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Original Article

Convergence Analysis of Galerkin and Multi-Galerkin Methods for Urysohn Integral Equations on the Half-Line Using Laguerre Polynomials

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Abstract - *This article discusses Galerkin, multi-Galerkin methods, and their iterated versions to approximate the solution of Urysohn-type integral equations on the half-line using Laguerre polynomials as basis functions. Here, we consider that the kernel function is sufficiently smooth. We have shown that the approximate solution in Galerkin and iterated Galerkin methods converges to the exact solution with the order* $\mathcal{O}\left(n^{-\frac{r}{2}}\right)$ *and order* $\mathcal{O}(n^{-r})$*, respectively. Here, r represents the smoothness of the solution, and n denotes the degree of the polynomials. We improve the result using the multi-Galerkin method and its iterative method. In fact, we prove that multi-Galerkin and iterated multi-Galerkin methods converge to the exact solution with the order* $\mathcal{O}\left(n^{-\frac{3r}{2}}\right)$ *and* $\mathcal{O}(n^{-2r})$*, respectively. Numerical results are presented to support theoretical results.*

Keywords - *Multi-Galerkin method, Superconvergence results, Laguerre polynomials, Urysohn integral equation.*

1. Introduction

The objective of this paper is to discuss the nonlinear Urysohn-type integral equations on $\mathbb{R}^+ = [0, \infty)$ given below:

$$u(\eta) - \mathcal{K}(u)(\eta) = y(\eta), \eta \in [0, \infty) \tag{1.1}$$

Integral equations of the type (1.1) are well documented, for instance (see [2,5,9,8, 23, 25, 28,30]). In [28], the author discussed the existence and approximation of the solution of the nonlinear-Urysohn equations (1.1) on the half-line by considering finite section approximations for (1.1). However, the author did not derive any error bounds therein. In [26], the authors discussed projection and multi-projection methods for the approximate solution of nonlinear-Hammerstein type integral equations using piecewise polynomials as basis functions. On the other hand, in the piecewise polynomial-based method, one has to increase the number of partition points. This leads to solving a large number of linear algebraic equations, which is computationally very expensive. To overcome this difficulty, in this article, we consider global polynomial (Laguerre polynomials) based projection and multi-projection methods and their iterated versions to approximate the solution of (1.1). This paper develops Laguerre polynomials-based projection and multi-projection methods and their iterated versions for solving Eq. (1.1). Throughout the article, c is assumed to be a generic constant.

This paper is organized as follows: In Section 2, we discuss Laguerre polynomials-based Galerkin and iterated Galerkin methods to approximate the solution of Eq. (1.1). In Section 3, we discuss convergence results for Galerkin and its iterated method. In Section 4, we discuss the superconvergence results for Eq. (1.1). In Section 5, numerical results are presented to support the theoretical prediction, and Section 6 concludes the paper.

2. Galerkin and its Iterated Method for Urysohn Integral Equations on the Half-line

Consider the weighted space $\mathcal{L}^2_\omega(\mathbb{R}^+) = \{v \mid \|v\|_{L^2_\omega} < \infty\}$, and associated inner product and norm as follows:

$$< y, x >_\omega = \int_0^\infty \omega(s) y(s) x(s) ds \text{ and } \|y\|_\omega = < y, y >_\omega^{1/2}$$

where $\omega(s) = e^{-s}$. Let $\mathcal{B}\left(\mathcal{L}_\omega^2(\mathbb{R}^+)\right)$ be the space of all bounded linear operators from $\mathcal{L}_\omega^2(\mathbb{R}^+)$ into $\mathcal{L}_\omega^2(\mathbb{R}^+)$. Let $\mathbf{X}^+ \subseteq \mathcal{L}_\omega^2(\mathbb{R}^+)$ be the Banach space of all bounded continuous functions on $\mathbb{R}^+ = [0,\infty)$.

For $\left(\int_0^\infty e^{-s} ds\right) = 1$, the following hold

$$\|v\|_\omega = \left(\int_0^\infty e^{-s}|v(s)|^2 ds\right)^{\frac{1}{2}} \leq \|v\|_\infty. (2.1)$$

Consider the following equation on the positive real numbers

$$u(\eta) - \int_0^\infty k\big(\eta, s, u(s)\big) ds = y(s), s \in \mathbb{R}^+, (2.2)$$

where the kernel $k(.,.u()),.y, are\ smooth, known\ functions\ and\ u \in \mathcal{L}_\omega^2(\mathbb{R}^+)$ is the unknown function to be approximated. Define

$$\mathcal{K}(u)(\eta) = \int_0^\infty k\big(\eta, s, u(s)\big) ds, u \in \mathcal{L}_\omega^2(\mathbb{R}^+). (2.3)$$

$\mathcal{K} : \mathcal{L}_\omega^2(\mathbb{R}^+) \to \mathcal{L}_\omega^2(\mathbb{R}^+)\ be\ a\ nonlinear\ integral\ operator$. Using (2.3) in (2.2), we obtain

$$u(\eta) - \mathcal{K}(u)(\eta) = y(\eta), \eta \in \mathbb{R}^+. (2.4)$$

The Frechet derivative $\mathcal{K}'(u)$ at u is a linear operator and is defined by

$$(\mathcal{K}'(u)v)(\eta) = \int_0^\infty \frac{\partial}{\partial u} k(\eta, s, u(s)) v(s) ds = \int_0^\infty k_u(\eta, s, u(s)) v(s) ds, v \in \mathcal{L}_\omega^2(\mathbb{R}^+)$$

Consider $\mathbb{T}: \mathcal{L}_\omega^2(\mathbb{R}^+) \to \mathcal{L}_\omega^2(\mathbb{R}^+)$ as

$$\mathbb{T}(u) = y + \mathcal{K}(u), u \in \mathcal{L}_\omega^2(\mathbb{R}^+), (2.5)$$

Then the equation (2.4) becomes

$$u = \mathbb{T}(u) \qquad (2.6)$$

Throughout the paper, we assume that E.q. (2.6) processes the solution u_0 (say) in $\mathbf{X}^+$and in $\mathcal{L}_\omega^2(\mathbb{R}^+)$ and $\left\|\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1}\right\|_\infty \leq C_1 < \infty$ and $\left\|\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1}\right\|_\omega \leq C_2 < \infty$, where C_1 and C_2 are constant independent of n.(see [1, 28]).

In the sequel, we make the following assumptions on $k(.,.,u().), for\ j = 0,1,2, ...$

A1. $\sup_{\eta,s\in\mathbb{R}^+} \left|e^s \frac{\partial^j}{\partial \eta^j} k_u(\eta, s, u_0(s))\right| \leq c_1 < \infty$,

A2. $h_j(\eta, s, u(s)) = \frac{\partial^j}{\partial \eta^j} e^s k_u(\eta, s, u(s))$, for $u \in \mathcal{L}_\omega^2(\mathbb{R}^+)$,

A3. $h_j(\eta, s, u(s))$ satisfies the Lipschitz condition in the third variable, i.e., $u_1, u_2 \in \mathcal{L}_\omega^2(\mathbb{R}^+)$, there exists a constant c_2 independent of n such that

$$\left|h_j(.,.,u_1(.)) - h_j(.,.,u_2(.))\right| \leq c_2|u_1 - u_2|.$$

Next, for solving equation (1.1), we choose the approximating space $\mathbf{X}_n = \mathrm{span}\{\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_n\}$, the space of all Laguerre polynomials of degree $\leq n$. The scaled Laguerre polynomials of degree n are defined by

$$\mathcal{L}_n(s) = \frac{1}{n!} e^s \partial_s^n (s^n e^{-s}), n = 0,1, \ldots$$

Also, Laguerre polynomials satisfy the following recurrence relation

$$(n+1)\mathcal{L}_{n+1}(s) = (2n+1-s)\mathcal{L}_n(s) - n\mathcal{L}_{n-1}(s), n \geq 1$$

and the orthogonality

$$< \mathcal{L}_n(s), \mathcal{L}_m(s) >_\omega = \delta_{nm}$$

Let $A^r(\mathbb{R}^+) = \left\{ v \mid \|v\|_{A^r(\mathbb{R}^+)} < \infty \right\}, r \geq 0$ (see [14]),

where the semi-norm and norm are defined by

$$|v|_{A^r(\mathbb{R}^+)} = \|\partial_s^r v\|_{\omega_r} \text{ and } \|v\|_{A^r(\mathbb{R}^+)} = \left(\sum_{j=0}^{r} |v|^2_{A^j(\mathbb{R}^+)} \right)^{\frac{1}{2}}$$

where

$$\|\partial_s^r v\|_{\omega_r} = \left(\int_0^\infty s^r e^{-s} |\partial_s^r v|^2 ds \right)^{\frac{1}{2}}$$

Clearly

$$\|\partial_s^r v\|_{\omega_r} \leq (\Gamma(r+1))^{\frac{1}{2}} \|\partial_s^r v\|_\infty, \text{ where } (\Gamma(r+1))^{\frac{1}{2}} = \left(\int_0^\infty s^r e^{-s} ds \right)^{\frac{1}{2}}. \tag{2.7}$$

Also, the following hold

$$\|v\|_{A^0(\mathbb{R}^+)} = \|v\|_{\omega}, v \in L^2_\omega(\mathbb{R}^+).$$

2.1. Orthogonal Projection

Let $\pi_{n,\omega}: L^2_\omega(\mathbb{R}^+) \to \mathbf{X}_n$ be the orthogonal projection defined by

$$< \pi_{n,\omega} u, \psi >_\omega = < u, \psi >_\omega \text{ , for all } \psi \in \mathbf{X}_n, u \in L^2_\omega(\mathbb{R}^+). \tag{2.8}$$

The following properties of $\pi_{n,\omega}$ (see [13,14]), It is very important to discuss the convergence results:

For $v \in A^r(\mathbb{R}^+)$,

(i) $\left\| \pi_{n,\omega} v - v \right\|_{A^{r'}(\mathbb{R}^+)} \leq c n^{\frac{r'-r}{2}} |v|_{A^r(\mathbb{R}^+)}, 0 \leq r' < r,$

(ii) $\left\| \pi_{n,\omega} \right\|_\omega \leq c.$

Consider the following equation

$$u_n - \pi_{n,\omega} \mathcal{K}(u_n) = \pi_{n,\omega} y \tag{2.9}$$

where $u_n \in \mathbf{X}_n$ be the approximate solution in the Galerkin method.

