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Original Article

Positivity Analysis of the Explicit Euler Method for the Scott Model with Transaction Costs

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Abstract *- This paper investigates the pricing of European options with transaction costs under the Scott stochastic volatility model. Based on the Delta hedging strategy, a nonlinear partial differential equation for option pricing incorporating transaction costs is established. To solve this equation, the finite difference method is employed for discretization, leading to the construction of an explicit Euler numerical scheme. Subsequently, conditions for the non-negativity of the numerical method are studied and rigorously proven using mathematical induction. Finally, numerical experiments validate the effectiveness of the theoretical findings: when the conditions of the lemma are satisfied, the numerical solutions remain non-negative.*

Keywords *- Scott model, Transaction costs, Positivity analysis, Explicit Euler method, Partial differential equation.*

1. Introduction

The Black-Scholes model [1] has long laid the fundamental theoretical framework for option pricing. However, this model adopts a series of simplified assumptions and fails to take into account the transaction costs and other market frictions that exist objectively in real trading environments. In the process of actual asset trading, transaction costs are an unavoidable objective existence. Therefore, studying option pricing equations that include transaction costs is of great theoretical value and practical guiding significance for accurately depicting market operation rules and reasonably evaluating the intrinsic value of financial products. Nevertheless, such pricing equations are usually difficult to obtain analytical solutions, and numerical methods have thus become the core means of option pricing. On this basis, to ensure that the option prices solved by numerical methods conform to basic economic logic, it is necessary to guarantee the positivity of the numerical solutions.

Leland [2] proposed that transaction costs are given by $TC = \kappa_{TC}|\nu|S$, where ν is the parameter of shares of the underlying asset bought ($\nu > 0$) or sold ($\nu < 0$) at price S, and κ_{TC} is a proportionality coefficient reflecting investor-specific preferences. Building on Leland's framework, Lu et al. [3] applied a Delta hedging strategy under the Heston stochastic volatility model and obtained a nonlinear option pricing equation that accounts for transaction costs, leading to a fully nonlinear parabolic partial differential equation. Later, Tan et al. [4] investigated European option pricing under the 3/2 non-affine stochastic volatility model with transaction costs and established a related nonlinear PDE system. Elham Mashayekhi [5] implemented numerical schemes for the HCIR model with transaction costs and conducted comparisons between numerical outputs and analytical solutions. These contributions are largely centered on the Heston model and its extended forms, with growing attention to developing reliable and efficient numerical approximation techniques.

Unlike the Heston model [6], the Scott model [7] characterizes the underlying asset using a mean-reverting stochastic process, which more accurately reflects how market volatility changes over time in real-world settings. This feature boosts the practical applicability of the Scott framework, but it also makes it quite difficult to derive semi-analytical closed-form solutions for its pricing partial differential equation—a challenge similar to what is seen with the Heston model [8]. As such, most solution methods for this model are restricted to Monte Carlo simulation, which tends to have relatively low computational efficiency. To date, only the European option PDE system proposed by Marín-Sánchez [9] has been reported for the Scott model; extensions that incorporate transaction costs to enhance realism have not yet been addressed. Based on this, this study focuses on analyzing the Scott option pricing model with transaction costs, constructing efficient numerical schemes, and examining the positivity preservation property of the discretization method.

This paper is structured as outlined below. The derivation of the Scott stochastic volatility option pricing model with transaction costs is provided in Section 2. Section 3 introduces the explicit Euler scheme used to solve the corresponding nonlinear pricing partial differential equation and establishes the positivity of the proposed numerical approach. The validity of the positivity constraints is further verified through numerical experiments detailed in Section 4. Concluding observations, along with recommendations for future research directions, are summarized in Section 5.

