

Original Article

# On a New Class of Nano *sgp* -Separation Axioms: Regular and Normal Spaces

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**Abstract** - This paper introduces nano *sgp*-regular and nano *sgp*-normal spaces defined via nano *sgp*-closed sets, and investigates their characterizations and relationships with nano *sgp*-regular and nano *sgp*-normal spaces. Preservation properties under subspaces and various classes of mappings are also established.

**Keywords** - Nano-topology, Nano *sgp*-closed sets, Nano *sgp*-regular space, Nano *sgp*-normal space.

## 1. Introduction

The theory of nano topological spaces was introduced as a mathematical framework inspired by rough set theory to model approximation and uncertainty in finite universes [5]. Since its introduction, nano topology has been extensively investigated, leading to the development of various generalized notions of open and closed sets, continuity, and separation axioms [3,4]. These generalizations play a crucial role in extending classical topological concepts to settings where standard assumptions such as infinite universes or classical openness are not applicable. Separation axioms are fundamental in topology, as they provide tools to distinguish points and sets using neighborhoods. Classical notions such as regular and normal spaces have been generalized in several directions using different types of closed sets. In the context of nano topology, nano regular and nano normal spaces have been studied using nano closed sets and their variants [6]. Further generalizations involving nano generalized closed sets and nano weak forms of openness have been explored to obtain finer classifications of nano topological spaces [1,4].

More recently, research has focused on separation axioms defined via stronger generalized closed sets, such as nano *gp*-closed and nano *sgp*-closed sets, which offer a richer and more flexible structural framework [6]. Motivated by these developments, the present paper introduces and studies the notions of nano *sgp*-regular and nano *sgp*-normal spaces, which extend the classical concepts of regularity and normality in nano topology. The main objective of this work is to examine their fundamental properties, establish equivalent characterizations, and analyze their relationships with existing nano *sgp*-separation axioms.

## 2. Preliminaries

In this section, we review the fundamental definitions and results of nano topological spaces. Throughout this paper,  $U$  denotes a non-empty finite universe and  $R$  an equivalence relation on  $U$ . The pair  $(U, R)$  is called an approximation space.

**Definition 2.1[5]:** Let  $X \subseteq U$ . The lower approximation, upper approximation, and boundary region of  $X$  with respect to  $R$  are defined as  $\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$ ,  $\bar{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$  and  $B_R(X) = \bar{R}(X) \setminus \underline{R}(X)$ , where  $[x]_R$  denotes the equivalence class of  $x$  under  $R$ .

**Definition 2.2[5]:** The collection  $\tau_R(X) = \{\emptyset, U, \underline{R}(X), \bar{R}(X), B_R(X)\}$  forms a topology on  $U$ , called the nano topology induced by  $X$ . The triple  $(U, \tau_R(X))$  is called a nano topological space. Elements of  $\tau_R(X)$  and they are called nano open sets, and their complements are called nano closed sets.

**Definition 2.3[4]:** Let  $A \subseteq U$ . The nano closure of  $A$ , denoted by  $\text{Cl}(A)$ , is the intersection of all nano closed sets containing  $A$ . The nano interior of  $A$ , denoted by  $\text{Int}(A)$ , is the union of all nano open sets contained in  $A$ .



**Definition 2.4[5]:** A subset  $A$  of a nano topological space  $U$  is said to be:

- (i) nano semi-open if  $A \subseteq NCl(NInt(A))$
- (ii) nano pre-open if  $A \subseteq NInt(NCl(A))$ ,

The complements of these sets are called nano pre-closed and nano  $s$ -closed sets, respectively.

Several generalized closed sets have been introduced in nano topology to extend classical results. In particular, nano  $sgp$ -closed sets play a significant role in defining weakened separation axioms.

