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Original Article

Computing Spectra and Product Eccentricity Energy of the Cartesian Product of $\mathcal{K}_2$ and Cycle Graphs

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Abstract - *This article investigates the product eccentricity energy of the Cartesian product of $\mathcal{K}_2$ and $\mathcal{C}_n$. While spectral graph theory has extensively covered adjacency and Laplacian energies of product graphs, there remains a research gap in understanding how eccentricity-based spectra behave under graph operations. By leveraging established results in spectral graph theory, product eccentricity energy of cycle graphs, and illustrations on the Cartesian product of $\mathcal{K}_2$ and $\mathcal{C}_n$, $n \geq 3$ are given.*

Keywords - *Cartesian Product, Complete Graph $\boldsymbol{\mathcal{K}_2}$, Cycle Graphs, Product Eccentricity Energy, Spectrum.*

1. Introduction

Simple, undirected, and connected graphs are all taken into consideration in this study. The distance $d(u, v)$ is the length of any shortest path connecting u and v in $\mathcal{P}$ for any vertices $u, v \in V(\mathcal{P})$. The greatest distance between a vertex and any other vertex of $\mathcal{P}$ is its eccentricity. The greatest and smallest eccentricities of all the vertices in a graph $\mathcal{P}$ are its diameter and radius.

The eigenvalues of graph $\mathcal{P}$ are $\eta_1, \eta_2, \cdots \eta_n$ of the matrix assumed in non-increasing order. The eigenvalues of $\mathcal{P}$ are real with sum equal to zero since the matrix is symmetric and real. The sum of the absolute values of the eigenvalues of $\mathcal{P}$ is the graph's energy. In areas of mathematics like linear algebra and combinatorial optimization, the spectrum and energy of a graph have important uses and relationships. The goal of this study is to use graph operations to determine the product eccentricity energy.

2. Literature Review

Every graph that is taken into consideration is a simple, finite, and undirected graph. Douglas B. West's Introduction to Graph Theory [1] is used for standard graph theory terminology and notations, and Lang [2] for algebra.

2.1. The Evolution of Graph Energy

Dragoš M. Cvetković [3] in 1980 studied the concept of spectra of graphs, which laid the groundwork for spectral graph theory. Historically, this field gained significant momentum through theoretical chemistry. Ivan Gutman [6], in 1978, computed the energy of a graph P as

$$E(P) = \sum_{i=1}^{n} |\zeta_i|.$$

Li. X and Y. Shi, along with I. Gutman in early 2012 [7] introduced the energy of a graph.

2.2. From Adjacency Energy to Distance and Eccentricity Energy

As the limitations of the adjacency matrix in capturing long-range structural information became apparent, researchers shifted focus toward distance-based matrices. Later, M. Aouchiche and P. Hansen made a detailed study on the distance matrix and its spectrum [4]. N. Prabhavathy in [5] proposed the concept of eccentricity energy of various graphs. Later, the graph's maximum degree energy, as described by C. Adiga et al. [8], depends on the maximum degree matrix of the corresponding graph.

M. Randic [9] presented a new distance-type graph matrix, which Wang et al. called the eccentricity matrix. Ahmed M. Naji et al. [10] developed the term "Maximum Eccentricity Matrix" in 2016. Later, Mohammad Issa Sowaity and B.Sharada [11] in 2017 developed the concept of sum-eccentricity energy of a graph.

In 2011, [12] Andries E. Brouwer and Willem H. Haemers found spectra of many graphs. In theoretical chemistry, using Huckel theory, the $\pi -$ electron energy of a conjugated carbon molecule was calculated. In 2025 [13], Priya Karen S and Arokia Lancy A defined the concept of product eccentricity energy of various graphs using their eigenvalues.