The iterated approximate solution for equation (2.9) is defined as follows:

$$\tilde{u}_n = y + \mathcal{K}(u_n) \tag{2.10}$$

Define $\mathbb{T}_n$ as

$$\mathbb{T}_n(u_n) = \pi_{n,\omega} y + \pi_{n,\omega} \mathcal{K}(u_n), \tag{2.11}$$

then (2.9) becomes

$$\mathbb{T}_n(u_n) = u_n. \tag{2.12}$$

Applying $\pi_{n,\omega}$ on both sides of (2.10)

$$\pi_{n,\omega}\tilde{u}_n = \pi_{n,\omega}y + \pi_{n,\omega}\mathcal{K}(u_n) \tag{2.13}$$

Combining (2.9) and (2.13)

$$\pi_{n,\omega}\tilde{u}_n = u_n \tag{2.14}$$

Substituting (2.14) in (2.10)

$$\tilde{u}_n = y + \mathcal{K}\left(\pi_{n,\omega}\tilde{u}_n\right) \tag{2.15}$$

Let $\mathbb{S}_n(u) = y + \mathcal{K}\left(\pi_{n,\omega}u\right),\ u \in L^2_\omega(0,\infty)$, so that the equation (2.15) becomes

$$\mathbb{S}_n(\tilde{u}_n) = \tilde{u}_n \tag{2.16}$$

The Frechet derivatives of $\mathbb{T}_n$ and $\mathbb{S}_n$ at u_0 are as follows

$$\mathbb{T}'_n(u_0) = \pi_{n,\omega}\mathcal{K}'(u_0) \tag{2.17}$$

$$\mathbb{S}'_n(u_0) = \mathcal{K}'\left(\pi_{n,\omega}u_0\right)\pi_{n,\omega} \tag{2.18}$$

Note that $\mathbb{T}'_n(u_0), \mathbb{S}'_n(u_0)$ are linear operators from $L^2_\omega(\mathbb{R}^+)$ into $L^2_\omega(\mathbb{R}^+)$. Next, we need to discuss the following results, which are crucial for the convergence analysis.

3. Convergence Analysis

Here, the existence and uniqueness of the approximate solution in the Galerkin method are discussed.

Lemma 3.1*:* Let 1 be not an eigenvalue of $\mathcal{K}'(u_0)$, where u_0 is the unique solution of Eq. (2.2). Then $\left(\mathcal{I} - \mathbb{T}'_n(u_0)\right)^{-1} \in L^2_\omega(\mathbb{R}^+)$and $\left\|\left(\mathcal{I} - \mathbb{T}'_n(u_0)\right)^{-1}\right\|_\omega \leq L_1 < \infty$, where L_1 is a constant and n is sufficiently large.

Proof: From (2.17), we have $\mathbb{T}'_n(u_0) = \pi_{n,\omega}\mathcal{K}'(u_0)$. Consider

$$\begin{aligned}
&\mathcal{I} + \left[\mathcal{K}'(u_0) - \pi_{n,\omega}\mathcal{K}'(u_0)\right]\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1} \\
&\qquad = \mathcal{I} + \left[\mathcal{I} - \left(\mathcal{I} - \mathcal{K}'(u_0)\right) - \pi_{n,\omega}\mathcal{K}'(u_0)\right]\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1} \\
&\qquad = \mathcal{I} + \left[\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1} - \mathcal{I} - \pi_{n,\omega}\mathcal{K}'(u_0)\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1}\right] \\
&\qquad = \left(\mathcal{I} - \pi_{n,\omega}\mathcal{K}'(u_0)\right)\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1} \ (2.22)
\end{aligned}$$

Now using (2.7), A1 and Cauchy-Schwarz inequality, for any $v \in L^2_\omega(\mathbb{R}^+)$, there holds

$$\begin{aligned}
&\left\|\left[\mathcal{K}'(u_0) - \pi_{n,\omega}\mathcal{K}'(u_0)\right]\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1}v\right\|_\omega \\
&\qquad = \left\|\left(\mathcal{I} - \pi_{n,\omega}\right)\left[\mathcal{K}'(u_0)\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1}v\right]\right\|_\omega \\
&\qquad \leq n^{-\frac{r}{2}}\left(\mathcal{K}'(u_0)\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1}v\right)\Big|_{A^r(\mathbb{R}^+)}
\end{aligned}$$

$$= n^{-\frac{r}{2}} \left\| \left(\mathcal{K}'(u_0) \left(\mathcal{I} - \mathcal{K}'(u_0) \right)^{-1} v \right)^{(r)} \right\|_{\omega_{(r)}}$$

$$\leq n^{-\frac{r}{2}} (\Gamma(r+1))^{\frac{1}{2}} \left\| \left(\mathcal{K}'(u_0) \left(\mathcal{I} - \mathcal{K}'(u_0) \right)^{-1} v \right)^{(r)} \right\|_{\infty}$$

$$= n^{-\frac{r}{2}} (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta \in \mathbb{R}^+} \left| \int_0^\infty \frac{\partial^r}{\partial \eta^r} k_u(\eta, s, u_0(s)) \left(\mathcal{I} - \mathcal{K}'(u_0) \right)^{-1} v(s) ds \right|$$

$$= n^{-\frac{r}{2}} (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta \in \mathbb{R}^+} \left| \int_0^\infty e^{-s} \frac{\partial^r}{\partial \eta^r} e^s k_u(\eta, s, u_0(s)) \left(\mathcal{I} - \mathcal{K}'(u_0) \right)^{-1} v(s) ds \right|$$

$$= n^{-\frac{r}{2}} (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta \in \mathbb{R}^+} \left| \int_0^\infty e^{-s} h_r(\eta, s, u_0(s)) \left(\mathcal{I} - \mathcal{K}'(u_0) \right)^{-1} v(s) ds \right|$$

$$\leq n^{-\frac{r}{2}} (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta \in \mathbb{R}^+} \int_0^\infty e^{-s} |h_r(\eta, s, u(s))| \left| \left(\mathcal{I} - \mathcal{K}'(u_0) \right)^{-1} v(s) \right| ds$$

$$\leq n^{-\frac{r}{2}} (\Gamma(r+1))^{\frac{1}{2}} \|h_r(.,.,u(.))\|_\omega \left\| \left(\mathcal{I} - \mathcal{K}'(u_0) \right)^{-1} \right\|_\omega \|v\|_\omega$$

$$\leq n^{-\frac{r}{2}} c_1 C_2 \|v\|_\omega \to 0 \text{ as } n \to \infty.$$

From above, it is clear that the operator $\mathcal{I} + \left[\mathcal{K}'(u_0) - \pi_{n,\omega} \mathcal{K}'(u_0) \right] \left(\mathcal{I} - \mathcal{K}'(u_0) \right)^{-1}$ is invertible. Hence, from (2.22), it follows that $\left(\mathcal{I} - \pi_{n,\omega} \mathcal{K}'(u_0) \right)^{-1}$ exists and is uniformly bounded in $L^2_\omega(\mathbb{R}^+)$, i.e.,

$$\left\| \left(\mathcal{I} - \mathbb{T}'_n(u_0) \right)^{-1} \right\|_\omega = \left\| \left(\mathcal{I} - \pi_{n,\omega} \mathcal{K}'(u_0) \right)^{-1} \right\|_\omega \leq L_1 < \infty.$$

Lemma 3.2: Let $x_1, x_2 \in L^2_\omega(\mathbb{R}^+)$, there hold

$$(i) \quad \|\mathbb{T}'(x_1) - \mathbb{T}'(x_2)\|_\omega = \|[\mathcal{K}'(x_1) - \mathcal{K}'(x_2)]\|_\omega \leq c_2 \|x_1 - x_2\|_\omega, \quad (2.23)$$

$$(ii) \quad \|\mathbb{T}'_n(x_1) - \mathbb{T}'_n(x_2)\|_\omega = \left\| \pi_{n,\omega} \left[\mathcal{K}'(x_1) - \pi_{n,\omega} \mathcal{K}'(x_2) \right] \right\|_\omega \leq b_1 \|x_1 - x_2\|_\omega. \quad (2.24)$$

Proof: (i) For any $x, x_1, x_2 \in L^2_\omega(\mathbb{R}^+)$, using A3 and Cauchy-Schwarz inequality, we have

$$\left\| \left(\mathbb{T}'(x_1) - \mathbb{T}'(x_2) \right) x \right\|_\omega = \left\| \left(\mathcal{K}'(x_1) - \mathcal{K}'(x_2) \right) x \right\|_\omega$$

$$\leq \left\| \left(\mathcal{K}'(x_1) - \mathcal{K}'(x_2) \right) x \right\|_\infty$$

$$= \sup_{\eta \in \mathbb{R}^+} |[\mathcal{K}'(x_1) - \mathcal{K}'(x_2)] x(\eta)|$$

$$= \sup_{\eta \in \mathbb{R}^+} \left| \int_0^\infty [k_u(\eta, s, x_1(s)) - k_u(\eta, s, x_2(s))] x(s) ds \right|$$

$$\leq \sup_{\eta \in \mathbb{R}^+} \int_0^\infty e^{-s} |e^s k_u(\eta, s, x_1(s)) - e^s k_u(\eta, s, x_2(s)) x(s)| ds$$

$$\leq \sup_{\eta \in \mathbb{R}^+} \int_0^\infty e^{-s} |l_0(\eta, s, x_1(s)) - l_0(\eta, s, x_2(s)) x(s)| ds$$

$$\leq \int_0^\infty e^{-s} c_2 |x_1(s) - x_2(s)\|x(s)| ds$$

$$\leq c_2 \|x_1 - x_2\|_\omega \|x\|_\omega$$

Hence, we obtain $\|[\mathbb{T}'(x_1) - \mathbb{T}'(x_2)]\|_\omega \leq c_2 \|x_1 - x_2\|_\omega$, which proves (2.23). In a similar procedure, one can obtain (2.24).