**Definition 2.5:** Let  $(U, \tau_R)$  be a nano topological space and let  $A \subseteq U$ . The set  $A$  is said to be nano  $sgp$ -closed if for every nano-open set  $G$  in  $U$  with  $A \subseteq G$ , the nano  $sgp$ -closure of  $A$  is contained in  $G$ . The complement of a nano  $sgp$ -closed set is called a nano  $sgp$ -open set.

**Definition 2.6:** For any subset  $A \subseteq U$ , the nano  $sgp$ -closure of  $A$ , denoted by  $pCl(A)$ , is the intersection of all nano  $sgp$ -closed sets containing  $A$ .

**Definition 2.7:** Let  $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$  be a function. Then  $f$  is said to be:

- (i) nano continuous if the inverse image of every nano open set in  $V$  is nano open in  $U$ ,
- (ii) nano open if the image of every nano open set in  $U$  is nano open in  $V$ .
- (iii) nano  $sgp$ -irresolute if the inverse image of every nano  $sgp$ -closed set in  $V$  is nano  $sgp$ -closed in  $U$ , almost nano  $sgp$ -irresolute if the inverse image of every nano  $sgp$ -closed set is nano closed.

### 3. Nano $sgp$ -Regular Spaces

**Definition 3.1:** Let  $(U, \tau_R)$  be a nano topological space. The space  $U$  is said to be nano  $sgp$ -regular if for every point  $x \in U$  and for every nano  $sgp$ -closed set  $F$  not containing  $x$ , there exist nano-open sets  $G$  and  $H$  such that  $x \in G, F \subseteq H$ , and  $G \cap H = \emptyset$ .

**Theorem 3.2:** Every nano regular space is a nano  $sgp$ -regular space.

**Proof:** Let  $(U, \tau_R)$  a nano-regular space, let  $x \in U$ , and let  $F$  be a nano  $sgp$ -closed set with  $x \notin F$ . Since the complement of  $F$  is nano  $sgp$ -open and contains  $x$ , and since nano-regularity guarantees the existence of disjoint nano-open sets separating a point from a nano-closed set, there exist nano-open sets  $G$  and  $H$  such that  $x \in G, F \subseteq H$ , and  $G \cap H = \emptyset$ . Hence,  $x$  and  $F$  can be separated by disjoint nano-open sets, and therefore the space is nano  $sgp$ -regular.

**Remark 3.3:** The reverse implication of the above Theorem need not be true can be seen from the following example.

For example, Let  $U = \{a, b, c\}$  with nano topology  $\tau_R = \{\emptyset, U, \{a\}, \{a, b\}\}$ . The nano closed sets are  $\emptyset, U, \{c\}$ , and  $\{b, c\}$ . The space is nano  $sgp$ -regular, since for any point  $x$  and any nano  $sgp$ -closed set  $F$  not containing  $x$ , suitable nano open sets can be chosen to separate  $x$  and  $F$ . However, the space is not nano-regular. Indeed, for the point  $b$  and the nano closed set  $F = \{c\}$ , the only nano open sets containing  $b$  are  $U$  and  $\{a, b\}$ , and any nano open set containing  $F$  must be  $U$ ; these are not disjoint. Hence, nano  $sgp$ -regularity does not imply nano regularity.

**Theorem 3.4:** Let  $(U, \tau_R)$  be a nano topological space. The following conditions are equivalent:

- (i)  $U$  is a nano  $sgp$ -regular space.
- (ii) For each point  $x \in U$  and for each nano  $sgp$ -open set  $A$  containing  $x$ , there exists a nano open set  $V$  such that  $x \in V \subseteq Cl(V) \subseteq A$ .

**Proof:** Assume that  $U$  is a nano  $sgp$ -regular space. Let  $x \in U$  and let  $A$  be a nano  $sgp$ -open set containing  $x$ . Then the complement  $F = U \setminus A$  is a nano  $sgp$ -closed set not containing  $x$ . By nano  $sgp$ -regularity, there exist nano open sets  $G$  and  $H$  such that  $x \in G, F \subseteq H$ , and  $G \cap H = \emptyset$ . Since  $G \cap H = \emptyset$ , it follows that  $Cl(G) \subseteq U \setminus H$ , and because  $F \subseteq H$ , we have  $U \setminus H \subseteq A$ . Hence,  $Cl(G) \subseteq A$ . Taking  $V = G$ , there exists a nano open set  $V$  such that  $x \in V \subseteq Cl(V) \subseteq A$ , which proves condition (ii).