3. Preliminaries

Definition 3.1 [5] An essential concept in this theory is a vertex's eccentricity, which determines the greatest distance between two vertices. The distance between two vertices a and b in $V(\mathcal{P})$ is the shortest $a - b$ path length in $\mathcal{P}$. Formally, it can be expressed as:

$$\xi(b) = \max\{d(b,a) | \forall\, a \in V(P)\} \tag{1}$$

Definition 3.2: [3] The spectrum of the graph is denoted by

$$Spec(\mathcal{P}) = \begin{bmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_n \\ m_1 & m_2 & \cdots & m_n \end{bmatrix} \tag{2}$$

where $\zeta_i\, , i = 1,2,\cdots n$ are the eigenvalues of the matrix and $m_1, m_2, \cdots m_n$ are its corresponding multiplicities.

Definition 3.3:[14] The cartesian product of two simple graphs $\mathcal{H}$ and $\mathcal{K}$ is the graph $\mathcal{P} = \mathcal{H} \times \mathcal{K}$ with $V(\mathcal{P}) = V(\mathcal{H}) \times V(\mathcal{K})$ in which vertices (h,k) and (h',k') are adjacent iff either

1. $h = h'$ and k, k' are adjacent in $\mathcal{K}$ or
2. $k = k'$ and h, h' are adjacent in $\mathcal{K}$

The following equations follow directly from definition [1]

$$d_{\mathcal{H}\times\mathcal{K}}\big((h,k),(h',k')\big) = d_{\mathcal{H}}(h,h') + d_{\mathcal{K}}(k,k') \tag{3}$$

and

$$\xi_{\mathcal{H}\times\mathcal{K}}\big((h,k),(h',k')\big) = \xi_{\mathcal{H}}(h,h') + \xi_{\mathcal{K}}(k,k') \tag{4}$$

A generalization of the concept and equations above to the Cartesian product of **n** graphs is possible, and it is denoted as $\mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_n$

Definition 3.4: [13] Let us consider a graph $\mathcal{P}$ with n vertices. Then, the product eccentricity matrix of the graph $\mathcal{P}$ is defined as

$$\mathcal{P}_{ij} = \begin{cases} \xi(v_i).\,\xi(v_j) & if\ v_i \sim v_j \\ 0 & otherwise \end{cases} \tag{5}$$

$P_\xi(\mathcal{P})$ denotes the product eccentricity energy of the graph. $E_{PE}(\mathcal{P})$ is defined to be the sum of the absolute eigenvalues. $E_{PE}(\mathcal{P}) = \sum_{i=1}^{n} |\eta_i|$ where $\eta_1, \eta_2, \cdots \eta_n$ Are the eigenvalues of the given product eccentricity matrix.

Definition 3.5: [15] If $\zeta_1, \zeta_2, \cdots \zeta_n$ are the eigenvalues of $A(\mathcal{C}_n)$ then for any $k.A(\mathcal{C}_n)$ The eigenvalues are $k.\zeta_1, k\,\zeta_2, \cdots, k\zeta_n$. Eigenvalues of the adjacency matrix of the cycle graph are,

$$\zeta_j = 2\cos\left(\frac{2\,\pi(j-1)}{n}\right), \qquad j = 1,2,\cdots n. \tag{6}$$

Definition 3.6:[15] For a complete graph, the eigenvalues computed from the adjacency energy are 1and $n - 1$ with multiplicity $(n-1)$ and $(n-1)$ with multiplicity 1. Eigen values of $\mathcal{K}_2$ is $-1{,}1$

4. Product Eccentricity Energy of Cycle Graphs

Theorem 4.1 *If $\eta_1, \eta_2, \cdots \eta_n$ Are the eigenvalues of the cycle graph $\mathcal{C}_n$ Then the product eccentricity energy is*

$$E_{PE}(\mathcal{C}_n) = 2\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^2 \sum_{j=1}^{n} \left|\cos\left(\frac{2\,\pi(j-1)}{n}\right)\right| \tag{7}$$

