = L(u_1) + \phi(u_0, u_2) = L(v_{2p}) + \phi(v_p, v_0) = p$. If $u_t = u_2$, then $u_{t+1} = u_3 = v_{p+1}$, and $u_{t+2} = u_4 = v_1$.

$$\begin{aligned} L(u_3) + L(u_4) &= L(v_{p+1}) + L(v_1) \\ &= 1 + (p-1) = p \end{aligned}$$

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L(u_3) + \phi(u_2, u_4) \\ &= L(v_{p+1}) + \phi(v_0, v_1) \\ &= 1 + (p-1) = p \end{aligned}$$

If $u_t = u_3 = v_{p+1}$, then $u_{t+1} = u_4 = v_1$, and $u_{t+2} = u_5 = u_{k+1} = v_{p+1}^1$.

$$\begin{aligned} L(u_4) + L(u_5) &= L(v_1) + L(v_{p+1}^1) \\ &= p - 1 + 2 = p + 1 \end{aligned}$$

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L(u_4) + \phi(u_3, u_5) \\ &= L(v_1) + \phi(v_{p+1}, v_{p+1}^1) \\ &= (p-1) + 1 = p \end{aligned}$$

If $u_t = u_4 = v_{p+1}$, then $u_{t+1} = u_5 = u_{k+1} = v_{p+1}^1$, and $u_{t+2} = u_{k+2} = v_2^1$.

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L(u_{k+1}) + \phi(u_k, u_{k+2}) \\ &= L(v_{p+1}^1) + \phi(v_1, v_2^1) \\ &= 2 + (p-2) = p \end{aligned}$$

Subcase 2: Let $m = 2p$. Then $k = 3$.
If $u_t = u_0$, then $u_{t+1} = u_1 = v_0$, and $u_{t+2} = u_2 = v_{p+1}$.

$$\begin{aligned} L(u_0) + L(u_1) &= L(v_p) + L(v_0) \\ &= p. \end{aligned}$$

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L(u_1) + \phi(u_0, u_2) \\ &= L(v_0) + \phi(v_p, v_{p+1}) \\ &= p \end{aligned}$$

If $u_t = u_2 = v_{p+1}$, then $u_{t+1} = u_3 = v_1$, and $u_{t+2} = u_4 = u_{k+1} = v_{p+1}^1$.

$$\begin{aligned} L(u_3) + L(u_4) &= L(v_1) + L(v_{p+1}^1) \\ &= p - 1 + 2 = p + 1 \end{aligned}$$

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L(u_3) + \phi(u_2, u_4) \\ &= L(v_1) + \phi(v_{p+1}, v_{p+1}^1) \\ &= (p-1) + 1 = p \end{aligned}$$

If $u_t = u_3 = v_1$, then $u_{t+1} = u_4 = u_{k+1} = v_2^1$, and $u_{t+2} = u_5 = u_{k+2} = v_2^1$.

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L(u_4) + \phi(u_3, u_5) \\ &= L(v_{p+1}^1) + \phi(v_1, v_2^1) \\ &= 2 + p - 2 = p \end{aligned}$$

We see that in both the sub-cases,

$$\begin{aligned} L(u_t) + L(u_{t+1}) &\le p + 1 \\ L(u_{t+1}) + \phi(u_t, u_{t+2}) &\le p \end{aligned}$$

hold for all $t \le k$, except when m is odd and $t = 1$.

When m is odd and $t = 1$,

$$L(u_t) + L(u_{t+1}) = m - 1$$
$$L(u_{t+1}) + \phi(u_t, u_{t+2}) = \frac{m+1}{2} = \left\lfloor \frac{m}{2} \right\rfloor + 1$$

Case-2: Let $t > k$.