Conversely, assume that for each point  $x \in U$  and for each nano  $sgp$ -open set  $A$  containing  $x$ , there exists a nano open set  $V$  such that  $x \in V \subseteq Cl(V) \subseteq A$ . Let  $F$  be a nano  $sgp$ -closed set with  $x \notin F$ . Then  $A = U \setminus F$  is a nano  $sgp$ -open set containing  $x$ . By the assumption, there exists a nano open set  $V$  with  $x \in V \subseteq Cl(V) \subseteq A$ . Let  $G = V$  and  $H = U \setminus Cl(V)$ . Then  $G$  and  $H$  are nano open sets such that  $x \in G, F \subseteq H$ , and  $G \cap H = \emptyset$ . Hence, the point  $x$  and the nano  $sgp$ -closed set  $F$  can be separated by disjoint nano open sets, and therefore  $U$  is a nano  $sgp$ -regular space.

**Theorem 3.5:** Every subspace of a nano *sgp*-regular space is also a nano *sgp*-regular space.

**Proof:** Let  $(U, \tau_R)$  be a nano *sgp*-regular space and let  $Y \subseteq U$ . Consider the subspace  $(Y, \tau_Y)$ , where  $\tau_Y = \{Y \cap G : G \in \tau_R\}$  is the nano subspace topology on  $Y$ . Let  $x \in Y$  and let  $F$  be a nano *sgp*-closed set in  $Y$  such that  $x \notin F$ . Then there exists a nano *sgp*-closed set  $F_1$  in  $U$  such that  $F = Y \cap F_1$ . Since  $x \notin F_1$  and  $U$  is nano *sgp*-regular, there exist nano open sets  $G$  and  $H$  in  $U$  such that  $x \in G$ ,  $F_1 \subseteq H$ , and  $G \cap H = \emptyset$ . Now  $G \cap Y$  and  $H \cap Y$  are nano open sets in the subspace  $Y$ , with  $x \in G \cap Y$ ,  $F \subseteq H \cap Y$ , and  $(G \cap Y) \cap (H \cap Y) = \emptyset$ . Hence, the point  $x$  and the nano *sgp*-closed set  $F$  in  $Y$  can be separated by disjoint nano open sets in  $Y$ . Therefore, every subspace of a nano *sgp*-regular space is itself nano *sgp*-regular.

**Theorem 3.6:** Let  $(U, \tau_R)$  be a nano topological space. The following conditions are equivalent:

- (i)  $U$  is a nano *sgp*-regular space.
- (ii) For each point  $x \in U$  and for each nano *sgp*-open set  $V$  of  $U$  containing  $x$ , there exists a nano open set  $W$  in  $U$  such that  $x \in W \subseteq \text{Cl}(W) \subseteq V$ .
- (iii) For each point  $x \in U$  and for each nano *sgp*-closed set  $A$  of  $U$  not containing  $x$ , there exists a nano open set  $W$  in  $U$  such that  $\text{Cl}(W) \cap A = \emptyset$ .

**Proof: (i)  $\Rightarrow$  (ii):** Assume first that  $U$  is a nano *sgp*-regular space. Let  $x \in U$  and let  $V$  be a nano *sgp*-open set containing  $x$ . Then the complement  $A = U \setminus V$  is a nano *sgp*-closed set not containing  $x$ . By the definition of nano *sgp*-regularity, there exist nano open sets  $G$  and  $H$  such that  $x \in G$ ,  $A \subseteq H$ , and  $G \cap H = \emptyset$ . Since  $G \cap H = \emptyset$ , it follows that  $\text{Cl}(G) \subseteq U \setminus H$ . Because  $A \subseteq H$ , we have  $U \setminus H \subseteq U \setminus A = V$ , and hence  $\text{Cl}(G) \subseteq V$ . Taking  $W = G$ , we obtain  $x \in W \subseteq \text{Cl}(W) \subseteq V$ , and therefore the second condition holds.