Proof: Let $v_1, v_2, \cdots v_n$ be the vertices of the cycle graph $\mathcal{C}_n$ Every vertex has the same eccentricity; thus, every cycle graph is

self-centered.
If $\mathfrak{n}$ is even, $\mathfrak{n} = 2k$, then the eccentricity of every vertex is $k = \frac{\mathfrak{n}}{2}$.
If $\mathfrak{n}$ is odd, $\mathfrak{n} = 2k+1$, then the eccentricity of every vertex is $k = \frac{\mathfrak{n}-1}{2}$
Thus, for every vertex $v \in V(\mathcal{C}_\mathfrak{n})$ The eccentricity is $\left\lfloor \frac{\mathfrak{n}}{2} \right\rfloor$.

$$\xi(v) = \left\lfloor \frac{\mathfrak{n}}{2} \right\rfloor \; \forall v \in V(\mathcal{C}_\mathfrak{n}) \tag{8}$$

By definition 3.4, the product eccentricity matrix of the graph $\mathcal{P}$ is defined as

$$\mathcal{P}_{ij} = \begin{cases} \xi(v_i).\xi(v_j) & if\ v_i \sim v_j \\ 0 & otherwise \end{cases}$$

$$P_\xi(\mathcal{C}_\mathfrak{n}) = \begin{bmatrix} 0 & c_{12} & c_{13} & \cdots & c_{1\mathfrak{n}} \\ c_{21} & 0 & c_{23} & \cdots & c_{2\mathfrak{n}} \\ c_{31} & c_{32} & 0 & \cdots & c_{3\mathfrak{n}} \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ c_{\mathfrak{n}1} & c_{\mathfrak{n}2} & c_{\mathfrak{n}3} & \cdots & c_{\mathfrak{n}\mathfrak{n}} \end{bmatrix}$$

For any two adjacent vertices v_i, v_j For any cycle graph, the product eccentricity matrix takes the value.

$$\mathcal{P}_{ij} = \begin{cases} \xi^2 & if\ v_i \sim v_j \\ 0 & otherwise \end{cases} \tag{9}$$

$$P_\xi(\mathcal{C}_\mathfrak{n}) = \xi^2 \begin{bmatrix} 0 & 1 & 0 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$P_\xi(\mathcal{C}_\mathfrak{n}) = \xi^2.A(\mathcal{C}_\mathfrak{n}) \tag{10}$$

The product eccentricity matrix $P_\xi(\mathcal{C}_\mathfrak{n})$ is ξ^2 times the adjacency matrix.

$$P_\xi(\mathcal{C}_\mathfrak{n}) = \xi^2.A(\mathcal{C}_\mathfrak{n})$$

The eigenvalues of $P_\xi(\mathcal{C}_\mathfrak{n})$ are $\eta_1, \eta_2 \cdots, \eta_\mathfrak{n}$. Using definition 3.4**,** the eigenvalues of the product eccentricity matrix can be computed as

$$\eta_j = \xi^2.2\cos\left(\frac{2\,\pi(j-1)}{\mathfrak{n}}\right), j = 1,2, \cdots \mathfrak{n} \tag{11}$$

Where $\xi = \left\lfloor \frac{\mathfrak{n}}{2} \right\rfloor$

By definition 3.4, the product eccentricity energy for $\mathcal{C}_\mathfrak{n}$ is given as,

$$\begin{aligned} E_{PE}(\mathcal{C}_\mathfrak{n}) &= \textstyle\sum_{j=1}^{\mathfrak{n}} |\eta_j| \\ &= \textstyle\sum_{j=1}^{\mathfrak{n}} \left|\xi^2.2\cos\left(\frac{2\,\pi(j-1)}{\mathfrak{n}}\right)\right| \\ &= \xi^2.2 \textstyle\sum_{j=1}^{\mathfrak{n}} \left|\cos\left(\frac{2\,\pi(j-1)}{\mathfrak{n}}\right)\right| \end{aligned}$$

$$E_{PE}(\mathcal{C}_\mathfrak{n}) = 2\left(\left\lfloor \frac{\mathfrak{n}}{2} \right\rfloor\right)^2 \textstyle\sum_{j=1}^{\mathfrak{n}} \left|\cos\left(\frac{2\,\pi(j-1)}{n}\right)\right| \tag{12}$$