Let

$$\mathbf{X}_r^+ = \{v \in \mathbf{X}^+ \mid \|v\|_{r,\infty} < \infty\}, \text{ where } \|v\|_{r,\infty} = \sup_{s \in \mathbb{R}^+} |D^r u(s)|$$

and $B(u_0, \epsilon) = \{u \in L_\omega^2(\mathbb{R}^+) : \|u - u_0\|_\omega < \epsilon\}, \tilde{B}(u_0, \epsilon) = \{u \in \mathbf{X}^+ : \|u - u_0\|_\infty < \epsilon\}$ for some $0 < \epsilon < 1$.
We next provide the following error estimate.

Theorem 3.3. Let the unique solution of Eq. (2.2), $u_0 \in \boldsymbol{X}_r^+, r \geq 1$, then the open ball $B(u_0, \epsilon)$ contains a unique solution u_n Eq. (2.9), where n is sufficiently large, and the following holds

$$\frac{Y_n}{1+s} \leq \|u_n - u_0\|_\omega \leq \frac{Y_n}{1-s}, \text{ for } 0 < s < 1$$

where,

$$Y_n = \left\| \left(\mathcal{I} - \mathbb{T}_n'(u_0)\right)^{-1} \left(\mathbb{T}_n(u_0) - \mathbb{T}(u_0)\right) \right\|_\omega$$

and

$$\|u_n - u_0\|_\omega = \mathcal{O}\left(n^{-\frac{r}{2}}\right)$$

Proof: From Lemma 4 and for any $v \in B(u_0, \epsilon)$, we have

$$\|\mathbb{T}_n'(v) - \mathbb{T}_n'(u_0)\|_\omega = \left\|\pi_{n,\omega}\mathcal{K}'(v) - \pi_{n,\omega}\mathcal{K}'(u_0)\right\|_\omega$$

$$= \left\|\pi_{n,\omega}\mathcal{K}'(v) - \pi_{n,\omega}\mathcal{K}'(u_0)\right\|_\omega$$

$$\leq c b_1 \|u - u_0\|_\omega$$

Using the inequality $\left\| \left(\mathcal{I} - \mathbb{T}_n'(u_0)\right)^{-1} \right\|_\omega \leq L_1$, we obtain

$$\sup_{\|u - u_0\|_\omega \leq \epsilon} \left\| \left(\mathcal{I} - \mathbb{T}_n'(u_0)\right)^{-1} \left(\mathbb{T}_n'(u) - \mathbb{T}_n'(u_0)\right) \right\|_\infty \leq L_1 c b_1 \|u - u_0\|_\omega \leq L_1 c b_1 \epsilon = s$$

Here ϵ has chosen sufficiently small s.t $s \in (0,1)$, which proves (2.19) of Theorem 1. Next from Eqs. (2.5) and (2.11), we have

$$Y_n = \left\| \left(\mathcal{I} - \mathbb{T}_n'(u_0)\right)^{-1} \left(\mathbb{T}_n(u_0) - \mathbb{T}(u_0)\right) \right\|_\omega$$

$$\leq L_1 \|\mathbb{T}_n(u_0) - \mathbb{T}(u_0)\|_\omega$$

$$= L_1 \left\|\pi_{n,\omega}\mathcal{K}(u_0) + \pi_{n,\omega} y - \mathcal{K}(u_0) + y\right\|_\omega$$

$$= \left\|\left(\pi_{n,\omega} - \mathcal{I}\right) u_0\right\|_\omega$$

$$\leq n^{-\frac{r}{2}} \left\|u_0^{(r)}\right\|_\omega \quad (2.25)$$

From the estimate (2.25), we observe that $Y_n \to 0$ as $n \to \infty$. So for an $k_0 \in \mathbb{N}$, $Y_n \leq \epsilon(1-s)$ for all $n \geq k_0$, which agrees with Eq. (2.20) of Theorem 1. Hence, from Theorem 1, it follows that

$$\frac{\gamma_n}{1+q} \leq \|u_n - u_0\|_\omega \leq \frac{\gamma_n}{1-q} \tag{2.26}$$

where $\gamma_n = \| \left(\mathcal{J} - \left(\mathbb{T}'_n(u_0)\right)^{-1}\left(\mathbb{T}_n(u_0) - \mathbb{T}(u_0)\right)\|_\omega$.
Hence, using the estimate (2.25) in (2.26), we have

$$\begin{aligned} \|u_n - u_0\|_\omega \leq \frac{\gamma_n}{1-q} &= \frac{\left\|\left(\mathcal{J} - \mathbb{T}'_n(u_0)\right)^{-1}\left(\mathbb{T}_n(u_0) - \mathbb{T}(u_0)\right)\right\|_\omega}{1-q} \\ &\leq \mathcal{L}_1\left\|\left(\mathbb{T}_n(u_0) - \mathbb{T}(u_0)\right)\right\|_\omega \\ &= \mathcal{O}\left(n^{-\frac{r}{2}}\right). \end{aligned}$$

Lemma 3.4: Let $v \in \boldsymbol{X}^+, for\ j = 0,1,2 \ldots$ the following hold

$$(i)\left\|(\mathbb{S}'_n(u_0)v)^{(j)}\right\|_\omega \leq M_1 < \infty. \quad (2.27)$$

$$(ii)\left\|(\mathbb{S}'_n(u_0)v)^{(j)}\right\|_\infty \leq M_1 < \infty \quad (2.28)$$

Proof: For A1, A3, and for any $v \in \mathbf{X}^+$and $j = 0,1,2$ ..., we have

$$\begin{aligned} &\left\|(\mathbb{S}'_n(u_0)v)^{(j)}\right\|_\omega \leq \left\|(\mathbb{S}'_n(u_0)v)^{(j)}\right\|_\infty \\ &= \sup_{\eta\in\mathbb{R}^+}\left|(\mathbb{S}'_n(u_0)v)^{(j)}(\eta)\right| \\ &= \sup_{\eta\in\mathbb{R}^+}\left|\left(\mathcal{K}'\left(\pi_{n,\omega}u_0\right)\left(\pi_{n,\omega}v\right)\right)^{(j)}(\eta)\right| \\ &= \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty \frac{\partial^j}{\partial\eta^j}k_u\left(\eta,s,\pi_{n,\omega}u_0(s)\right)\left(\pi_{n,\omega}\right)v(s)ds\right| \\ &= \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}\frac{\partial^j}{\partial\eta^j}e^s k_u\left(\eta,s,\pi_{n,\omega}u_0(s)\right)\left(\pi_{n,\omega}\right)v(s)ds\right| \\ &= \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}l_j\left(\eta,s,\pi_{n,\omega}u_0(s)\right)\left(\pi_{n,\omega}\right)v(s)ds\right| \\ &= \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}\left[l_j\left(\eta,s,\pi_{n,\omega}u_0(s)\right) - l_j(\eta,s,u_0(s)) + l_j(\eta,s,u_0(s))\right]\left(\pi_{n,\omega}\right)v(s)ds\right| \\ &\leq \sup_{\zeta\in\mathbb{R}^+}\left[\int_0^\infty e^{-s}\left|l_j\left(\eta,s,\pi_{n,\omega}u_0(s)\right) - l_j(\eta,s,u_0(s))\right|\left|\left(\pi_{n,\omega}v\right)(s)\right|ds + \int_0^\infty e^{-s}\left|l_j(\eta,s,u_0(s))\right|\left|\left(\pi_{n,\omega}v\right)(s)\right|ds\right] \end{aligned}$$

hence, we obtain

$$\left\|(\mathbb{S}_n'(u_0)v)^{(j)}\right\|_\omega \leq \sup_{\eta\in\mathbb{R}^+}\left[\int_0^\infty e^{-s}c_2\left|(\pi_{n,\omega}-\mathcal{I})u_0(s)\right|\left|(\pi_{n,\omega}v)(s)\right|ds + \left\|l_j(.,.,u_0)\right\|_\omega\left\|\pi_{n,\omega}v\right\|_\omega\right]$$

$$\leq c_2\left\|(\pi_{n,\omega}-\mathcal{I})u_0\right\|_\omega\left\|\pi_{n,\omega}v\right\|_\omega + c_1c\|v\|_\omega$$

$$\leq cc_2(1+c)\|u_0\|_\omega\|v\|_\omega + c_1c\|v\|_\omega$$

$$= [cc_2(1+c)\|u_0\|_\omega + c_1c]\|v\|_\omega$$

$$= M_1\|v\|_\omega \leq M_1\|v\|_\infty \quad (2.29)$$

where $M_1 = [cc_2(1+c)\|u_0\|_\omega + c_1c]$, which proves (2.28). Now using (2.1) and (2.29), we have

$$\left\|(\mathbb{S}_n'(u_0)v)^{(j)}\right\|_\omega \leq \left\|(\mathbb{S}_n'(u_0)v)^{(j)}\right\|_\infty \leq M_1\|v\|_\omega$$

Hence (2.27) follows.
From now onwards, we assume that for $i,j = 0,1,2,\ldots$

$$\left(\int_0^\infty s^i e^{-s}\left|\frac{\partial^i}{\partial s^i}l_j(\eta,s,u_0(s))\right|^2 ds\right)^{\frac{1}{2}} \leq c_3 < \infty (2.30)$$

