International Journal of Mathematics Trends and Technology
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V72I4P104

Volume 72 Issue 4, 23-30, April 2026
© 2026 Seventh Sense Research Group®

Original Article

Variable Control Chart based on Xgamma Distribution

Chandralekha I[1], Shalini K[2], Sangeetha[3]

PG & Department of Statistics, Salem Sowdeswari College, Salem, Tamil Nadu, India.

[2]Corresponding Author : shalini.stat@gmail.com

Received: 21 February 2026 Revised: 25 March 2026 Accepted: 14 April 2026 Published: 26 April 2026

Abstract - *Statistical Process Control (SPC) plays a vital role in monitoring and improving industrial processes. However, many quality characteristics, such as failure times, defect counts, waiting times, and repair durations, are strictly positive and highly skewed, violating the normality assumption underlying traditional Shewhart charts. The Xgamma distribution is a versatile and flexible mixture distribution suitable for modeling positive, right-skewed data in reliability, lifetime, and process monitoring studies, particularly when classical exponential or gamma models fail to adequately describe the tail behavior and variability. This study proposes an Individual (I) control chart based on Xgamma distribution, a suitable probability model capable of handling skewed and heavy-tailed data. A fourth-root transformation is employed to reduce skewness and achieve normality. Control limits are constructed using a percentile-based approach. The performance of the proposed chart is evaluated through a simulation study in terms of both in-control Average Run Length (ARL_0) and out-of-control Average Run Length (ARL_1). Simulation study results indicate that the proposed chart maintains the in-control ARL_0 while achieving smaller ARL_1 values under various shift magnitudes, demonstrating improved sensitivity to small and moderate process shifts compared with the exponential-based Individuals chart.*

Keywords - *Quality control, Statistical process control, Exponential distribution, Gamma distribution, Xgamma distribution, Control chart, Average run length.*

1. Introduction

Manufacturing industries place great emphasis on delivering consistent, high-quality products, as customer satisfaction depends heavily on reliability and performance. Quality refers to the degree to which a product or process meets established specifications or customer expectations. To achieve this, organizations integrate quality planning, quality control, and continuous improvement into their production systems to reduce variation and enhance process capability.

Statistical Quality Control (SQC) provides a set of statistical techniques for monitoring, controlling, and improving quality. It consists primarily of Statistical Process Control (SPC) and Acceptance Sampling. SPC focuses on understanding and analysing process variation to distinguish between normal, inherent fluctuations and unusual deviations that signal potential problems. Among the tools used in SPC, control charts play a vital role. Control charts are graphical monitoring tools used to evaluate whether a manufacturing process remains statistically stable and free from assignable causes of variation over time. Originally developed by Shewhart [1], control charts help detect shifts, trends, and abnormal variations so that corrective actions can be taken promptly to maintain process stability.

Many industrial quality characteristics, such as lifetimes, waiting times, inter-arrival times, and repair times, are strictly positive and often right-skewed. When normality assumptions are violated, traditional Shewhart control charts may produce misleading results, leading to excessive false alarms or failure to detect actual process shifts. To address skewness, power transformations such as the fourth-root transformation are applied to achieve normality. Kittlitz [2] constructed an individual's chart based on the fourth-root transformation, using percentile-based control limits for exponential data. Zheng and Chakraborti [3] examined the robustness of the quantile control chart when the underlying distribution has different shapes and when both the number and positions of selected quantiles vary. Park *et al.* [4] proposed control charts based on randomized quantile residuals from regression models, and demonstrated their performance through simulation studies and real-life data collected from real quality-control processes.

An EWMA chart using the weighted exponential distribution was proposed by Iqbal *et al.* [5], showing improved shift detection and reduced ARLs relative to the conventional exponential chart. Ozsan *et al*. [6] analyzed parameter estimation effects

This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/)

on exponential EWMA charts for low-defect processes, showing estimation error adds variability that distorts performance, especially with small samples and shifts, while larger Phase-I samples stabilize results. Saghir *et al.* [7] proposed a transformed modified EWMA control chart for gamma-distributed data, showing superior ARL performance and effective shift detection, supported by an industrial application. Zhang *et al*. [8] developed a Gamma chart with a random-shift ATS model, improving accuracy and sensitivity over fixed-shift and exponential charts.

The Xgamma distribution is used in many real-world applications where data show positive skewness and heavy tails. Sen *et al.* [9] developed the Xgamma distribution, which blends exponential and gamma components to achieve greater modeling flexibility. Its ability to represent diverse patterns of skewness makes the Xgamma distribution an attractive alternative for constructing control charts for positively skewed process data. Sen *et al.* [10] introduced a quasi Xgamma distribution to study the distribution properties, survival properties, and order statistics, applying it to real bladder-cancer data for lifetime modeling. Sen *et al.* [11] developed a two-parameter generalization of the Xgamma distribution and investigated its structural, survival, and reliability properties through estimation methods and simulation studies. Bicer [12] introduced a new class of Transmuted Xgamma (TXG) distribution, deriving key statistical properties such as hazard function, survival functions, mean residual life, moments, order statistics, inequality curves, and studied the parameter estimation via simulations.