When $\mathfrak{n}$ is even, $\xi = \frac{\mathfrak{n}}{2}$

$$E_{PE}(\mathcal{C}_\mathfrak{n}) = 2\left(\frac{\mathfrak{n}}{2}\right)^2 \sum_{j=1}^{\mathfrak{n}} \left|\cos\left(\frac{2\,\pi(j-1)}{\mathfrak{n}}\right)\right| \tag{13}$$

When $\boldsymbol{\mathfrak{n}}$ is odd, $\xi = \frac{\mathfrak{n}-1}{2}$

$$E_{PE}(\mathcal{C}_n) = 2\left(\frac{n-1}{2}\right)^2 \sum_{j=1}^{n} \left|\cos\left(\frac{2\,\pi(j-1)}{n}\right)\right| \tag{14}$$

Thus, in general, the product eccentricity energy of cycle graphs $\mathcal{C}_n$ is given as

$$E_{PE}(\mathcal{C}_n) = 2\left(\left\lfloor\frac{n}{2}\right\rfloor\right)^2 \sum_{j=1}^{n} \left|\cos\left(\frac{2\,\pi(j-1)}{n}\right)\right| \tag{15}$$

Illustration 4.2

Product Eccentricity Energy of $\mathcal{C}_5$ is 25.888

$$P_\xi(\mathcal{C}_5) = \begin{bmatrix} 0 & 4 & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 4 \\ 4 & 0 & 0 & 4 & 0 \end{bmatrix}$$

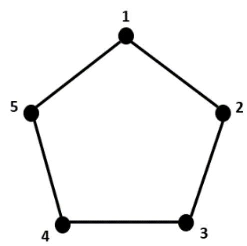

Fig. 1 Cycle Graph with 5 vertices

$$\left|P_\xi(\mathcal{C}_5) - \eta I\right| = \begin{vmatrix} -\eta & 4 & 0 & 0 & 4 \\ 4 & -\eta & 4 & 0 & 0 \\ 0 & 4 & -\eta & 4 & 0 \\ 0 & 0 & 4 & -\eta & 4 \\ 4 & 0 & 0 & 4 & -\eta \end{vmatrix}$$

$$\left|P_\xi(\mathcal{C}_5) - \eta I\right| = (\eta - 8)\left(\eta - \left(2\left(-1-\sqrt{5}\right)\right)^2\right)\left(\eta - \left(2\left(\sqrt{5}-1\right)\right)^2\right)$$

$$Spec(\mathcal{C}_5) = \begin{pmatrix} 8 & 2\left(-1-\sqrt{5}\right) & 2\left(\sqrt{5}-1\right) \\ 1 & 2 & 2 \end{pmatrix}$$

$E_{PE}(\mathcal{C}_5) = 1|8| + 2\left|\left(2\left(-1-\sqrt{5}\right)\right)\right| + 2\left|\left(2\left(\sqrt{5}-1\right)\right)\right|$

Hence, $E_{PE}(\mathcal{C}_5)$ =25.8885

Using (7) of Theorem 4.1, we compute the energy of $\mathcal{C}_5$,

Eccentricity of any vertex in $\mathcal{C}_5$ is 2, therefore we have $\xi(\mathcal{C}_5) = \left\lfloor\frac{5}{2}\right\rfloor = 2$

For $j = 1, cos(0) = 1$

For $j = 1$, $\left|\cos\left(\frac{2\pi}{5}\right)\right| = \left|\frac{\sqrt{5}-1}{4}\right|$