2. Pricing Model

Throughout this paper, it is assumed that the underlying asset price process $\{\boldsymbol{S_t}: \boldsymbol{t} \geq \boldsymbol{0}\}$ follows the following Scott's stochastic volatility model under the risk-neutral measure $\mathbb{Q}$:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_1, \\ dV_t = \kappa[\theta - (\sqrt{V_t} - \eta)^2]dt + \sigma\sqrt{V_t} dW_2. \end{cases} \tag{2.1}$$

Let V_t denotes the instantaneous variance of the asset follows an Ornstein-Uhlenbeck process, $\{W_1: t \geq 0\}$ and $\{W_2: t \geq 0\}$ are two correlated standard Brownian motions with a correlation coefficient $\rho \in [-1,1]$, μ the risk-free interest rate, κ the mean-reversion rate, θ the long-run variance, σ the volatility-of-variance, and ρ the correlation between the two Brownian motions, i.e., $\langle dW_1, dW_2 \rangle = \rho dt$. All parameters are assumed to be positive and to satisfy standard model constraints.

Following Lu's method [3], we derive the pricing model by constructing a delta-hedging portfolio, yielding the option pricing model that satisfies the following nonlinear PDE:

$$\frac{\partial U}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 V \frac{\partial^2 U}{\partial V^2} + \rho\sigma V S \frac{\partial^2 U}{\partial S \partial V} + \kappa[\theta - (\sqrt{V_t} - \eta)^2]\frac{\partial U}{\partial V} + rS\frac{\partial U}{\partial S} - rU - \mathfrak{I} = 0. \tag{2.2}$$

where $\mathfrak{I}$ is given by

$$\mathfrak{I} = \sqrt{\frac{2}{\pi \delta t}} \kappa_{TC} S \sqrt{V S^2 (\frac{\partial^2 U}{\partial S^2})^2 + 2\rho\sigma V S \frac{\partial^2 U}{\partial S^2}\frac{\partial^2 U}{\partial S \partial V} + \sigma^2 V (\frac{\partial^2 U}{\partial S \partial V})^2}.$$

For a European call option with strike price E and expiration T, the corresponding terminal and boundary conditions take the following forms:

$$\begin{cases} U(S, V, T) = max(S - E, 0), \\ \lim_{S \to 0} U(S, V, t) = 0, \\ \lim_{S \to \infty} \frac{\partial U}{\partial S} = 1, \\ \lim_{V \to 0} \frac{\partial U}{\partial t} + rS\frac{\partial U}{\partial S} - rU = 0, \\ \lim_{V \to \infty} U(S, V, t) = S. \end{cases} \tag{2.3}$$

Given the presence of the nonlinear term induced by transaction costs, deriving an analytical solution for PDE (2.2) presents significant challenges. Accordingly, Section 3 is dedicated to developing an approximate numerical method.

3. Numerical Scheme

3.1. Explicit Euler Method

First, the two backward parabolic partial differential equations are converted to forward parabolic equations through the introduction of the time variable $\tau = T - t$. Assume that the time $\tau \in [0, T]$, stock price $S \in [0, S_{max}]$, and the variance $V \in [V_{min}, V_{max}]$, then discretize the domain $\{(S, V, \tau): 0 \leq S \leq S_{max}, V_{min} \leq V \leq V_{max}, 0 \leq \tau \leq T\}$ with the following uniform grids:

$$S_i = (i - 1)\Delta S, i = 1, \cdots N_s + 1, \Delta S = \frac{S_{max}}{N_s},$$

$$V_j = (j - 1)\Delta V, j = 1, \cdots N_V + 1, \Delta V = \frac{V_{max} - V_{min}}{N_V},$$

$$\tau_n = (n - 1)\Delta \tau, n = 1, \cdots N_t + 1, \Delta \tau = \frac{T}{N_t}.$$