**(ii)  $\Rightarrow$  (iii):** assume that for each point  $x \in U$  and for each nano *sgp*-open set  $V$  containing  $x$ , there exists a nano open set  $W$  such that  $x \in W \subseteq \text{Cl}(W) \subseteq V$ . Let  $A$  be any nano *sgp*-closed set of  $U$  not containing  $x$ . Then  $V = U \setminus A$  is a nano *sgp*-open set containing  $x$ . By the assumption, there exists a nano open set  $W$  such that  $x \in W \subseteq \text{Cl}(W) \subseteq V$ . Since  $V = U \setminus A$ , it follows that  $\text{Cl}(W) \cap A = \emptyset$ , which proves the third condition.

**(iii)  $\Rightarrow$  (i):** assume that for each point  $x \in U$  and for each nano *sgp*-closed set  $A$  of  $U$  not containing  $x$ , there exists a nano open set  $W$  such that  $\text{Cl}(W) \cap A = \emptyset$ . Define  $H = U \setminus \text{Cl}(W)$ . Then  $H$  is a nano open set containing  $x$ , and clearly  $W \cap H = \emptyset$  with  $x \in W$ . Hence, the point  $x$  and the nano *sgp*-closed set  $A$  are separated by disjoint nano open sets, which shows that  $U$  is a nano *sgp*-regular space. Therefore, all three conditions are equivalent.

**Theorem 3.7:** Let  $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$  be a bijective function. If  $f$  is nano *sgp*-irresolute and nano open, and if  $(U, \tau_R(X))$  is a nano *sgp*-regular space, then  $(V, \tau_R(Y))$  is also a nano *sgp*-regular space.

**Proof:** Let  $y \in V$  and let  $A$  be a nano *sgp*-closed set in  $V$  such that  $y \notin A$ . Since  $f$  is bijective, there exists a unique point  $x \in U$  such that  $f(x) = y$ . Because  $f$  is nano *sgp*-irresolute, the inverse image  $f^{-1}(A)$  is a nano *sgp*-closed set in  $U$ . Moreover,  $x \notin f^{-1}(A)$ . Since  $(U, \tau_R(X))$  is nano *sgp*-regular, there exist nano open sets  $G$  and  $H$  in  $U$  such that  $x \in G$ ,  $f^{-1}(A) \subseteq H$ , and  $G \cap H = \emptyset$ . As  $f$  is nano open, the images  $f(G)$  and  $f(H)$  are nano open sets in  $V$ . Clearly,  $y = f(x) \in f(G)$  and  $A \subseteq f(H)$ . Furthermore, since  $G \cap H = \emptyset$  and  $f$  is injective, it follows that  $f(G) \cap f(H) = \emptyset$ . Hence, the point  $y$  and the nano *sgp*-closed set  $A$  can be separated by disjoint nano open sets in  $V$ . Therefore,  $(V, \tau_R(Y))$  is a nano *sgp*-regular space.

**Theorem 3.8:** Let  $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$  be an injective function which is nano *sgp*-continuous and nano closed. If  $(V, \tau_R(Y))$  is a nano *sgp*-regular space, then  $(U, \tau_R(X))$  is also a nano *sgp*-regular space.