For $j = 2$, $\left|\cos\left(\frac{4\pi}{5}\right)\right| = \left|\frac{\sqrt{5}+1}{4}\right|$

For $j = 3$, $\left|\cos\left(\frac{6\pi}{5}\right)\right| = \left|\frac{\sqrt{5}+1}{4}\right|$

For $j = 4$, $\left|\cos\left(\frac{8\pi}{5}\right)\right| = \left|\frac{\sqrt{5}-1}{4}\right|$

$$E_{PE}(\mathcal{C}_5) = 2^2\left(2\left(\frac{\sqrt{5}-1}{4}\right) + 2\left(\frac{\sqrt{5}-1}{4}\right) + 2\left(\frac{\sqrt{5}-1}{4}\right) + 2\left(\frac{\sqrt{5}-1}{4}\right)\right) = 4\left(2+2\sqrt{5}\right) = 25.885$$

Python Program for Computing $E_{PE}(\mathcal{C}_5)$

```
import networkx as nx
import numpy as np
import sympy as sp
G = nx.Graph()
edges = [(1, 2),(2,3),(3,4),(4,5),(5,1)]
G.add_edges_from(edges)
ecc_dict = nx.eccentricity(G)
nodes = sorted(G.nodes())
n = len(nodes)
P = np.zeros((n, n), dtype=int)
for i in range(n):
    for j in range(n):
        if G.has_edge(nodes[i], nodes[j]):
            P[i][j] = ecc_dict[nodes[i]] * ecc_dict[nodes[j]]
print("Eccentricities:", ecc_dict)
print("\nProduct Eccentricity Matrix (P):")
print(P)
eigenvalues = np.linalg.eigvals(P)
energy = np.sum(np.abs(eigenvalues))
print("\nEigenvalues of Product Eccentricity Matrix:")
print(eigenvalues)
print(f"\nEnergy of the Product Eccentricity Matrix: {energy:.4f}")
P_sym = sp.Matrix(P)
x = sp.symbols('x')
char_poly = P_sym.charpoly(x)
print("\nCharacteristic Polynomial of P:")
print(char_poly.as_expr())
```

Output:

```
Eccentricities: {1: 2, 2: 2, 3: 2, 4: 2, 5: 2}
Product Eccentricity Matrix (P):
[[0 4 0 0 4]
 [4 0 4 0 0]
 [0 4 0 4 0]
 [0 0 4 0 4]
 [4 0 0 4 0]]

Eigenvalues of Product Eccentricity Matrix:
[-6.47213595  2.47213595  8    -6.47213595  2.47213595]
Energy of the Product Eccentricity Matrix: 25.8885
Characteristic Polynomial of P:
x**5 - 80*x**3 + 1280*x – 2048
```

Product Eccentricity Energy of $\mathcal{K}_2 \times \mathcal{C}_n$

Lemma 4.3: The eccentricity of any vertex (u, v) in $\mathcal{K}_2 \times \mathcal{C}_n$ is given as $\xi = 1 + \left\lfloor \frac{n}{2} \right\rfloor$ (16)

Proof: By definition 3.3**, a** fundamental property of the Cartesian product of graphs is that the distance between any two vertices (u, v) and (u', v') in $\mathcal{H} \times \mathcal{K}$ is the sum of the distances in the component graphs

$$d_{\mathcal{H}\times\mathcal{K}}\left((u, v), (u', v')\right) = d_{\mathcal{H}}(u, u') + d_{\mathcal{K}}(v, v')$$

Using the idea of definition 3.3, as eccentricity evaluates the maximum distance between a specific vertex and any other vertices in a graph, it becomes

$$\xi_{\mathcal{H}\times\mathcal{K}}\big((u,v)\big) = \max_{(u',v')\in V(\mathcal{H}\times\mathcal{K})} d_{\mathcal{H}\times\mathcal{K}}\big((u,v),(u',v')\big) \tag{17}$$