Subcase 1: All u_t, u_{t+1} and u_{t+2} are in the same S_j' for some j with $1 \le j \le p-2$. Then $u_t = u_{k+n_j+i}$ for some i with $1 \le i \le 2r_j$. Note that t and i are simultaneously odd or even because n_j is even. First, we assume $1 \le i \le 2r_j - 2$. Then

$$\begin{aligned} u_t &= u_{4+n_j+i} \\ &= \begin{cases} v_{p+j}^{\frac{i+1}{2}}, & \text{if } t \text{ is odd} \\ v_{j+1}^{\frac{i}{2}}, & \text{if } t \text{ is even ;} \end{cases} \\ u_{t+1} &= u_{4+n_j+(i+1)} \\ &= \begin{cases} v_{j+1}^{\frac{i+1}{2}}, & \text{if } t \text{ is odd} \\ v_{j+1}^{\frac{i+1}{2}}, & \text{if } t \text{ is even ;} \end{cases} \\ u_{t+2} &= u_{4+n_j+(i+2)} \\ &= \begin{cases} v_{p+j}^{\frac{i+3}{2}}, & \text{if } t \text{ is odd ;} \\ v_{p+j}^{\frac{i+3}{2}}, & \text{if } t \text{ is even.} \end{cases} \end{aligned}$$

Therefore, we get the following when t is odd.

$$\begin{aligned} L(u_t) + L(u_{t+1}) &= L\left(v_{p+j}^{\frac{i+1}{2}}\right) + L\left(v_{j+1}^{\frac{i+1}{2}}\right) \\ &= j + 1 + p - (j+1) + 1 \\ &= p + 1, \end{aligned}$$

and

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L\left(v_{j+1}^{\frac{i+1}{2}}\right) + \phi\left(v_{p+j}^{\frac{i+1}{2}}, v_{p+j}^{\frac{i+3}{2}}\right) \\ &= p - (j+1) + 1 + j = p. \end{aligned}$$

Also, for even integer t, we have

$$\begin{aligned} L(u_t) + L(u_{t+1}) &= L\left(v_{j+1}^{\frac{i}{2}}\right) + L\left(v_{p+j}^{\frac{i+2}{2}}\right) \\ &= p - j + j + 1 \\ &= p + 1, \end{aligned}$$

and

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L\left(v_{p+j}^{\frac{i+2}{2}}\right) + \phi\left(v_{j+1}^{\frac{i}{2}}, v_{j+1}^{\frac{i+2}{2}}\right) \\ &= (j+1) + p - (j+1) = p. \end{aligned}$$

Now we take $i = 2r_j - 1$. Then we have

$$\begin{aligned} u_t &= u_{4+n_j+2r_j-1} = v_{p+j}^{r_j} \\ u_{t+1} &= u_{4+n_j+2r_j} = v_{j+1}^{r_j} \\ u_{t+2} &= u_{4+n_j+2r_j+1} = v_{p+j+1} \end{aligned}$$

and consequently

$$\begin{aligned} L(u_t) + L(u_{t+1}) &= L\left(v_{p+j}^{r_j}\right) + L\left(v_{j+1}^{r_j}\right) \\ &= j + 1 + p - (j+1) + 1 \\ &= p + 1, \\ L(u_{t+1}) + L(u_{t+2}) &= L\left(v_{j+1}^{r_j}\right) + L\left(v_{p+j+1}\right) \\ &= p - j + j + 1 \\ &= p + 1, \\ L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L\left(v_{j+1}^{r_j}\right) + \phi\left(v_{p+j}^{r_j}, v_{p+j+1}\right. \\ &= p - (j+1) + 1 + j = p. \end{aligned}$$

Again for $i = 2r_j$,

$$\begin{aligned} u_t &= u_{4+n_j+2r_j} = v_{j+1}^{r_j} \\ u_{t+1} &= u_{4+n_j+2r_j+1} = v_{p+j+1} \\ u_{t+2} &= u_{4+n_j+2r_j+2} = v_{j+1} \end{aligned}$$

and then

$$\begin{aligned} L(u_t) + L(u_{t+1}) &= L\left(v_{j+1}^{r_j}\right) + L\left(v_{p+j+1}\right) \\ &= p - j + j + 1 \\ &= p + 1 \end{aligned}$$

Also,

$$\begin{aligned} L(u_{t+1}) + L(u_{t+2}) &= L\left(v_{p+j+1}\right) + L\left(v_{j+1}\right) \\ &= j + 1 + p - (j+1) \\ &= p \end{aligned}$$

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L\left(v_{p+j+1}\right) + \phi\left(v_{j+1}^{r_j}, v_{j+1}\right) \\ &= (j+1) + p - (j+1) = p \end{aligned}$$