**Proof:** Let  $x \in U$  and let  $F$  be a nano *sgp*-closed set in  $U$  such that  $x \notin F$ . Since  $f$  is nano *sgp*-continuous, the image  $f(F)$  is a nano *sgp*-closed set in  $V$ . Moreover, since  $f$  is injective, we have  $f(x) \notin f(F)$ . As  $(V, \tau_R(Y))$  is nano *sgp*-regular, there exist nano open sets  $G$  and  $H$  in  $V$  such that  $f(x) \in G$ ,  $f(F) \subseteq H$ , and  $G \cap H = \emptyset$ . Because  $f$  is nano closed, the inverse images  $f^{-1}(G)$  and  $f^{-1}(H)$  are nano open sets in  $U$ . Clearly,  $x \in f^{-1}(G)$ ,  $F \subseteq f^{-1}(H)$ , and  $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ . Hence, the point  $x$  and the nano *sgp*-closed set  $F$  can be separated by disjoint nano open sets in  $U$ . Therefore,  $(U, \tau_R(X))$  is a nano *sgp*-regular space.

**Theorem 3.9:** Let  $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$  be an injective function which is nano *sgp*-irresolute and nano closed. If  $(V, \tau_R(Y))$  is a nano *sgp*-regular space, then  $(U, \tau_R(X))$  is also a nano *sgp*-regular space.

**Proof:** Let  $x \in U$  and let  $F$  be a nano *sgp*-closed set in  $U$  with  $x \notin F$ . Since  $f$  is nano *sgp*-irresolute,  $f(F)$  is nano *sgp*-closed in  $V$ , and injectivity gives  $f(x) \notin f(F)$ . As  $(V, \tau_R(Y))$  is nano *sgp*-regular, there exist disjoint nano open sets  $G$  and  $H$  in  $V$  such that

$f(x) \in G$  and  $f(F) \subseteq H$ . Because  $f$  is nano closed,  $f^{-1}(G)$  and  $f^{-1}(H)$  are nano open in  $U$ , with  $x \in f^{-1}(G)$ ,  $F \subseteq f^{-1}(H)$ , and  $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ . Hence,  $U$  is nano *sgp*-regular.

#### 4. Nano *sgp*-Normal Spaces

**Definition 4.1:** Let  $(U, \tau_R)$  be a nano topological space. The space  $U$  is said to be a nano *sgp*-normal space if for every pair of disjoint nano *sgp*-closed sets  $F$  and  $G$  in  $U$ , there exist disjoint nano open sets  $H$  and  $K$  such that  $F \subseteq H$ ,  $G \subseteq K$ , and  $H \cap K = \emptyset$ .

**Theorem 4.2:** Every nano normal space is a nano *sgp*-normal space

**Proof:** Let  $(U, \tau_R)$  be a nano normal space, and let  $A$  and  $B$  be two disjoint nano *sgp*-closed subsets of  $U$ . By nano normality, there exist disjoint nano-open sets  $G$  and  $H$  such that  $A \subseteq G$  and  $B \subseteq H$ . Hence, the disjoint nano *sgp*-closed sets  $A$  and  $B$  can be separated by disjoint nano-open sets, and therefore the space is nano *sgp*-normal.

**Remark 4.3:** The reverse implication of the above Theorem need not be true can be seen from the following example.

For example, Let  $U = \{a, b, c\}$  with the nano topology  $\tau_R = \{\emptyset, \{a\}, \{a, b\}, U\}$ . The corresponding nano-closed sets are  $U$ ,  $\{b, c\}$ ,  $\{c\}$ , and  $\emptyset$ . In this space, any two disjoint nano *sgp*-closed sets can be separated by disjoint nano-open sets, and hence the space is nano *sgp*-normal. However, the disjoint nano-closed sets  $\{a\}$  and  $\{b, c\}$  cannot be separated by disjoint nano-open sets, since the only nano-open set containing  $\{b, c\}$  is  $U$ . Therefore, the space fails to be nano-normal.