Consider any arbitrary vertex. (u,v), the eccentricity of any vertex in the Cartesian product of $\mathcal{K}_2$ and $\mathcal{C}_n$ is given as

$$\xi_{\mathcal{K}_2\times\mathcal{C}_n}\big((u,v)\big) = \max_{(u',v')\in V(\mathcal{K}_2\times\mathcal{C}_n)} d_{\mathcal{K}_2\times\mathcal{C}_n}\big((u,v),(u',v')\big)$$

$$= \max_{u'\in V(\mathcal{K}_2)} d_{\mathcal{K}_2}(u,u') + \max_{v'\in V(\mathcal{C}_n)} d_{\mathcal{C}_n}(v,v')$$

$$\xi_{\mathcal{K}_2\times\mathcal{C}_n}\big((u,v)\big) = \xi_{\mathcal{K}_2}(u) + \xi_{\mathcal{C}_n}(v) \tag{18}$$

The maximum distance in $\mathcal{K}_2$ is 1 and the maximum distance in $\mathcal{C}_n$ is $\left\lfloor\frac{n}{2}\right\rfloor$.

Thus, the eccentricity of every vertex in $\mathcal{K}_2\times\mathcal{C}_n$ is given as $\xi = 1 + \left\lfloor\frac{n}{2}\right\rfloor$

Theorem 4.4: *The Product eccentricity energy of* $\mathcal{K}_2\times\mathcal{C}_n$ *is*

$$E_{PE}(\mathcal{K}_2\times\mathcal{C}_n) = \left(1+\left\lfloor\frac{n}{2}\right\rfloor\right)^2\left[\sum_{j=1}^{n}\left|1+2\cos\left(\frac{2\pi(j-1)}{n}\right)\right| + \sum_{j=1}^{n}\left|1-2\cos\left(\frac{2\pi(j-1)}{n}\right)\right|\right] \tag{19}$$

Proof: Let $\mathcal{P} = \mathcal{K}_2\times\mathcal{C}_n$ be the graph with $2n$ vertices. From Theorem 4.2, the eccentricity of a cycle graph is $\left\lfloor\frac{n}{2}\right\rfloor$.
The eccentricity of $\mathcal{K}_2$ is 1.

$$\xi(\mathcal{P}) = \xi_{\mathcal{K}_2}(u) + \xi_{\mathcal{C}_n}(v) \tag{20}$$

By Lemma 4.3 $\xi(\mathcal{P}) = 1 + \left\lfloor\frac{n}{2}\right\rfloor$
By the definition of 2.4, the product eccentricity matrix of $\mathcal{K}_2\times\mathcal{C}_n$ is obtained

$$\mathcal{P}_{ij} = \begin{cases}\xi^2 & if\ v_i \sim v_j\\ 0 & otherwise\end{cases}$$

$$P_\xi(\mathcal{K}_2\times\mathcal{C}_n) = \xi^2.A(\mathcal{K}_2\times\mathcal{C}_n) \tag{21}$$

The Product Eccentricity matrix of $\mathcal{K}_2\times\mathcal{C}_n$ is obtained as a direct scalar multiple of the adjacency matrix $A(\mathcal{K}_2\times\mathcal{C}_n)$

$$P_\xi(\mathcal{K}_2\times\mathcal{C}_n) = \left(1+\left\lfloor\frac{n}{2}\right\rfloor\right)^2.A(\mathcal{K}_2\times\mathcal{C}_n) \tag{22}$$