Subcase 2: Let u_t, u_{t+1}, u_{t+2} are in different S_j' for some j with $1 \le j \le p-2$.
This is possible only when $u_t \in S_j'$ and $u_{t+2} \in S_{j+1}'$ with $1 \le j \le p-3$.
Now, if $u_{t+1} \in S_j'$, then

$$\begin{aligned} u_t &= u_{4+n_j+2r_j+1} = v_{p+j+1}, \\ u_{t+1} &= u_{4+n_j+2r_j+2} = v_{j+1}, \\ u_{t+2} &= u_{4+n_{j+1}+1} = v_{p+j+1}^{1}, \end{aligned}$$

and hence,

$$\begin{aligned} L(u_{t+1}) + L(u_{t+2}) &= L(v_{j+1}) + L(v_{p+j+1}^1) \\ &= p - (j+1) + j + 1 + 1 \\ &= p + 1, \end{aligned}$$

$$\begin{aligned} L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L(v_{j+1}) + \phi(v_{p+j+1}, v_{p+j+1}^1) \\ &= p - (j+1) + j + 1 \\ &= p \end{aligned}$$

If $u_{t+1} \in S'_{j+1}$, then

$$\begin{aligned} u_t &= u_{4+n_j+2r_j+2} = v_{j+1}, \\ u_{t+1} &= u_{4+n_{j+1}+1} = v_{p+j+1}^1, \\ u_{t+2} &= u_{4+n_{j+1}+2} = v_{j+2}^1, \end{aligned}$$

and

$$\begin{aligned} L(u_{t+1}) + L(u_{t+2}) &= L(v_{p+j+1}^1) + L(v_{j+2}^1) \\ &= (j+1) + 1 + p - (j+2) + 1 \\ &= p + 1 \\ L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L(v_{p+j+1}^1) + \phi(v_{j+1}, v_{j+2}^1) \\ &= (j+1) + 1 + p - (j+2) \\ &= p. \end{aligned}$$

From all the above cases, we conclude that

$$L(u_t) + L(u_{t+1}) \leq (p+1) = \frac{m+1}{2}$$

and

$$L(u_{t+1}) + \phi(u_t, u_{t+2}) \leq p = \frac{m-1}{2}.$$

If $m = 2p$, then $k = 3$. In a similar way, we can show that,

$$L(u_t) + L(u_{t+1}) \leq \frac{m}{2} + 1$$

and

$$L(u_{t+1}) + \phi(u_t, u_{t+2}) \leq \frac{m}{2}.$$

Combining all these, we get,

$$\begin{aligned} L(u_t) + L(u_{t+1}) &\leq \left\lfloor \frac{m}{2} \right\rfloor + 1 \\ L(u_{t+1}) + \phi(u_t, u_{t+2}) &\leq \left\lfloor \frac{m}{2} \right\rfloor \end{aligned}$$

(b) Let us consider $t = 1$.

If $m = 2p + 1$, then $u_t = u_1, u_{t+1} = u_2 = v_0$, and $u_{t+2} = u_3 = v_{p+1}$.

$$\begin{aligned} L(u_1) + L(u_2) &= L(v_{2p}) + L(v_0) \\ &= p + p = 2p. \\ L(u_{t+1}) + \phi(u_t, u_{t+2}) &= L(u_2) + \phi(u_1, u_3) \\ &= L(v_0) + \phi(v_{2p}, v_{p+1}) \\ &= p + 1. \end{aligned}$$

If $m = 2p$, then $u_t = u_1 = v_0, u_{t+1} = u_2 = v_{p+1}$, and $u_{t+2} = u_3 = v_1$.

$$L(u_1) + L(u_2) = L(v_0) + L(v_{p+1}) = p + 1.$$
$$L(u_{t+1}) + \phi(u_t, u_{t+2}) = L(u_2) + \phi(u_1, u_3)$$
$$= L(v_{p+1}) + \phi(v_0, v_1) = p.$$

So,

$$L(u_1) + L(u_2) \leq m - 1,$$
$$L(u_{t+1}) + \phi(u_t, u_{t+2}) \leq \left\lfloor \frac{m}{2} \right\rfloor + 1.$$

Main Results and Discussion

Now we prove our main theorem, which gives the radio number of the caterpillar $\mathcal{C}(m; r_1, r_2, \ldots, r_{p-2})$ with an optimal radio labelling.