Theorem 3.5: For sufficiently large n, the following result holds

$$\begin{aligned}(i)\ \left\|\left(\mathcal{I}-\mathbb{S}_n'(u_0)\right)^{-1}\right\|_\omega \leq L_2 < \infty\\ (ii)\ \left\|\left(\mathcal{I}-\mathbb{S}_n'(u_0)\right)^{-1}\right\|_\infty \leq L_2 < \infty\end{aligned}\quad (2.32)$$

where L_2 is a constant that does not depend on n.
Proof: Let $v \in \mathbf{X}^+$, consider

$$\left\|\left(\mathcal{T}'(u_0)-\mathbb{S}_n'(u_0)\right)v\right\|_\omega = \left\|\left[\mathcal{K}'(u_0)-\mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega}\right]v\right\|_\omega$$

$$\leq \left\|\left[\mathcal{K}'(u_0)-\mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega}\right]v\right\|_\omega$$

$$= \left\|\mathcal{K}'(u_0)v-\mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega}v\right\|_\infty$$

$$= \left\|\mathcal{K}'(u_0)\left[\pi_{n,\omega}+(\mathcal{I}-\pi_{n,\omega})\right]v-\mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega}v\right\|_\infty$$

$$\leq \left\|\left[\mathcal{K}'(u_0)-\mathcal{K}'(\pi_{n,\omega}u_0)\right]\pi_{n,\omega}v\right\|_\infty + \left\|\mathcal{K}'(u_0)(\mathcal{I}-\pi_{n,\omega})v\right\|_\infty \quad (2.33)$$

Using the assumption A3 and the Cauchy-Schwarz inequality, the first term of the estimate (2.33) becomes

$$\left\|\left[\mathcal{K}'(u_0)-\mathcal{K}'(\pi_{n,\omega}u_0)\right]\pi_{n,\omega}v\right\|_\infty$$

$$= \sup_{\eta\in\mathbb{R}^+}\left|\left[\mathcal{K}'(u_0)-\mathcal{K}'(\pi_{n,\omega}u_0)\right]\pi_{n,\omega}v(\eta)\right|$$

$$= \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}\left[e^s k_u(\eta,s,u_0(s))-e^s k_u(\eta,s,\pi_{n,\omega}u_0(s))\right]\pi_{n,\omega}v(s)ds\right|$$

$$\leq \sup_{\eta\in\mathbb{R}^+}\int_0^\infty e^{-s}\left|l_0(\eta,s,u_0(s))-l_0(\eta,s,\pi_{n,\omega}u_0(s))\right|\left|\pi_{n,\omega}v(s)\right|ds$$

$$\leq \int_0^\infty c_2 \left|(\mathcal{J}-\pi_{n,\omega})u_0(s)\right|\left|\pi_{n,\omega}v(s)\right|ds$$

$$\leq c_2\left\|(\mathcal{J}-\pi_{n,\omega})u_0\right\|_\omega \left\|\pi_{n,\omega}v\right\|_\omega.$$

Hence, from above, we obtain

$$\left\|\left[\mathcal{K}'(u_0)-\mathcal{K}'(\pi_{n,\omega}u_0)\right]\pi_{n,\omega}v\right\|_\omega \leq \left\|\left[\mathcal{K}'(u_0)-\mathcal{K}'(\pi_{n,\omega}u_0)\right]\pi_{n,\omega}v\right\|_\infty$$

$$\leq cc_2 n^{-\frac{r}{2}}|u_0|_{A^r(\mathbb{R}^+)}\|v\|_\omega \leq cc_2 n^{-\frac{r}{2}}|u_0|_{A^r(\mathbb{R}^+)}\|v\|_\infty \quad (2.34)$$

Again, the second term of (2.33) and the orthogonality of $\pi_{n,\omega}$ leads to

$$\left\|\mathcal{K}'(u_0)(\mathcal{J}-\pi_{n,\omega})v\right\|_\omega \leq \left\|\mathcal{K}'(u_0)(\mathcal{J}-\pi_{n,\omega})v\right\|_\infty$$

$$= \sup_{\eta\in\mathbb{R}^+}\left|\mathcal{K}'(u_0)(\mathcal{J}-\pi_{n,\omega})v(\eta)\right|$$

$$= \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty k_u(\eta,s,u_0(s))(\mathcal{J}-\pi_{n,\omega})v(s)ds\right|$$

$$= \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}e^{s}k_u(\eta,s,u_0(s))(\mathcal{J}-\pi_{n,\omega})v(s)ds\right|$$

$$= \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}l_0(\eta,s,u_0(s))(\mathcal{J}-\pi_{n,\omega})v(s)ds\right|$$

$$= \sup_{\eta\in\mathbb{R}^+} < l_0(\eta.,u_0(.)),(\mathcal{J}-\pi_{n,\omega})v(.) >_\omega$$

$$= \sup_{\eta\in\mathbb{R}^+} < (\mathcal{J}-\pi_{n,\omega})l_0(\eta.,u_0(.)),v(.) >_\omega$$

$$\leq \sup_{\eta\in\mathbb{R}^+} n^{-\frac{r}{2}}|l_0(\eta.,u_0(.))|_{A^r(\mathbb{R}^+)}\|v\|_\omega$$

$$= n^{-\frac{r}{2}}\left\|\left(l_0(\eta.,u_0(.))\right)^{(r)}\right\|_{\omega_r}\|v\|_\omega$$

$$\leq n^{-\frac{r}{2}}c_3\|v\|_\omega \quad (2.35)$$

$$\leq n^{-\frac{r}{2}}c_3\|v\|_\infty \quad (2.36)$$

Using (2.33), (2.34), and (2.35), we see that

$$\|\mathbb{T}'(u_0)-\mathbb{S}_n'(u_0)\|_\infty = \mathcal{O}\left(n^{-\frac{r}{2}}\right) \to 0 \text{ as } n\to\infty.$$

Also using (2.1), (2.33), (2.34), and (2.36), we have

$$\|\mathbb{T}'(u_0)-\mathbb{S}_n'(u_0)\|_\omega \leq \|\mathbb{T}'(u_0)-\mathbb{S}_n'(u_0)\|_\infty$$

$$\leq n^{-\frac{r}{2}}\left[cc_2|u_0|_{A^r(0,\infty)}+c_3\right]\|v\|_\omega.$$

This implies

$$\|\mathbb{T}'(u_0) - \mathbb{S}'_n(u_0)\|_\omega = \mathcal{O}\left(n^{-\frac{r}{2}}\right) \to 0 \text{ for } n \to \infty.$$

Now $\left\|\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1}\right\|_\infty \leq C_1 < \infty$, and $\left\|\left(\mathcal{I} - \mathcal{K}'(u_0)\right)^{-1}\right\|_\omega \leq C_2$, using Lemma 2, the estimates (2.31) and (2.32) follow.

Lemma 3.6: Let $v_1, v_2 \in \boldsymbol{X}^+$, we have

$$(i)\|\mathbb{S}'_n(v_1) - \mathbb{S}'_n(v_2)\|_\omega \leq b_2\|v_1 - v_2\|_\omega, (2.37)$$
$$(ii)\|\mathbb{S}'_n(v_1) - \mathbb{S}'_n(v_2)\|_\infty \leq b_2\|v_1 - v_2\|_\infty (2.38)$$

where b_2 is a constant.

Proof: For any $v \in L^2_\omega(\mathbb{R}^+)$and using A3, we have

$$\begin{aligned}
\|[\mathbb{S}'_n(v_1) - \mathbb{S}'_n(v_2)]v\|_\omega &\leq \|[\mathbb{S}'_n(v_1) - \mathbb{S}'_n(v_2)]v\|_\infty \\
&= \left\|\left[\mathcal{K}'\left(\pi_{n,\omega}v_1\right) - \mathcal{K}'\left(\pi_{n,\omega}v_2\right)\right]\pi_{n,\omega}v\right\|_\infty \\
&= \sup_{\eta\in\mathbb{R}^+}\left|\left[\mathcal{K}'\left(\pi_{n,\omega}v_1\right) - \mathcal{K}'\left(\pi_{n,\omega}v_2\right)\right]\left(\pi_{n,\omega}v\right)(\eta)\right| \\
&= \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}\left[e^s k_u\left(\eta, s, \pi_{n,\omega}v_1(s)\right) - e^s k_u\left(\eta, s, \pi_{n,\omega}v_1(s)\right)\right]\left(\pi_{n,\omega}v\right)(s)ds\right| \\
&\leq \sup_{\eta\in\mathbb{R}^+}\int_0^\infty e^{-s}\left|l_0\left(\eta, s, \pi_{n,\omega}v_1(s)\right) - l_0\left(\eta, s, \pi_{n,\omega}v_1(s)\right)\right|\left|\left(\pi_{n,\omega}v\right)(s)\right|ds \\
&\leq \int_0^\infty e^{-s}c_2\left|\pi_{n,\omega}(v_1 - v_2)(s)\right|\left|\left(\pi_{n,\omega}v\right)(s)\right|ds \\
&\leq c_2\left\|\pi_{n,\omega}(v_1 - v_2)\right\|_\omega\left\|\pi_{n,\omega}v\right\|_\omega \\
&\leq c_2c^2\|v_1 - v_2\|_\omega\|v\|_\omega \\
&= b_2\|v_1 - v_2\|_\omega\|v\|_\omega \ (2.39) \\
&\leq b_2\|v_1 - v_2\|_\infty\|v\|_\infty \ (2.40)
\end{aligned}$$

where $b_2 = c_2c^2$. From (2.40), we have

$$\|\mathbb{S}'_n(v_1) - \mathbb{S}'_n(v_2)\|_\omega \leq b_2\|v_1 - v_2\|_\omega$$