The sum of the eigenvalues of $\mathcal{K}_2$ and $\mathcal{C}_n$ is the eigenvalue of the adjacency matrix of $A(\mathcal{K}_2\times\mathcal{C}_n)$. $\{-1,1\}$ are the eigenvalues of the adjacency matrix of $\mathcal{K}_2$.
By definition 3.3,

$$\mu_j = 2\cos\left(\frac{2\pi(j-1)}{n}\right), j = 1,2,\cdots n. \tag{23}$$

(23) are the eigenvalues of the adjacency matrix of the cycle graph.
Therefore, the $2n$ eigenvalues of $A(\mathcal{K}_2\times\mathcal{C}_n)$ are

$$\eta_j^1 = 1 + 2\cos\left(\frac{2\pi(j-1)}{n}\right), for\ j = 1,2,\cdots n \tag{24}$$

$$\eta_j^2 = -1 + 2\cos\left(\frac{2\pi(j-1)}{n}\right), for\ j = 1,2,\cdots n \tag{25}$$

Thus, the product eccentricity energy of $\mathcal{K}_2\times\mathcal{C}_n$ is given as

$$E_{PE}(\mathcal{K}_2\times\mathcal{C}_n) = \left(1+\left\lfloor\frac{n}{2}\right\rfloor\right)^2\left[\sum_{j=1}^{n}\left|1+2\cos\left(\frac{2\pi(j-1)}{n}\right)\right| + \sum_{j=1}^{n}\left|1-2\cos\left(\frac{2\pi(j-1)}{n}\right)\right|\right]$$

This completes the proof.

Illustration 4.5 :
Product Eccentricity Energy of $\mathcal{K}_2 \times \mathcal{C}_3$ is 32

$$P_\xi(\mathcal{K}_2 \times \mathcal{C}_3) = \begin{bmatrix} 0 & 4 & 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 & 0 & 4 \\ 0 & 4 & 0 & 4 & 4 & 0 \\ 4 & 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 4 & 4 & 0 & 4 \\ 4 & 4 & 0 & 0 & 4 & 0 \end{bmatrix}$$

$$|P_\xi(\mathcal{K}_2 \times \mathcal{C}_3) - \eta I| = \begin{vmatrix} -\eta & 4 & 0 & 4 & 0 & 4 \\ 4 & -\eta & 4 & 0 & 0 & 4 \\ 0 & 4 & -\eta & 4 & 4 & 0 \\ 4 & 0 & 4 & -\eta & 4 & 0 \\ 0 & 0 & 4 & 4 & -\eta & 0 \\ 4 & 4 & 0 & 0 & 4 & -\eta \end{vmatrix}$$

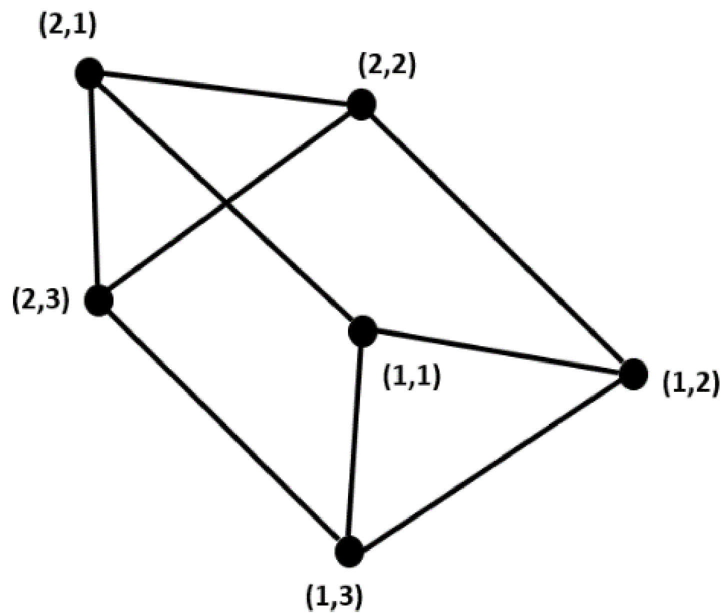

Fig. 2 Cartesian Product Complete Graph with 2 Vertices and Cycle Graph with 3 Vertices