Theorem 3.1. $rn\left(\mathcal{C}(m; r_1, r_2, \ldots, r_{p-2})\right) = \begin{cases} \frac{1}{2}n(m-2) + 1, & \text{when } m = 2p \\ \frac{1}{2}(n-1)(m-1) + 2, & \text{when } m = 2p + 1. \end{cases}$

Proof. It is enough to prove that there exists a radio labeling f whose Span is equal to the lower bound given in Theorem 2.1.

Define $f: V\left(\mathcal{C}(m; r_1, r_2, \ldots, r_{p-2})\right) \rightarrow \{0,1,2,3, \ldots\}$ as follows: $f(u_0) = 0$ and $f(u_{i+1}) = f(u_i) + m - L(u_i) - L(u_{i+1}) + \delta_i$, where,

$$\delta_i = \begin{cases} 1, & m \text{ is odd and } i = 2 \\ 0, & \text{otherwise} \end{cases}$$

For any two integers i and j with $0 \leq i, j \leq n - 1$, we prove that $f(u_j) - f(u_i) \geq m - d(u_i, u_j)$. From the definition of f, we have

$$\begin{aligned} f(u_j) - f(u_i) &= \sum_{t=i}^{j-1} \{f(u_{t+1}) - f(u_t)\} \\ &= \sum_{t=i}^{j-1} \{m - L(u_t) - L(u_{t+1}) + \delta_t\} \\ &= (j - i)m - \sum_{t=i}^{j-1} \{L(u_t) + L(u_{t+1})\} + \sum_{t=i}^{j-1} \delta_t \end{aligned}$$

First, let us take $j = i + 1$. Then

$$\begin{aligned} f(u_{i+1}) - f(u_i) &= m - \{L(u_i) + L(u_{i+1})\} + \delta_i \\ &= m - d(u_i, u_{i+1}) + \delta_i \\ &\geq m - d(u_i, u_{i+1}), \text{ as } \delta_i \geq 0 \end{aligned}$$

Now let $j = i + 2$. Then

$$f(u_{i+2})-f(u_i)=2m-\sum_{t=i}^{i+1}\{L(u_t)+L(u_{t+1})\}+\sum_{t=i}^{i+1}\delta_i$$
$$=2m-\{L(u_i)+L(u_{i+2})\}-2L(u_{i+1})+\sum_{t=i}^{i+1}\delta_i$$
$$=2m-\{d(u_i,u_{i+2})+2\phi(u_i,u_{i+2})\}-2L(u_{i+1})+\sum_{t=i}^{i+1}\delta_i$$
$$=m-d(u_i,u_{i+2})+m-2\{L(u_{i+1})+\phi(u_i,u_{i+2})\}+\sum_{t=i}^{i+1}\delta_i$$

If m is odd and $i=1$, then we have

$$\begin{aligned}f(u_3)-f(u_1)&=m-d(u_1,u_3)+m-2\{L(u_2)+\phi(u_1,u_3)\}+\delta_1+\delta_2\\&=m-d(u_1,u_3)+m-2\{L(v_0)+\phi(v_{2p},v_{p+1})\}+1\\&=m-d(u_1,u_3)+m+1-2\left\{\frac{m-1}{2}+1\right\}\\&=m-d(u_1,u_3)+m+1-(m+1)\\&=m-d(u_1,u_3).\end{aligned}$$

Now, for all other value of i and for any m, we have $L(u_{i+1})+\phi(u_i,u_{i+2})\le\left\lfloor\frac{m}{2}\right\rfloor$,

$$f(u_{i+2})-f(u_i)=m-d(u_i,u_{i+2})+m-2\{L(u_{i+1})+\phi(u_i,u_{i+2})\}+\sum_{t=i}^{i+1}\delta_i$$
$$=m-d(u_i,u_{i+2})+m-2\left\lfloor\frac{m}{2}\right\rfloor+\sum_{t=i}^{i+1}\delta_i$$
$$\ge m-d(u_i,u_{i+2}),\text{ as } m-2\left\lfloor\frac{m}{2}\right\rfloor\ge 0,\sum_{t=i}^{i+1}\delta_i\ge 0$$