**Theorem 4.4:** Let  $U$  be a nano topological space. The following conditions are equivalent:

- (i)  $U$  is a nano *sgp*-normal space.
- (ii) For each nano *sgp*-closed set  $A$  and each nano *sgp*-open set  $V$  with  $A \subseteq V$ , there exists a nano open set  $W$  such that  $A \subseteq W \subseteq pCl(W) \subseteq V$ .
- (iii) For each pair of distinct nano *sgp*-closed sets  $A$  and  $B$ , there exists a nano open set  $W$  such that  $A \subseteq W$  and  $pCl(W) \cap B = \emptyset$ .
- (iv) For each pair of distinct nano *sgp*-closed sets  $A$  and  $B$ , there exist nano open sets  $V$  and  $W$  such that  $A \subseteq V$ ,  $B \subseteq W$ , and  $pCl(V) \cap pCl(W) = \emptyset$ .

**Proof: (i)  $\Rightarrow$  (ii):** Assume first that  $U$  is a nano *sgp*-normal space. Let  $A$  be a nano *sgp*-closed set and  $V$  be a nano *sgp*-open set containing  $A$ . Then the complement  $B = U \setminus V$  is a nano *sgp*-closed set disjoint from  $A$ . By nano *sgp*-normality, there exist disjoint nano open sets  $W$  and  $H$  such that  $A \subseteq W$  and  $B \subseteq H$ . Since  $W \cap H = \emptyset$ , it follows that  $pCl(W) \subseteq U \setminus H \subseteq V$ .

**(ii)  $\Rightarrow$  (iii):** Next, assume that condition (ii) holds. Let  $A$  and  $B$  be disjoint nano *sgp*-closed sets. Then  $V = U \setminus B$  is a nano *sgp*-open set containing  $A$ , and by (ii) there exists a nano open set  $W$  such that  $A \subseteq W \subseteq pCl(W) \subseteq V$ . Since  $pCl(W) \subseteq V = U \setminus B$ , we have  $pCl(W) \cap B = \emptyset$ .

**(iii)  $\Rightarrow$  (iv):** Now assume that condition (iii) holds. Let  $A$  and  $B$  be disjoint nano *sgp*-closed sets. By (iii), there exists a nano open set  $V$  such that  $A \subseteq V$  and  $pCl(V) \cap B = \emptyset$ . Similarly, exchanging the roles of  $A$  and  $B$ , there exists a nano open set  $W$  such that  $B \subseteq W$  and  $pCl(W) \cap A = \emptyset$ . Then  $pCl(V) \cap pCl(W) \subseteq (pCl(V) \cap B) \cup (pCl(W) \cap A) = \emptyset$ ,

**(iv)  $\Rightarrow$  (i):** Finally, assume condition (iv) holds. Let  $A$  and  $B$  be any pair of disjoint nano *sgp*-closed sets. Then by (iv) there exist nano open sets  $V$  and  $W$  such that  $A \subseteq V$ ,  $B \subseteq W$ , and  $pCl(V) \cap pCl(W) = \emptyset$ . Clearly,  $V$  and  $W$  are disjoint in the sense of their closures, so  $A$  and  $B$  can be separated by disjoint nano open sets, which shows that  $U$  is a nano *sgp*-normal space.

**Theorem 4.5:** Let  $U$  be a nano topological space. Then  $U$  is a nano *sgp*-normal space if and only if for any two disjoint nano *sgp*-closed sets  $A$  and  $B$  in  $U$ , there exist nano open sets  $V$  and  $W$  in  $U$  such that  $A \subseteq V$ ,  $B \subseteq W$ , and  $pCl(V) \cap pCl(W) = \emptyset$ .