$$\left|P_\xi(\mathcal{K}_2 \times \mathcal{C}_3) - \eta I\right| = (\eta - 12)(\eta - 4)\eta^2(\eta + 8)^2$$

$$Spec(\mathcal{K}_2 \times \mathcal{C}_3) = \begin{pmatrix} -8 & 8 & 4 & 12 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Hence, $E_{PE}(\mathcal{K}_2 \times \mathcal{C}_3) = 32$
Using (19) of Theorem 4.4, we compute the energy of $\mathcal{K}_2 \times \mathcal{C}_3$,
Eccentricity of any vertex in $\mathcal{K}_2 \times \mathcal{C}_3$ is 2, therefore we have $\xi(\mathcal{K}_2 \times \mathcal{C}_3) = 1 + \left\lfloor \frac{3}{2} \right\rfloor = 2$
For $j = 1, cos(0) = 1$

$$|1 + 2(1)| + |1 - 2(1)| = 4$$

For $j = 1$, $\cos\left(\frac{2\pi}{3}\right) = -0.5$

$$|1 + 2(-0.5)| + |1 - 2(-0.5)| = 2$$

For $j = 2, \cos\left(\frac{4\pi}{3}\right) = -0.5$

$$|1 + 2(-0.5)| + |1 - 2(-0.5)| = 2$$

$E_{PE}(\mathcal{K}_2 \times \mathcal{C}_3) = 2^2(4 + 2 + 2) = 32$

Python Program for Computing $E_{PE}(\mathcal{K}_2 \times \mathcal{C}_3)$

```
import networkx as nx
import numpy as np
import sympy as sp
G = nx.Graph()
edges = [(1, 2), (1,4),(1,6),(2,3),(2,6),(3,4),(3,5),(4,5),(5,6)]
G.add_edges_from(edges)
ecc_dict = nx.eccentricity(G)
nodes = sorted(G.nodes())
n = len(nodes)
P = np.zeros((n, n), dtype=int)
for i in range(n):
for j in range(n):
    if G.has_edge(nodes[i], nodes[j]):
       P[i][j] = ecc_dict[nodes[i]] * ecc_dict[nodes[j]]
print("Eccentricities:", ecc_dict)
print("\nProduct Eccentricity Matrix (P):")
print(P)
eigenvalues = np.linalg.eigvals(P)
energy = np.sum(np.abs(eigenvalues))
print("\nEigenvalues of Product Eccentricity Matrix:")
print(eigenvalues)
print(f"\nEnergy of the Product Eccentricity Matrix: {energy:.4f}")
P_sym = sp.Matrix(P)
x = sp.symbols('x')
char_poly = P_sym.charpoly(x)
print("\nCharacteristic Polynomial of P:")
print(char_poly.as_expr())
```

Output:

```
Eccentricities: {1: 2, 2: 2, 4: 2, 6: 2, 3: 2, 5: 2}
Product Eccentricity Matrix (P):
[[0 4 0 4 0 4]
 [4 0 4 0 0 4]
 [0 4 0 4 4 0]
 [4 0 4 0 4 0]
 [0 0 4 4 0 4]
 [4 4 0 0 4 0]]

Eigenvalues of Product Eccentricity Matrix:
[ 1.20000000e+01  4.00000000e+00 -8.00000000e+00 -8.00000000e+00  3.63273796e-16 -
6.58294986e-16]

Energy of the Product Eccentricity Matrix: 32.0000
Characteristic Polynomial of P:
x**6 - 144*x**4 - 256*x**3 + 3072*x**2
```

5. Conclusion

Finding the energy of a graph is one of the emerging concepts in graph theory that bridges chemistry and mathematical concepts and leads researchers into many applications. Here, the problem of finding the product eccentricity energy of $\mathcal{C}_n$ and the Cartesian product of $\mathcal{K}_2$ and $\mathcal{C}_n$ are investigated. The product eccentricity energy of other products of graphs can be studied.

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