Let $\boldsymbol{U_{i,j}^n}$ be the numerical approximation of $\boldsymbol{U(S_i, V_j, \tau_n)}$ and $\boldsymbol{\mathfrak{I}_{i,j}^n}$ be the numerical approximation of $\boldsymbol{\mathfrak{I}}$ at $\boldsymbol{U(S_i, V_j, \tau_n)}$. Then the partial derivatives at the mesh grid $(\boldsymbol{i, j, n})$ are approximated by

$$\frac{\partial U}{\partial \tau} \approx \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta\tau}, \frac{\partial U}{\partial S} \approx \frac{U_{i+1,j}^n - U_{i-1,j}^n}{2\Delta S}, \frac{\partial U}{\partial V} \approx \frac{U_{i,j+1}^n - U_{i,j-1}^n}{2\Delta V},$$
$$\frac{\partial^2 U}{\partial S^2} \approx \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{(\Delta S)^2}, \frac{\partial^2 U}{\partial V^2} \approx \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{(\Delta V)^2},$$

$$\frac{\partial^2 U}{\partial S \partial V} \approx \frac{U_{i+1,j+1}^n - U_{i+1,j-1}^n - U_{i-1,j+1}^n + U_{i-1,j-1}^n}{4\Delta S \Delta V}.$$

Now, replacing the above expressions in (2.2). The discounting term ($-\boldsymbol{rU}$), being a zero-order term (containing no derivatives), is typically approximated directly using the value from the current time step. It is obtained the explicit euler method given by:

$$U_{i,j}^{n+1} = \frac{1}{1 + r\Delta\tau}[a_{i,j}U_{i,j}^n + b_{i,j}U_{i-1,j}^n + c_{i,j}U_{i+1,j}^n + d_{i,j}U_{i,j-1}^n + e_{i.j}U_{i,j+1}^n$$
$$+(g_{i,j}U_{i+1,j+1}^n + h_{i,j}U_{i+1,j-1}^n + k_{i,j}U_{i-1,j+1}^n + m_{i,j}U_{i-1,j-1}^n) - \Delta\tau\mathfrak{I}_{i,j}^n]. \quad (3.1)$$

where

$$a_{i,j} = 1 - \Delta\tau((i-1)^2(j-1)\Delta V + \frac{\sigma^2(j-1)}{\Delta V}),$$
$$b_{i,j} = \frac{1}{2}\Delta\tau((i-1)^2(j-1)\Delta V - (i-1)r),$$
$$c_{i,j} = \frac{1}{2}\Delta\tau((i-1)^2(j-1)\Delta V + (i-1)r),$$
$$d_{i,j} = \frac{1}{2}\Delta\tau(\frac{(j-1)\sigma^2}{\Delta V} - \frac{\kappa(\theta - (\sqrt{(j-1)\Delta V} - \eta)^2)}{\Delta V}),$$
$$e_{i,j} = \frac{1}{2}\Delta\tau(\frac{(j-1)\sigma^2}{\Delta V} + \frac{\kappa(\theta - (\sqrt{(j-1)\Delta V} - \eta)^2)}{\Delta V}),$$
$$g_{i,j} = m_{i,j} = -h_{i,j} = -k_{i,j} = \frac{\Delta\tau\rho\sigma(i-1)(j-1)}{4}.$$

3.2. Positivity of the Numerical Method

A valid option pricing model must yield nonnegative prices in realistic financial scenarios. Accordingly, ensuring the positivity of the proposed numerical scheme is essential, and this property is established through the lemma presented below.

Lemma 3.1: There exists a constant $C_1 > 0$ and $C_2 > 0$, independent of variables in $\mathfrak{I}$ such that

$$|\mathfrak{I}| \le C_1|\frac{\partial^2 U}{\partial S^2}| + C_2|\frac{\partial^2 U}{\partial S \partial V}|.$$

The proof can be found in reference [10].