If $j=i+3$. Then

$$f(u_{i+3})-f(u_i)=3m-\sum_{t=i}^{i+2}\{L(u_t)+L(u_{t+1})\}+\sum_{t=i}^{i+2}\delta_i$$
$$=3m-\{L(u_i)+L(u_{i+3})\}-2\{L(u_{i+1})+L(u_{i+2})\}+\sum_{t=i}^{i+2}\delta_i$$
$$=m-d(u_i,u_{i+3})+2m-2\left\{L(u_{i+1})+L(u_{i+2})+\sum_{t=i}^{i+1}\delta_i\right.$$
$$\ge m-d(u_i,u_{i+3})+2m-4\left\lfloor\frac{m}{2}\right\rfloor+\sum_{t=i}^{i+1}\delta_i$$
$$=m-d(u_i,u_{i+3})+2\left(m-2\left\lfloor\frac{m}{2}\right\rfloor\right)+\sum_{t=i}^{i+1}\delta_i$$
$$\ge m-d(u_i,u_{i+3}),\text{ as } m-2\left\lfloor\frac{m}{2}\right\rfloor\ge 0$$

Now let $j \geq i + 4$. Then two cases may arise. Either $i > 1$ or $i = 0,1$.
If $i > 1$, then
In either case $m \geq 4$, so

$$f(u_j) - f(u_i) \geq m - d(u_i, u_j).$$

If $i = 0$, or 1, then

$$\begin{aligned} f(u_j) - f(u_i) &= (j-i)m - \sum_{t=i}^{j-1} \{L(u_t) + L(u_{t+1})\} + \sum_{t=i}^{j-1} \delta_t \\ &\geq (j-i)m - \{L(u_1) + L(u_2)\} - (j-i-1)\left\{\left\lfloor \frac{m}{2} \right\rfloor + 1\right\} + \sum_{t=i}^{j-1} \delta_t \\ &\geq m - (m-1) + (j-i-1)\left\{m - \left\lfloor \frac{m}{2} \right\rfloor - 1\right\}, \text{ as } \sum_{t=i}^{j-1} \delta_t \geq 0 \\ &\geq 1 + 3\left\{m - \left\lfloor \frac{m}{2} \right\rfloor - 1\right\}, \text{ as } j - i \geq 4 \\ &= \begin{cases} \frac{3m}{2} - \frac{1}{2}, & \text{if } m \text{ is odd;} \\ \frac{3m}{2} - 2, & \text{if } m \text{ is even;} \end{cases} \\ &\geq m - d(u_i, u_j). \end{aligned}$$

The radio labeling of $C(11{:}\,4,3,5)$ and $C(12;4,3,4,4)$, giving the radio number, is shown in Figures 3 and 4, respectively.

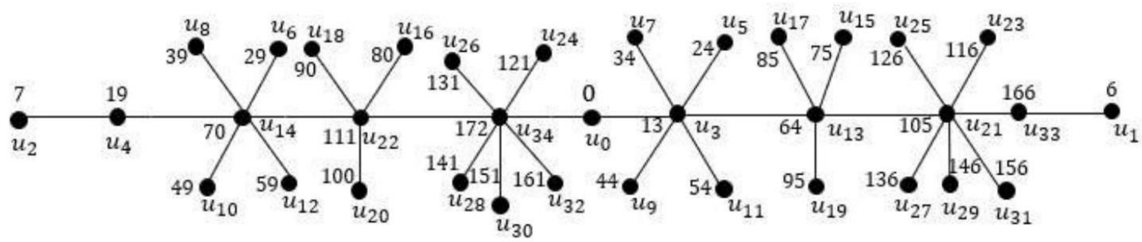

Fig. 3 The radio number of $C(11;4,3,5)$.