Using (2.1) and (2.40) for any $v \in \mathrm{X}^+$, we obtain

$$\|[\mathbb{S}'_n(v_1) - \mathbb{S}'_n(v_2)]v\|_\infty \leq b_2\|v_1 - v_2\|_\omega\|v\|_\omega \leq b_2\|v_1 - v_2\|_\infty\|v\|_\infty$$

which proves (2.38).
Hereafter, we assume, for $j = 0,1,2, \ldots$
B1. $g_j(\eta, s, u(s)) = \frac{\partial^j}{\partial\eta^j}e^s k(\eta, s, u(s))$, for $u \in L^2_\omega(\mathbb{R}^+)$,
B2. $\frac{\partial}{\partial u}g_j(\eta, s, u(s)) = g_{j,u}(\eta, s, u(s))$,

B3. $\sup_{\eta,s\in\mathbb{R}^+}\left|\frac{\partial^j}{\partial\eta^j}g_{j,u}(\eta,s,u_0(s))\right|\le c_4$,

B4. $g_{j,u}(\eta,s,u(s))$ satisfies the Lipshitz condition in the third variable, i.e., $v_1,v_2\in L^2_\omega(\mathbb{R}^+)$, $\exists c_5$ independent of n such that

$$\left|g_{j,u}(.,.,v_1(.))-g_{j,u}(.,.,v_2(.))\right|\le c_5|v_1-v_2|$$

Lemma 3.7 For $j=0,1,2,\ldots$, the following result holds

$$(i)\ \left\|\left(\mathcal{K}(\pi_{n,\omega}u_0)-\mathcal{K}(u_0)\right)^{(j)}\right\|_\omega=\mathcal{O}(n^{-r})\qquad(2.41)$$

$$(ii)\ \left\|\left(\mathcal{K}(\pi_{n,\omega}u_0)-\mathcal{K}(u_0)\right)^{(j)}\right\|_\infty=\mathcal{O}(n^{-r})\qquad(2.42)$$

Proof: Using the Mean-value Theorem, the Cauchy-Schwarz inequality, (2.30), and B3, we obtain

$$\|\left(\mathcal{K}(\pi_{n,\omega}u_0)-\mathcal{K}(u_0)\right)^{(j)}\|_\omega$$

$$\le\left\|\left(\mathcal{K}(\pi_{n,\omega}u_0)-\mathcal{K}(u_0)\right)^{(j)}\right\|_\infty$$

$$=\sup_{\eta\in\mathbb{R}^+}\left|\left(\mathcal{K}(\pi_{n,\omega}u_0)-\mathcal{K}(u_0)\right)^{(j)}(\eta)\right|$$

$$=\sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty\left[\frac{\partial^j}{\partial\eta^j}k(\eta,s,\pi_{n,\omega}u_0(s))-\frac{\partial^j}{\partial\eta^j}k(\eta,s,u_0(s))\right]ds\right|$$

$$=\sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}\left[\frac{\partial^j}{\partial\eta^j}e^sk(\eta,s,\pi_{n,\omega}u_0(s))-\frac{\partial^j}{\partial\eta^j}e^sk(\eta,s,u_0(s))\right]ds\right|$$

$$=\sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}\left[g_j(\eta,s,\pi_{n,\omega}u_0(s))-g_j(\eta,s,u_0(s))\right]ds\right|$$

$$=\sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}g_{j,u}\left(\eta,s,\left(u_0+\theta(\mathcal{I}-\pi_{n,\omega})u_0\right)(s)\right)(\mathcal{I}-\pi_{n,\omega})u_0(s)\Big)ds\right|$$

$$=\sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}\left[g_{j,u}\left(\eta,s,\left(u_0+\theta(\mathcal{I}-\pi_{n,\omega})u_0\right)(s)\right)-g_{j,u}(\eta,s,u_0(s))+g_{j,u}(\eta,s,u_0(s))\right](\mathcal{I}-\pi_{n,\omega})u_0)(s)\right|$$

$$\le\sup_{\eta\in\mathbb{R}^+}\left[\int_0^\infty e^{-s}\left|g_{j,u}\left(\eta,s,\left(u_0+\theta(\mathcal{I}-\pi_{n,\omega})u_0\right)(s)\right)-g_{j,u}(\eta,s,u_0(s))\right|\left|\left((\mathcal{I}-\pi_{n,\omega})u_0\right)(s)\right|ds+\left|\int_0^\infty e^{-s}g_{j,u}(\eta,s,u_0(s))\left((\mathcal{I}-\pi_{n,\omega})u_0\right)(s)ds\right|\right]$$

$$\le\int_0^\infty e^{-s}c_5\left|\left((\mathcal{I}-\pi_{n,\omega})u_0\right)(s)\right|^2ds+\sup_{\eta\in\mathbb{R}^+}<g_{j,u}(\eta.,u_0(.),(\mathcal{I}-\pi_{n,\omega})u_0)>_\omega$$

$$=c_5\left\|(\mathcal{I}-\pi_{n,\omega})u_0\right\|_\omega^2+\sup_{\eta\in\mathbb{R}^+}<(\mathcal{I}-\pi_{n,\omega})g_{j,u}(\eta,.,u_0(.),(\mathcal{I}-\pi_{n,\omega})u_0)>\omega$$

$$\le c_5n^{-r}|u_0|^2_{A^r(\mathbb{R}^+)}+n^{-r}\sup_{\eta\in\mathbb{R}^+}\left|g_{j,u}(\eta,.,u_0(.))\right|_{A^r(\mathbb{R}^+)}|u_0|_{A^r(\mathbb{R}^+)}$$

$$= c_5 n^{-r} |u_0|^2_{A^r(\mathbb{R}^+)} + n^{-r} \sup_{\eta\in\mathbb{R}^+} \left\| \partial_s^r g_{j,u}(\eta, ., u_0(.)) \right\|_{\omega_r} |u_0|_{A^r(\mathbb{R}+)}$$

$$\leq n^{-r} |u_0|_{A^r(\mathbb{R}^+)} \left(c_5 |u_0|_{A^r(\mathbb{R}^+)} + c_4 \right)$$

$$= \mathcal{O}(n^{-r})$$

where $0 < \theta < 1$. Similarly using $\left\| \left(\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0) \right)^{(j)} \right\|_{\omega} \leq \left\| \left(\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0) \right)^{(j)} \right\|_{\infty}$, we can obtain (2.42).

Theorem 3.8: Let $u_0 \in \boldsymbol{X}_r^+, r \geq 1$ be the unique solution of Eq. (2.4), then the following holds

$$\frac{\gamma_n}{1+q} \leq \|\tilde{u}_n - u_0\|_\infty \leq \frac{\gamma_n}{1-q}, \text{ for } 0 < q < 1$$

where $\gamma_n = \|\mathcal{I} - \mathbb{S}_n'(u_0))^{-1}(\mathbb{S}_n(u_0) - \mathbb{T}(u_0))\|_\infty$

and

(i) $\|\tilde{u}_n - u_0\|_\infty = \mathcal{O}(n^{-r})$,

(ii) $\|\tilde{u}_n - u_0\|_\omega = \mathcal{O}(n^{-r})$ 。

Proof: Using Theorem 7 and Lemma 8, we have that

$$\sup_{\|u-u_0\|_\infty \leq \epsilon} \left\| \left(\mathcal{I} - \mathbb{S}_n'(u_0) \right)^{-1} \left(\mathbb{S}_n'(u) - \mathbb{S}_n'(u_0) \right) \right\|_\infty \leq L_2 b_2 \|u - u_0\|_\infty \leq L_2 b_2 \epsilon = q$$

where we choose ϵ small enough s.t $q \in (0,1)$, which satisfies Eq. (2.19) of Theorem 1 (see [26]).

Now using Lemma 9 for $j = 0$, we obtain

$$\gamma_n = \left\| \left(\mathcal{I} - \mathbb{S}_n'(u_0) \right)^{-1} \left(\mathbb{S}_n(u_0) - \mathbb{T}(u_0) \right) \right\|_\infty$$

$$\leq L_2 \left\| \left(\mathbb{S}_n(u_0) - \mathbb{T}(u_0) \right) \right\|_\infty$$

$$= L_2 \left\| \mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0) \right\|_\infty$$

$$= \mathcal{O}(n^{-r}) \to 0 \text{ as n} \to \infty \quad (2.45)$$

Hence, we obtain that there exists an integer $k_0 \in \mathbb{N}$ s.t $\gamma_n \leq \epsilon(1-q)$ for all $n \geq k_0$, that agree with (2.20) of Theorem 1(see [26]). Hence, it follows from Theorem 1 (see [26]) that

$$\frac{\gamma_n}{1+q} \leq \|\tilde{u}_n - u_0\|_\infty \leq \frac{\gamma_n}{1-q} \qquad (2.46)$$

where $\gamma_n = \left\| \left(\mathcal{I} - \mathbb{S}_n'(u_0) \right)^{-1} \left(\mathbb{S}_n(u_0) - \mathbb{T}(u_0) \right) \right\|_\infty$.

Now using (2.45) and (2.46)

$$\|\tilde{u}_n - u_0\|_\infty \leq \frac{\gamma_n}{1-q} = \frac{\left\| \left(\mathcal{I} - \mathbb{S}_n'(u_0) \right)^{-1} \left(\mathbb{S}_n(u_0) - \mathcal{T}(u_0) \right) \right\|_\infty}{1-q}$$

$$\leq \mathcal{L}_2 \left\| \left(\mathbb{S}_n(u_0) - \mathbb{T}(u_0) \right) \right\|_\infty$$

$$= \mathcal{O}(n^{-r})$$

Using $\|\tilde{u}_n - u_0\|_\omega \leq \|\tilde{u}_n - u_0\|_\infty$, (2.44) obtained.