From the above lemma, it is obtained:

$$\Delta\tau|\mathfrak{I}_{i,j}^n| \le \Delta\tau C_1|\frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{(\Delta S)^2}| + \Delta\tau C_2|\frac{U_{i+1,j+1}^n - U_{i+1,j-1}^n - U_{i-1,j+1}^n + U_{i-1,j-1}^n}{4\Delta S \Delta V}|,$$
$$\le p_{i,j}|U_{i+1,j}^n| + 2p_{i,j}|U_{i,j}^n| + p_{i,j}|U_{i-1,j}^n| + q_{i,j}|U_{i+1,j+1}^n - U_{i+1,j-1}^n - U_{i-1,j+1}^n + U_{i-1,j-1}^n|,$$

where

$$p_{i,j} = \Delta\tau\kappa_{TC}\sqrt{\frac{2}{\pi\delta t}}(i-1)^2\sqrt{(j-1)\Delta V},$$

$$q_{i,j} = \frac{\Delta\tau\sigma\kappa_{TC}\sqrt{\frac{2}{\pi\delta t}}(i-1)\sqrt{(j-1)\ \Delta V}(\rho+\sqrt{1-\rho^2})}{4\Delta V}.$$

Lemma 3.2: For the numerical method (3.1), if $0 < \rho, \Delta\tau \leq \frac{1}{VS^2\frac{\sigma^2(N_V+1)}{\Delta V}{}_{TC}\sqrt{\frac{2}{\pi\delta\tau}}_{max}}$ and

$\frac{2\kappa^2{}_{TC}}{\pi\Delta\tau}(\frac{1}{\rho^2}+2\sqrt{\frac{1}{\rho^2}-1}) \leq V_{min}\sqrt{\frac{2}{\pi\Delta\tau}}\sqrt{\frac{2\kappa^2{}_{TC}}{\pi\Delta\tau}+r}^{-2}$ then $\forall 2 \leq n \leq N_t+1, 2 \leq i \leq N_S+1, 2 \leq j \leq N_V+1$,
We have that $a_{i,j} \geq 2p_{i,j}, b_{i,j} \geq p_{i,j}, g_{i,j} \geq q_{i,j}$, and $c_{i,j}, |d_{i,j}|, e_{i,j}, m_{i,j}, |h_{i,j}|, |k_{i,j}| \geq 0$.

Note that if $\rho < 0$, this lemma still holds, but instead of considering $g_{i,j}, |h_{i,j}|, m_{i,j}$ and $|k_{i,j}|$, we consider $|g_{i,j}|, h_{i,j}, |m_{i,j}|$ and $k_{i,j}$.

To streamline the algebraic manipulations in the positivity verification, we define $\Delta^n_{j,i+1} = U^n_{i+1,j+1} - U^n_{i+1,j-1}$, and $\Delta^n_{j,i-1} = U^n_{i-1,j+1} - U^n_{i-1,j-1}$, $\forall 1 \leq n \leq N_t, 1 \leq i \leq N_S, 1 \leq j \leq N_V$. Hence, we can rewrite the numerical scheme (3.1) as:

$$U^{n+1}_{i,j} = \frac{1}{1+r\Delta\tau}[a_{i,j}U^n_{i,j} + b_{i,j}U^n_{i-1,j} + c_{i,j}U^n_{i+1,j} + d_{i,j}U^n_{i,j-1} + e_{i.j}U^n_{i,j+1} + g_{i,j}(\Delta^n_{j,i+1} - \Delta^n_{j,i-1}) - \Delta\tau\Im^n_{i,j}]. \quad (3.2)$$

The conditions identified remain valid for scenarios involving small initial values and mild noise disturbances. Under the given constraints, the boundedness of stochastic increments and near-zero volatility regimes is maintained, which eliminates the occurrence of nonpositive numerical solutions in boundary cases.

Theorem 3.1: Suppose the hypotheses of Lemma 3.1 and Lemma 3.2 are satisfied, and that $\Delta^n_{j,i+1} - \Delta^n_{j,i-1} \geq 0$, are satisfied. Then the numerical scheme is positive for $\forall 1 \leq n \leq N_t, 1 \leq i \leq N_S, 1 \leq j \leq N_V$.