**Proof:** The proof follows from Theorem 4.4.

**Theorem 4.6:** Every nano *sgp*-closed subspace of a nano *sgp*-normal space is a nano *sgp*-normal space.

**Proof:** Let  $U$  be a nano *sgp*-normal space and let  $Y \subseteq U$  be a nano *sgp*-closed subspace with the subspace topology  $\tau_Y = \{Y \cap G : G \in \tau_N\}$ . Let  $A$  and  $B$  be disjoint nano *sgp*-closed sets in  $Y$ . Then there exist nano *sgp*-closed sets  $A_1$  and  $B_1$  in  $U$  such that  $A = A_1 \cap Y$  and  $B = B_1 \cap Y$ . Since  $U$  is nano *sgp*-normal, there exist disjoint nano open sets  $V$  and  $W$  in  $U$  such that  $A_1 \subseteq V$  and  $B_1 \subseteq W$ . Then  $V \cap Y$  and  $W \cap Y$  are nano open sets in  $Y$ ,  $A \subseteq V \cap Y$ ,  $B \subseteq W \cap Y$ , and  $(V \cap Y) \cap (W \cap Y) = \emptyset$ . Hence, the disjoint nano *sgp*-closed sets  $A$  and  $B$  in  $Y$  can be separated by disjoint nano open sets, showing that  $Y$  is a nano *sgp*-normal space.

**Theorem 4.7:** Let  $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$  be a bijective function. If  $f$  is nano open and nano  $sgp$ -irresolute, and if  $(U, \tau_R(X))$  is a nano  $sgp$ -normal space, then  $(V, \tau_R(Y))$  is also a nano  $sgp$ -normal space.

**Proof:** Let  $A$  and  $B$  be disjoint nano  $sgp$ -closed sets in  $V$ . Since  $f$  is bijective, there exist points in  $U$  corresponding to  $A$  and  $B$ . Because  $f$  is nano  $sgp$ -irresolute, the inverse images  $f^{-1}(A)$  and  $f^{-1}(B)$  are nano  $sgp$ -closed sets in  $U$ . Since  $(U, \tau_R(X))$  is nano  $sgp$ -normal, there exist disjoint nano open sets  $G$  and  $H$  in  $U$  such that  $f^{-1}(A) \subseteq G$  and  $f^{-1}(B) \subseteq H$ . As  $f$  is nano open, the images  $f(G)$  and  $f(H)$  are nano open sets in  $V$ . Clearly,  $A \subseteq f(G)$ ,  $B \subseteq f(H)$ , and  $f(G) \cap f(H) = \emptyset$ . Hence, the disjoint nano  $sgp$ -closed sets  $A$  and  $B$  in  $V$  can be separated by disjoint nano open sets, which shows that  $(V, \tau_R(Y))$  is a nano  $sgp$ -normal space.

**Theorem 4.8:** Let  $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$  be a function. If  $f$  is strongly nano  $sgp$ -open, continuous, and almost nano  $sgp$ -irresolute, and if  $(U, \tau_R(X))$  is a nano  $sgp$ -normal space, then  $(V, \tau_R(Y))$  is also a nano  $sgp$ -normal space.

**Proof:** Let  $A$  and  $B$  be disjoint nano  $sgp$ -closed sets in  $V$ . Since  $f$  is almost nano  $sgp$ -irresolute, the inverse images  $f^{-1}(A)$  and  $f^{-1}(B)$  are almost nano  $sgp$ -closed in  $U$ . Because  $(U, \tau_R(X))$  is nano  $sgp$ -normal, there exist disjoint nano open sets  $G$  and  $H$  in  $U$  such that  $f^{-1}(A) \subseteq G$  and  $f^{-1}(B) \subseteq H$ . Since  $f$  is strongly nano  $sgp$ -open, the images  $f(G)$  and  $f(H)$  are nano open sets in  $V$ . Clearly,  $A \subseteq f(G)$ ,  $B \subseteq f(H)$ , and  $f(G) \cap f(H) = \emptyset$ . Hence, the disjoint nano  $sgp$ -closed sets  $A$  and  $B$  in  $V$  can be separated by disjoint nano open sets, showing that  $(V, \tau_R(Y))$  is a nano  $sgp$ -normal space.

## 5. Conclusion

This paper examined nano  $sgp$ -regular and nano  $sgp$ -normal spaces. Their characterizations and relationships with nano regular and nano normal spaces were established, showing that the  $sgp$  versions are weaker. Preservation properties under subspaces and suitable mappings were also discussed.

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