4. Superconvergence Results for Nonlinear-Urysohn Integral Equations on $\mathbb{R}^+$.

Here we discuss the results for the multi-Galerkin and its iterated approximate solutions. To do this, define the multi-projection operator (see [4,6, 7, 16-18, 21, 22]) given by

$$\mathcal{K}_n^M(u) = \pi_{n,\omega}\mathcal{K}(u) + \mathcal{K}(\pi_{n,\omega}u) - \pi_{n,\omega}\mathcal{K}(\pi_{n,\omega}u), u \in L_\omega^2(\mathbb{R}^+) \quad (3.1)$$

In this method, we approximate Eq. (2.4) by

$$u_n^M - \mathcal{K}_n^M(u_n^M) = y \quad (3.2)$$

The iterated multi-Galerkin solution is defined as

$$\tilde{u}_n^M = \mathcal{K}(u_n^M) + y \quad (3.3)$$

Define

$$\mathcal{T}_n^M(u) = \mathcal{K}_n^M(u) + y, u \in L_\omega^2(\mathbb{R}^+) \quad (3.4)$$

and using this, Eq. (3.2) becomes

$$u_n^M = \mathcal{T}_n^M(u_n^M) \quad (3.5)$$

Define the Frechet derivative of $\mathcal{T}_n^M$ at $u = u_0$ by

$$\mathcal{T}_n^{M'}(u_0) = \mathcal{K}_n^{M'}(u_0) = \pi_{n,\omega}\mathcal{K}'(u_0) + \mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega} - \pi_{n,\omega}\mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega}. \quad (3.6)$$

Now we develop the following Lemmas and Theorems to establish the superconvergence results for the multi-Galerkin and its iterated method.

Lemma 4.1 Let u_0 be the unique solution of Eq. (2.4), then for any $\in L_\omega^2(0,\infty)$,

$$(i) \| \left((\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0) v \right)^{(r)} \|_{\omega_r} = \mathcal{O}\left(n^{-\frac{r}{2}}\right), \quad (3.7)$$

$$(ii) \left\| \left(\mathcal{K}'(\pi_{n,\omega}u_0)(\pi_{n,\omega} - \mathcal{I})v\right)^{(r)} \right\|_{\omega_r} = \mathcal{O}\left(n^{-\frac{r}{2}}\right). \quad (3.8)$$

Proof: Using Cauchy-Schwarz inequality, A3 and (2.7), for any $v \in L_\omega^2(\mathbb{R}^+)$, there holds

$$\begin{aligned}
&\left\| \left(\left(\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0) \right) v \right)^{(r)} \right\|_{\omega_r} \\
&\leq (\Gamma(r+1))^{\frac{1}{2}} \left\| \left(\left(\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0) \right) v \right)^{(r)} \right\|_\infty \\
&= (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta\in\mathbb{R}^+} \left| \left(\left(\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0) \right) v \right)^{(r)} (\eta) \right| \\
&= (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta\in\mathbb{R}^+} \left| \int_0^\infty \left[\frac{\partial^r}{\partial\eta^r} k_u(\eta, s, u_0(s)) - \frac{\partial^r}{\partial\eta^r} k_u(\eta, s, \pi_{n,\omega}u_0(s)) \right] v(s) ds \right| \\
&= (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta\in\mathbb{R}^+} \left| \int_0^\infty e^{-s} \left[\frac{\partial^r}{\partial\eta^r} e^s k_u(\eta, s, u_0(s)) - \frac{\partial^r}{\partial\eta^r} e^s k_u(\eta, s, \pi_{n,\omega}u_0(s)) \right] v(s) ds \right|
\end{aligned}$$

$$= (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta\in\mathbb{R}^+} \int_0^\infty e^{-s}\left|h_r(\eta,s,u_0(s)) - h_r\left(\eta,s,\pi_{n,\omega}u_0(s)\right)\right||v(s)|ds$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}} \int_0^\infty e^{-s}c_2\left|\left(\mathcal{I}-\pi_{n,\omega}\right)u_0(s)\right||v(s)|ds$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}} c_2\left\|\left(\mathcal{I}-\pi_{n,\omega}\right)u_0\right\|_\omega \|v\|_\omega$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}} c_2 n^{-\frac{r}{2}}|u_0|_{A^r(\mathbb{R}^+)}\|v\|_\omega$$

$$= \mathcal{O}\left(n^{-\frac{r}{2}}\right)$$

which proves (3.7).
Now for any $v \in L^2_\omega(\mathbb{R}^+)$, using A3 and the orthogonality of $\pi_{n,\omega}$ we have

$$\left\|\left(\mathcal{K}\left(\pi_{n,\omega}u_0\right)\left(\pi_{n,\omega}-\mathcal{I}\right)v\right)^{(r)}\right\|_{\omega_r}$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}} \left\|\left(\mathcal{K}'\left(\pi_{n,\omega}u_0\right)\left(\pi_{n,\omega}-\mathcal{I}\right)v\right)^{(r)}\right\|_\infty$$

$$= (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty \frac{\partial^r}{\partial\eta^r}k_u\left(\eta,s,\pi_{n,\omega}u_0(s)\right)\left(\pi_{n,\omega}-\mathcal{I}\right)v(s)ds\right|$$

$$= (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta\in\mathbb{R}^+}\left|\int_0^\infty e^{-s}\left[\frac{\partial^r}{\partial\eta^r}e^s k_u\left(\eta,s,\pi_{n,\omega}u_0(s)\right) - \frac{\partial^r}{\partial\eta^r}e^s k_u(\eta,s,u_0(s)) + \frac{\partial^r}{\partial\eta^r}e^s k_u(\eta,s,u_0(s))\right]\left(\pi_{n,\omega} - \mathcal{I}\right)v(s)ds\right|$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}} \sup_{\eta\in\mathbb{R}^+}\left[\int_0^\infty e^{-s}\left|h_r\left(\eta,s,\pi_{n,\omega}u_0(s)\right) - h_r(\eta,s,u_0(s))\right|\left|\left(\pi_{n,\omega}-\mathcal{I}\right)v(s)\right|ds + \left|\int_0^\infty e^{-s}h_r(\eta,s,u_0(s))\left|\left(\pi_{n,\omega}-\mathcal{I}\right)v(s)ds\right|\right]$$

hence using Cauchy-Schwarz inequality and Eq. (2.30), we obtain

$$\left\|\left(\mathcal{K}\left(\pi_{n,\omega}u_0\right)\left(\pi_{n,\omega}-\mathcal{I}\right)v\right)^{(r)}\right\|_{\omega_r}$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}}\left[\int_0^\infty e^{-s}c_2\left|\left(\pi_{n,\omega}-\mathcal{I}\right)u_0(s)\right|\left|\left(\pi_{n,\omega}-\mathcal{I}\right)v(s)\right|ds + \sup_{\eta\in\mathbb{R}^+}\left|< h_r(\eta,.,u_0(.)),\left(\pi_{n,\omega}-\mathcal{I}\right)v >_\omega\right|\right]$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}}\left[c_2\left\|\left(\pi_{n,\omega}-\mathcal{I}\right)u_0\right\|_\omega\left\|\left(\pi_{n,\omega}-\mathcal{I}\right)v\right\|_\omega + \sup_{\eta\in\mathbb{R}^+}\left|< \left(\pi_{n,\omega}-\mathcal{I}\right)h_r(\eta,.,u_0(.)),v >_\omega\right|\right]$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}}\left[c_2 n^{-\frac{r}{2}}|u_0|_{A^r(\mathbb{R}^+)}(1+c)\|v\|_\omega + \sup_{\eta\in\mathbb{R}^+}\left\|\left(\pi_{n,\omega}-\mathcal{I}\right)h_r(\eta,.,u_0(.))\right\|_\omega\|v\|_\omega\right]$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}}\left[c_2 n^{-\frac{r}{2}}|u_0|_{A^r(\mathbb{R}^+)}(1+c)\|v\|_\omega + n^{-\frac{r}{2}}\sup_{\eta\in\mathbb{R}^+}|h_r(\eta,.,u_0(.))|_{A^r(\mathbb{R}^+)}\|v\|_\omega\right]$$

$$\leq (\Gamma(r+1))^{\frac{1}{2}} n^{-\frac{r}{2}}\left[c_2(1+c)|u_0|_{A^r(\mathbb{R}^+)} + c_3\right]\|v\|_\omega$$

$$= \mathcal{O}\left(n^{-\frac{r}{2}}\right)$$

Hence, we obtain (3.8).

Theorem 4.2 For sufficiently large n, $\left\|\left(\mathcal{I} - \mathcal{T}_n^{M'}(u_0)\right)^{-1}\right\|_\omega \le L_3 < \infty$, where L_3 is a constant that does not depend on n.