Proof: The theorem is employed by the mathematical induction method.

Step 1: (n=1) In this case, we have that $U^0_{i+1,j+1} = U^0_{i+1,j-1} = max\{(i+1)\Delta S - E, 0\}$, hence, $\Delta^0_{j+1,i+1} = 0$. Similarly, we can obtain that $\Delta^0_{j-1,i+1} = 0$, and thus $\Delta^0_{j+1,i+1} - \Delta^0_{j-1,i+1} = 0$. Moreover, for the same reason, we conclude that $U^1_{i,j+1} = U^1_{i,j-1} = max\{i\Delta S - E, 0\}$, therefore $d_{i,j}U^0_{i,j+1} + e_{i,j}U^0_{i,j-1} = (d_{i,j}+e_{i,j})\,max\{i\Delta S - E, 0\} \geq 0$ since $d_{i,j} + e_{i,j} = \frac{\Delta\tau\sigma^2(j-1)}{\Delta V} \geq 0$. Finally, since the coefficients $a_{i,j}, b_{i,j}, c_{i,j} \geq 0$ for $1 \leq n \leq N_t, 1 \leq i \leq N_S, 1 \leq j \leq N_V$ and the payment function $U^0_{i,j} \geq 0$ for all i, j, then $U^1_{i,j} \geq 0$.

Step 2: (n=k) It is assumed that the positivity property is satisfied for n=k.

Step 3: (n=k+1) Note that $d_{i,j}U^k_{i,j-1} + e_{i,j}U^k_{i,j+1} \geq d_{i,j}U^k_{i,j+1} + e_{i,j}U^k_{i,j+1} = \frac{\Delta\tau\sigma^2(j-1)}{\Delta V}U^k_{i,j+1} \geq 0$. Besides, by hypothesis, $\Delta^k_{j,i+1} - \Delta^k_{j,i-1} \geq 0$ and $U^k_{i,j} \geq 0$, for $1 \leq i \leq N_S, 1 \leq j \leq N_V$, then according to Equation (3.2), we have $U^{k+1}_{i,j} \geq 0$ for all i, j.

For small volatility values and bounded stochastic increments, the coefficients of the scheme remain nonnegative under the lemma conditions. Thus, nonpositive values cannot arise even in near-degenerate volatility scenarios. The proof is completed.

4. Numerical Experiments

In this section, the positivity of the numerical method is verified under the conditions specified in Lemma 3.2. The parameters are selected $\boldsymbol{S_{max} = 200, V_{max} = 0.2, V_{min} = 0.01, T = 1, E = 100, r = 0.05, \theta = 0.01, \sigma = 0.05, \eta = 0.05, \rho = 0.7, \kappa = 2, \delta t = \frac{1}{6}}$.

Under the aforementioned parameter configuration, the parameters N_t and κ_{TC} are varied. For the first set of experiments, parameters are selected $N_t = 2000, \kappa_{TC} = 0.01$, which satisfies Lemma 3.2; for the second set of experiments, parameters are selected $N_t = 500, \kappa_{TC} = 0.05$, which violates Lemma 3.2. Subsequently, based on the numerical method (3.1), option price curves under these two parameter settings are plotted in Figure 1.

It is evident that when the conditions outlined in Lemma 3.2 are met, the option price U remains nonnegative for any given stock price S and volatility V; in contrast, negative values arise if these conditions are not satisfied.

In comparison with conventional finite difference methods applicable to the Scott model and transaction cost-inclusive frameworks, the scheme developed herein guarantees the nonnegativity of option prices under clearly defined conditions. Traditional discretization methods often yield negative values in coarse grids or periods of low volatility, but the present approach resolves this issue by integrating structural constraints specifically designed to preserve positivity in the discrete parameters. Numerical experiment results confirm that the proposed method achieves enhanced stability and greater robustness across various test scenarios.