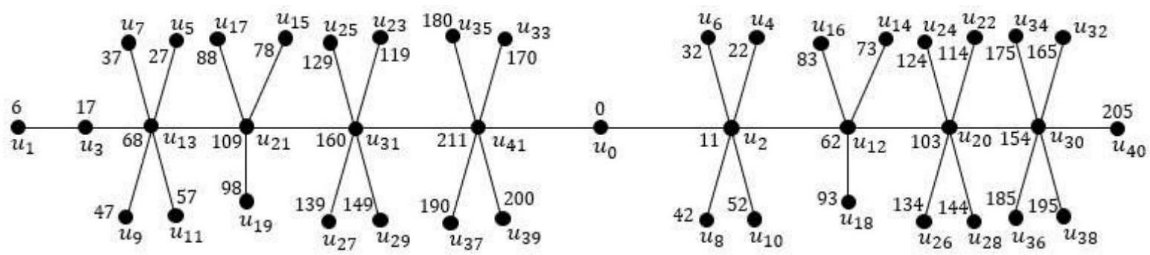

Fig. 4 The radio number of $C(12;4,3,4,4)$

References

[1] Devsi Bantva, Samir Vaidya, and Sanming Zhou, "Radio Number of Trees," *Discrete Applied Mathematics*, vol. 217, no. 2, pp. 110-122, 2016. [CrossRef] [Google Scholar] [Publisher Link]

[2] Daphne Der-Fen Liu, and Melanie Xie, "Radio Number for Square Cycles," *Congressus Numerantium*, vol. 169, 2004. [Google Scholar] [Publisher Link]

[3] Daphne Der-Fen Liu, "Radio Number of Trees," *Discrete Mathematics*, vol. 308, no. 7, pp. 1153-1164, 2008. [CrossRef] [Google Scholar] [Publisher Link]

[4] Daphne Der-Fen Liu, and Melanie Xie, "Radio Number for Square Paths," *Ars Combinatoria*, vol. 90, pp. 1-13, 2009. [Google Scholar] [Publisher Link]

[5] Daphne Der-Fen Liu, and Xuding Zhu, "Multi-Level Distance Labeling for Paths and Cycles," *SIAM Journal on Discrete Mathematics*, vol. 19, no. 3, pp. 610-621, 2005. [CrossRef] [Google Scholar] [Publisher Link]
[6] S.K. Vaidya, and D.D. Bantva, "Radio Number for Total Graph of Paths," *International Scholarly Research Notices*, vol. 2013, no. 1, pp. 1-5, 2013. [CrossRef] [Google Scholar] [Publisher Link]
[7] Douglas Brent West, *Introduction to Graph Theory*, Pearson, 2020. [Google Scholar] [Publisher Link]
[8] G.T Chartrand, D. Erwin, and Ping Zhang, "A Graph Labeling Suggested by FM Channel Restrictions," *Bulletin of the Institute of Combinatorics and its Applications*, vol. 43, pp. 43-57, 2005. [Google Scholar]
[9] Laxman Saha, and Pratima Panigrahi, "A Lower Bound for Radio *k*-Chromatic Number," *Discrete Applied Mathematics*, vol. 192, pp. 87-100, 2015. [CrossRef] [Google Scholar] [Publisher Link]
[10] Paul Martinez et al., "Radio Numbers for Generalized Prism Graphs," *Discussiones Mathematicae Graph Theory*, vol. 31, pp. 45-62, 2011. [Google Scholar] [Publisher Link]
[11] Ping Zhang, "Radio Labelings of Cycles," *Ars Combinatoria*, vol. 65, pp. 21-32, 2002. [Google Scholar] [Publisher Link]
[12] Ruxandra Marinescu-Ghemeci, "Radio Number for Some Thorn Graphs," *Discussiones Mathematicae Graph Theory*, vol. 30, pp. 201-222, 2010. [Google Scholar] [Publisher Link]
[13] Shin Min Kang et al., "Radio Number of Caterpillar Graphs," *Wulfenia Journal*, vol. 22, no. 5, pp. 48-58, 2015. [Google Scholar]
[14] Srinivasa Rao Kola, and Pratima Panigrahi, "Radio Numbers of Some Caterpillars," *Electronic Notes in Discrete Mathematics*, vol. 48, pp. 289-296, 2015. [CrossRef] [Google Scholar] [Publisher Link]
[15] W.K Hale, "Frequency Assignment: Theory and Applications," *Proceedings of the IEEE*, vol. 68, no. 12, pp. 1497-1514, 1980. [CrossRef] [Google Scholar] [Publisher Link]
[16] Xiangwen Li, Vicky Mak, and Sanming Zhou, "Optimal Radio Labelings of Complete M-Ary Trees," *Discrete Applied Mathematics*, vol. 158, pp. 507-515, 2010. [CrossRef] [Google Scholar] [Publisher Link]