Proof: Using Lemma 4.1, and $v \in L^2_\omega(\mathbb{R}^+)$, we obtain

$$\left\|\left[\mathcal{T}_n^{M'}(u_0) - (\mathcal{T})'(u_0)\right]v\right\|_\omega$$

$$= \left\|\left(\mathcal{K}_n^{M'}(u_0) - \mathcal{K}'(u_0)\right)v\right\|_\omega$$

$$= \left\|\left[\pi_{n,\omega}\mathcal{K}'(u_0) + \mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega} - \pi_{n,\omega}\mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega} - \mathcal{K}'(u_0)\right]v\right\|_\omega$$

$$= \left\|\left[(\pi_{n,\omega} - \mathcal{I})\mathcal{K}'(u_0) - (\pi_{n,\omega} - \mathcal{I})\mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega}\right]v\right\|_\omega$$

$$= \left\|(\pi_{n,\omega} - \mathcal{I})\left[\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega}\right]v\right\|_\omega$$

$$= \|(\pi_{n,\omega} - \mathcal{I})\left[(\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0) + \mathcal{K}'(\pi_{n,\omega}u_0) - \mathcal{K}'(\pi_{n,\omega}u_0)\pi_{n,\omega}\right]v\|_\omega$$

$$\le \left\|(\pi_{n,\omega} - \mathcal{I})\left[\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0)\right]v\right\|_\omega + \left\|(\pi_{n,\omega} - \mathcal{I})\mathcal{K}'(\pi_{n,\omega}u_0)(\pi_{n,\omega} - \mathcal{I})v\right\|_\omega$$

$$= \left\|(\pi_{n,\omega} - \mathcal{I})\left[\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0)\right]v\right\|_{A^0(\mathbb{R}^+)} + \left\|(\pi_{n,\omega} - \mathcal{I})\mathcal{K}'(\pi_{n,\omega}u_0)(\pi_{n,\omega} - \mathcal{I})v\right\|_{A^0(\mathbb{R}^+)}$$

$$\le n^{-\frac{r}{2}}\left|\left(\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0)\right)v\right|_{A^r(\mathbb{R}^+)} + n^{-\frac{r}{2}}\left|\mathcal{K}'(\pi_{n,\omega}u_0)(\pi_{n,\omega} - \mathcal{I})v\right|_{A^r(\mathbb{R}^+)}$$

$$= n^{-\frac{r}{2}}\left[\left\|\left(\left(\mathcal{K}'(u_0) - \mathcal{K}'(\pi_{n,\omega}u_0)\right)v\right)^{(r)}\right\|_{\omega_r} + \left\|\left(\mathcal{K}'(\pi_{n,\omega}u_0)(\pi_{n,\omega} - \mathcal{I})v\right)^{(r)}\right\|_{\omega_r}\right] \quad (3.9)$$

$$= \mathcal{O}(n^{-r}) \qquad (3.10)$$

From above, it follows that

$$\left\|\left[\mathcal{T}_n^{M'}(u_0) - \mathcal{T}'(u_0)\right]\right\|_\omega \to 0 \text{ as } n \to \infty$$

Thus $\mathcal{T}_n^{M'}(u_0)$ is norm convergent to $\mathcal{T}'(u_0)$. Hence, the Theorem proved.

Lemma 4.3 For $v_1, v_2 \in L^2_\omega(\mathbb{R}^+)$ The following holds

$$\|(\mathcal{T}_n^M)'(v_1) - (\mathcal{T}_n^M)'(v_2)\|_\omega \le b_5\|v_1 - v_2\|_\omega,$$

where b_5 is a constant.

Proof: For any $v_1, v_2 \in L^2_\omega(\mathbb{R}^+)$, we have

$$\mathcal{T}_n^{M'}(v_1) - \mathcal{T}_n^{M'}(v_2)\|_\omega$$

$$= \left\|\mathcal{K}_n^{M'}(v_1) - \mathcal{K}_n^{M'}(v_2)\right\|_\omega$$

$$= \|\left[\pi_{n,\omega}\mathcal{K}'(v_1) + (\mathcal{I} - \pi_{n,\omega})\mathcal{K}'(\pi_{n,\omega}v_1)\pi_{n,\omega} - \pi_{n,\omega}\mathcal{K}'(v_2) - (\mathcal{I} - \pi_{n,\omega})\mathcal{K}'(\pi_{n,\omega}v_2)\pi_{n,\omega}\right]\|_\omega$$

$$\leq \left\|\pi_{n,\omega}[\mathcal{K}'(v_1) - \mathcal{K}'(v_2)]\right\|_\omega + \left\|(\mathcal{I} - \pi_{n,\omega})\left[\mathcal{K}'(\pi_{n,\omega}v_1) - \mathcal{K}'(\pi_{n,\omega}v_2)\right]\pi_{n,\omega}\right\|_\omega \quad (3.11)$$

Hence, using Lemma 3.2, we obtain

$$\|\mathcal{T}_n^{M'}(v_1) - \mathcal{T}_n^{M'}(v_2)\|_\omega$$

$$\leq cb_1\|v_1 - v_2\|_\omega + \left(1 + \left\|\pi_{n,\omega}\right\|_\omega\right)\left\|\left[\mathcal{K}'(\pi_{n,\omega}v_1) - \mathcal{K}'(\pi_{n,\omega}v_2)\right]\pi_{n,\omega}\right\|_\omega$$

$$\leq cb_1\|v_1 - v_2\|_\omega + \left(1 + \left\|\pi_{n,\omega}\right\|_\omega\right)\left\|\pi_{n,\omega}\right\|_\omega\left[\mathcal{K}'(\pi_{n,\omega}v_1) - \mathcal{K}'(\pi_{n,\omega}v_2)\right]\|_\omega$$

$$\leq cb_1\|v_1 - v_2\|_\omega + (1 + c)\left\|\pi_{n,\omega}\right\|_\omega^2 b_1\|v_1 - v_2\|_\omega$$

$$\leq cb_1\|v_1 - v_2\|_\omega + (1 + c)c^2 b_1\|v_1 - v_2\|_\omega$$

$$= [cb_1 + (1 + c)c^2 b_1]\|v_1 - v_2\|_\omega$$

where $b_5 = [cb_1 + (1 + c)c^2 b_1]$.

In the following Theorems, we discuss the superconvergence results for the multi-Galerkin and iterated multi-Galerkin methods.

Theorem 4.4 Let $u_0 \in \boldsymbol{X}_r^+, r \geq 1$ be the unique solution of Eq. (2.4), the following hold

$$\frac{\gamma_n}{1+q} \leq \|u_n^M - u_0\|_\omega \leq \frac{\gamma_n}{1-q}, \text{ for } 0 < q < 1$$

where $\gamma_n = \left\|\left(\mathcal{I} - \mathcal{T}_n^{M'}(u_0)\right)^{-1}\left(\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0)\right)\right\|_\omega$,

and $\|u_n^M - u_0\|_\omega = \mathcal{O}\left(n^{-\frac{3r}{2}}\right)$

Proof: It follows from Theorem 4.2 and Lemma 4.3 that

$$\sup_{\|u-u_0\|\leq\epsilon}\|\left(\mathcal{I} - \left(\mathcal{T}_n^{M'}(u_0)\right)^{-1}\left(\left(\mathcal{T}_n^{M'}(u_0)\right)(u_0) - \left(\mathcal{T}_n^{M'}(u_0)\right)(u)\right)\|_\omega \leq L_3 b_5\|u - u_0\|_\omega \leq L_3 b_5\epsilon = q(say)$$

Now using (2.7), Lemma 3.7, and Theorem 4.2, we obtain

$$\gamma_n = \|\left(\mathcal{I} - \left(\mathcal{T}_n^{M'}(u_0)\right)^{-1}\left(\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0)\right)\|_\omega$$

$$\leq L_3\|\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0)\|_\infty$$

$$= L_3\left\|\pi_{n,\omega}\mathcal{K}(u_0) + \mathcal{K}(\pi_{n,\omega}u_0) - \pi_n\mathcal{K}(\pi_{n,\omega}u_0) - \mathcal{K}(u_0)\right\|_\omega$$

$$= L_3\left\|(\pi_{n,\omega} - \mathcal{I})\left[\mathcal{K}(u_0) - \mathcal{K}(\pi_{n,\omega}u_0)\right]\right\|_\omega$$

$$\leq L_3 n^{-\frac{r}{2}}\left|\mathcal{K}(u_0) - \mathcal{K}(\pi_{n,\omega}u_0)\right|_{A^r(\mathbb{R}^+)}$$

$$= L_3 n^{-\frac{r}{2}}\left\|\left(\mathcal{K}(u_0) - \mathcal{K}(\pi_{n,\omega}u_0)\right)^{(r)}\right\|_{\omega_r}$$

$$\leq L_3 n^{-\frac{r}{2}}(\Gamma(r+1))^{\frac{1}{2}} \left\| \left(\mathcal{K}(u_0) - \mathcal{K}(\pi_{n,\omega} u_0)\right)^{(r)} \right\|_\infty$$

$$= \mathcal{O}\left(n^{-\frac{3r}{2}}\right) \to 0 \text{ as } n \to \infty \quad (3.12),$$

which agrees with Eq. (2.20) of Theorem 1 (see [26]). Hence, it follows from Theorem 1 that

$$\frac{\gamma_n}{1+q} \leq \|u_n^M - u_0\|_\omega \leq \frac{\gamma_n}{1-q} \, (3.13)$$

where $\gamma_n = \left\| \left(\mathcal{I} - \mathcal{T}_n^{M'}(u_0)\right)^{-1} \left(\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0)\right) \right\|_\omega$.
Next, using (3.12) and (3.13), we have

$$\begin{aligned} \|u_n^M - u_0\|_\omega \leq \frac{\gamma_n}{1-q} &= \frac{\left\| \left(\mathcal{I} - \mathcal{T}_n^{M'}(u_0)\right)^{-1} \left(\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0)\right) \right\|_\omega}{1-q} \\ &\leq \mathcal{L}_3 \left\| \left(\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0)\right) \right\|_\omega \\ &= \mathcal{O}\left(n^{-\frac{3r}{2}}\right) \end{aligned}$$

Theorem 4.5 Let $u_0 \in \boldsymbol{X}_r^+, r \geq 1$ be the unique solution of Eq. (2.4), then for sufficiently large n, there holds

$$\begin{aligned} &\|u_0 - \tilde{u}_n^M\|_\omega \\ &\leq b(1 + c_4 L_3)\|u_n^M - u_0\|_\omega^2 + (1 + cc_4 L_3)\left\|\mathcal{K}'(u_0)(\mathcal{I} - \pi_{n,\omega})\left[\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0)\right]\right\|_\omega. \end{aligned}$$

Lemma 4.6: The following holds

$$\left\|\mathcal{K}'(u_0)(\mathcal{I} - \pi_{n,\omega})\left[\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0)\right]\right\|_\omega = \mathcal{O}(n^{-2r})$$