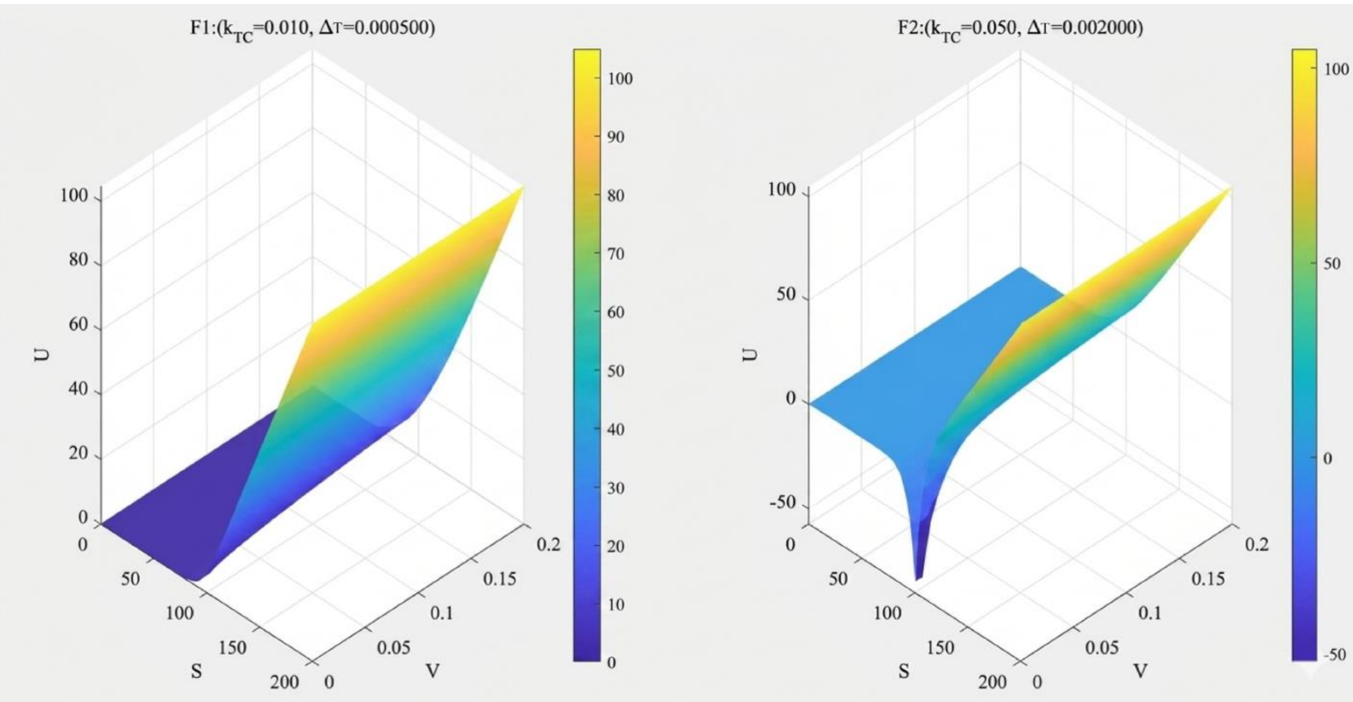

Fig. 1 Surface plots of the numerical method under different parameters. F1 satisfies the positivity condition, while F2 does not

5. Conclusion

This study derives the pricing partial differential equation for the Scott stochastic volatility model with transaction costs incorporated, and explores an explicit Euler numerical method for its solution. Furthermore, we establish the positivity conditions for this method and provide a rigorous proof via mathematical induction. Finally, numerical experiments are conducted to validate the positivity conditions of the numerical scheme. In future research, we plan to explore alternative numerical methods for solving this pricing PDE and pursue more accurate numerical solutions.

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