Proof: From the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} &\left\|\mathcal{K}'(u_0)(\mathcal{I} - \pi_{n,\omega})\left[\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0)\right]\right\|_\omega \\ &\leq \|\mathcal{K}'(u_0)(\mathcal{I} - \pi_{n,\omega})\left[\left(\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0)\right]\right\|_\infty \\ &= \sup_{\eta \in \mathbb{R}^+} \left|\mathcal{K}'(u_0)(\mathcal{I} - \pi_{n,\omega})\left(\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0)\right)(\eta)\right| \\ &= \sup_{\eta \in \mathbb{R}^+} \left| \int_0^\infty k_u(\eta, s, u_0(s))(\mathcal{I} - \pi_{n,\omega})\left(\left(\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0)\right)(s) ds \right| \\ &= \sup_{\eta \in \mathbb{R}^+} \left| \int_0^\infty e^{-s} e^s k_u(\eta, s, u_0(s))(\mathcal{I} - \pi_{n,\omega})\left(\left(\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0)\right)(s) ds \right| \\ &= \sup_{\eta \in \mathbb{R}^+} \left| \int_0^\infty e^{-s} l_0(\eta, s, u_0(s))(\mathcal{I} - \pi_{n,\omega})\left(\left(\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0)\right)(s) ds \right| \\ &= \sup_{\eta \in \mathbb{R}^+} \left| < l_0(\eta, ., u_0(.)), (\mathcal{I} - \pi_{n,\omega})\left(\mathcal{K}(\pi_{n,\omega} u_0) - \mathcal{K}(u_0)\right)(.) >_\omega \right| \end{aligned}$$

$$= \sup_{\eta\in\mathbb{R}^+} \left| < (\mathcal{J}-\pi_{n,\omega}) l_0(\eta,.,u_0(.)), (\mathcal{J}-\pi_{n,\omega}) \left(\mathcal{K}(\pi_{n,\omega}u_0) - \mathcal{K}(u_0)\right)(.) >_\omega \right|$$

$$\leq \sup_{\eta\in\mathbb{R}^+} \|(\mathcal{J}-\pi_{n,\omega}) l_0(\eta,.,u_0(.))\|_\omega \left\|(\mathcal{J}-\pi_{n,\omega}) \left(\mathcal{K}(\pi_{n,\omega}u_0) - \mathcal{K}(u_0)\right)\right\|_\omega$$

$$\leq \sup_{\eta\in\mathbb{R}^+} n^{-\frac{r}{2}} |l_0(\eta,.,u_0(.))|_{A^r(\mathbb{R}^+)} n^{-\frac{r}{2}} \left|\left(\mathcal{K}(\pi_{n,\omega}u_0) - \mathcal{K}(u_0)\right)\right|_{A^r(\mathbb{R}^+)}$$

$$= \sup_{\eta\in\mathbb{R}^+} n^{-r} |l_0(\eta,.,u_0(.))|_{A^r(\mathbb{R}^+)} \left|\left(\mathcal{K}(\pi_{n,\omega}u_0) - \mathcal{K}(u_0)\right)\right|_{A^r(\mathbb{R}^+)}$$

$$= \sup_{\eta\in\mathbb{R}^+} n^{-r} |l_0(\eta,.,u_0(.))|_{A^r(\mathbb{R}^+)} \left\|\left(\mathcal{K}(\pi_{n,\omega}u_0) - \mathcal{K}(u_0)\right)^{(r)}\right\|_{\omega_r}.$$

It follows from (2.7) and Lemma 9 that

$$\|\mathcal{K}'(u_0)(\mathcal{J}-\pi_{n,\omega})[\mathcal{K}(\pi_{n,\omega}u_0) - \mathcal{K}(u_0)]\|_\omega$$

$$= \sup_{\eta\in\mathbb{R}^+} n^{-r} \|l_0(\eta,.,u_0(.))^{(r)}\|_{\omega_r} \left\|\left(\mathcal{K}(\pi_{n,\omega}u_0) - \mathcal{K}(u_0)\right)^{(r)}\right\|_{\omega_r}$$

$$\leq c_3 n^{-r} (\Gamma(r+1))^{\frac{1}{2}} \left\|\left(\mathcal{K}(\pi_{n,\omega}u_0) - \mathcal{K}(u_0)\right)^{(r)}\right\|_{\infty}$$

$$= \mathcal{O}(n^{-2r})$$

Theorem 4.7 Let $u_0 \in L^2_\omega(\mathbb{R}^+)$be the unique solution of Eq. (2.4), then the following hold

$$\|u_0 - \tilde{u}_n^M\|_\omega = \mathcal{O}(n^{-2r})$$

Proof: Using Theorems 4.4, 4.5, and Lemma 4.6, the result follows.

5. Numerical Results

Here we present numerical results. We discuss the error for the approximate solutions of Galerkin, multi-Galerkin, and their iterated methods.

Example 5.1 Consider the following integral equation of Urysohn type.

$$u(\eta) + \int_0^\infty \frac{e^{-(\eta+s)}}{1+u(s)+u^2(s)} ds = \left(1 - \frac{\pi}{3\sqrt{3}}\right) e^{-\eta},$$

where the exact solution $u(\zeta) = 3e^{-\eta}$. The errors for Galerkin, multi-Galerkin, iterated Galerkin, and iterated multi-Galerkin methods are given in the following Table 1, 2 :

Table 1. Error in Galerkin and iterated Galerkin Methods

n	$\|u-u_n\|_\omega$ (Error in Galerkin)	$\|u-\tilde{u}_n\|_\omega$ (Error in iterated Galerkin)	$\|u-\tilde{u}_n\|_\infty$ (Error in iterated Galerkin)
3	3.96283×10^{-2}	4.99834×10^{-3}	4.33848×10^{-3}
4	4.00781×10^{-3}	1.5466231×10^{-3}	2.3913006×10^{-3}
5	7.124416×10^{-4}	2.398162×10^{-4}	3.041939×10^{-4}
6	1.023312×10^{-4}	7.0219316×10^{-5}	6.020016×10^{-5}
7	5.092360×10^{-5}	6.9819231×10^{-6}	5.778782×10^{-6}
8	8.7436192×10^{-6}	8.8131061×10^{-7}	8.2213678×10^{-7}

Table 2. Error in multi-Galerkin and iterated multi-Galerkin Methods

n	$\|u - u_n^M\|_\omega$ (Error in the multi-Galerkin method)	$\|u - \tilde{u}_n^M\|_\omega$ (Error in iterated multi-Galerkin)	$\|u - \tilde{u}_n^M\|_\infty$ (Error in iterated multi-Galerkin)
3	$1.00370216 \times 10^{-3}$	3.3918062×10^{-4}	2.0031213×10^{-4}
4	$5.83778109 \times 10^{-4}$	5.7086323×10^{-5}	5.7233161×10^{-5}
5	$6.66321316 \times 10^{-5}$	3.0328161×10^{-6}	4.2930678×10^{-6}
6	$4.09393786 \times 10^{-6}$	$7.38231617 \times 10^{-7}$	$5.112909913 \times 10^{-7}$
7	5.7022316×10^{-7}	$6.09561381 \times 10^{-8}$	$4.23163316 \times 10^{-8}$
8	$7.22183261 \times 10^{-8}$	$4.19338172 \times 10^{-9}$	$2.77823106 \times 10^{-9}$

Example 5.2: Consider another Example of Urysohn-type integral equation

$$u(\eta) - \int_0^\infty e^{-s(\eta+1)} \arctan\,(\eta + u(\eta))u^2(s)ds = \eta e^{-4\eta^2}$$

where the actual solution is unknown. The errors for Galerkin, multi-Galerkin, iterated Galerkin, and iterated multi-Galerkin methods are given in the following Table 3, 4:

Table 3. Error in Galerkin and iterated Galerkin Methods

n	$\|u - u_n\|_\omega$ (Error in Galerkin)	$\|u - \tilde{u}_n\|_\omega$ (Error in iterated Galerkin)	$\|u - \tilde{u}_n\|_\infty$ (Error in iterated Galerkin)
3	8.44104×10^{-1}	3.65790×10^{-2}	1.39669×10^{-2}
4	1.82182×10^{-1}	4.94580×10^{-3}	3.54817×10^{-3}
5	5.28160×10^{-2}	8.88476×10^{-4}	8.87562×10^{-4}
6	7.98750×10^{-3}	1.54583×10^{-4}	1.33643×10^{-4}
7	2.40955×10^{-4}	6.62305×10^{-5}	5.06331×10^{-5}
8	4.88379×10^{-5}	4.11200×10^{-6}	3.66690×10^{-6}

Table 4. Error in multi-Galerkin and iterated multi-Galerkin Methods

n	$\|u - u_n^M\|_\omega$ (Error in the multi-Galerkin method)	$\|u - \tilde{u}_n^M\|_\omega$ (Error in iterated multi-Galerkin)	$\|u - \tilde{u}_n^M\|_\infty$ (Error in iterated multi-Galerkin)
3	$1.00370216 \times 10^{-3}$	3.3918062×10^{-4}	2.0031213×10^{-4}
4	$5.83778109 \times 10^{-4}$	5.7086323×10^{-5}	5.7233161×10^{-5}
5	$6.66321316 \times 10^{-5}$	3.0328161×10^{-6}	4.2930678×10^{-6}
6	$4.09393786 \times 10^{-6}$	$7.38231617 \times 10^{-7}$	$5.112909913 \times 10^{-7}$
7	5.7022316×10^{-7}	$6.09561381 \times 10^{-8}$	$4.23163316 \times 10^{-8}$
8	$7.22183261 \times 10^{-8}$	$4.19338172 \times 10^{-9}$	$2.77823106 \times 10^{-9}$

6. Conclusion

From the above results, it is clear that the iterated Galerkin method improves over the Galerkin method in weighted norm and in infinity norm. Also, the iterated multi-Galerkin method improves over the iterated Galerkin and multi-Galerkin methods in the weighted norm. The two numerical results presented above support the theoretical findings

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of the paper.

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