Similarly, Yadav *et al.* [13] introduced the inverse Xgamma Distribution (IXGD) and derived its reliability properties, moments, inverse moments, random ordering, and order statistics, and estimated the parameters by various methods such as Maximum Likelihood Estimation (MLE), Least Square Estimation (LSE), and Weighted Least Square Estimation (WLSE). Yadav *et al.* [14] defined the Exponentiated Xgamma distribution (EXGD), derived its moments, quantiles, generating functions, order statistics, and reliability curves; they further illustrated its applicability by fitting the model to lifetime data. Tripathi *et al.* [15] proposed a flexible extension of Xgamma, derived raw moments, generating functions, conditional moments, quantiles, survival, and hazard functions. Abulebda *et a.* [16] proposed a bivariate Xgamma (BXG) distribution using the Farlie–Gumbel–Morgenstern copula.

Ashok Kumar and Muthukumar [17] used Xgamma as a baseline distribution in frailty models for survival data; they demonstrated that Xgamma-based frailty models perform well on real datasets compared to conventional baselines. Sen *et al.* [18] studied the distribution of the ratio of two independent Xgamma variables, derived moments, entropy measures, and provided a characterization through truncated incomplete moments useful for ratio-type statistical modelling. Kadri *et al.* [19] developed the distributional properties of the sum of independent Xgamma random variables.

Alomani *et al.* [20] developed modified acceptance sampling plans specifically designed for the Two-Parameter Xgamma Distribution (TPXGD) when lifetimes were truncated at a fixed point. Using the mean of the TPXGD as the primary quality metric, they analyzed acceptance numbers, required sample sizes, operating characteristic curves, and producer's risk for various parameter values. Sen *et al.* [21] introduced the group Acceptance Sampling Inspection Plan (GASIP) and double acceptance sampling inspection plan (DASIP) for truncated life tests based on the quasi-xgamma distribution. The parameters of these plans were determined by ensuring that the consumer's risk was controlled for specified ratios of true mean life, stated life, and termination time.

The Xgamma distribution can be applied in manufacturing and quality control for process monitoring, ensuring products meet specifications through failure time, waiting times, and product lifetimes, which are purely positive and often positively skewed. Even conventional normality-based or simple exponential models may fail to adequately account for this skew, resulting in less reliable control limits and higher false alarm rates. The Xgamma distribution provides greater flexibility through the ability to model skewed lifetime data and varying risk structures, often leading to better fitting performance than conventional models. Studies on Xgamma and its extensions have shown strong reliability and stability modeling capabilities for real-life engineering and failure time data. Therefore, the use of the Xgamma framework in control chart design allows for more realistic control limits, improved transient detection, reduced false alarms, and improved reliability assessment during continuous process monitoring. Although extensive research has been conducted on the theoretical properties, generalizations, and reliability applications of the Xgamma distribution, its application in real-time statistical process control remains unexplored. In particular, no control chart has been developed specifically for the Xgamma framework with a formal evaluation of run-length performance characteristics such as Average Run Length (ARL). Furthermore, the suitability of the Xgamma distribution for monitoring positively skewed industrial and reliability data has not been investigated in comparison with existing skewed alternatives. The lack of rigorous derivations for control limits, model spread behavior, and ARL performance under the Xgamma framework creates a significant gap in the statistical process control literature. This unresolved issue motivates the development of a dedicated Xgamma-based control chart framework for efficient real-time monitoring and early shift detection.

This paper presents the construction of an individual control chart based on the Xgamma distribution utilizing the fourth-root transformation. In Section 2, the Xgamma distribution is described. In Section 3, the control limits for the suggested control chart are evaluated for their Average Run Length (ARL) performance. Section 4 presents a simulation study for developing the control chart. The summary of results is presented in Section 5.

2. Xgamma Distribution

The Xgamma distribution is widely used across industries such as manufacturing, quality control, insurance, risk management, healthcare, and reliability engineering. In manufacturing and quality control, the Xgamma distribution models failure times and supports predictive maintenance and risk assessment. The Xgamma (xg) distribution is a special finite mixture of exponential and gamma distributions. The probability density function (pdf) of the Xgamma distribution (Sen *et al.,* [9]) with parameter θ is given as

$$\boldsymbol{f(x) = \frac{\theta^2}{(1+\theta)}\left(1 + \frac{\theta}{2}x^2\right)e^{-\theta x}; x, \theta > 0} \tag{1}$$

The Cumulative Distribution Function (cdf) is

$$F(x) = 1 - \frac{(1+\theta+\theta x+\frac{1}{2}\theta^2 x^2)}{(1+\theta)}e^{-\theta x}; x, \theta > 0 \tag{2}$$

The mean and variance of the Xg distribution are

$$\mu = \frac{\theta+3}{\theta(1+\theta)} \tag{4}$$

$$\sigma^2 = \frac{\theta^2+8\theta+3}{\theta^2(1+\theta)^2} \tag{5}$$

The parameter θ is estimated using the method of moments

$$\hat{\theta} = \frac{-(\bar{X}-1)\sqrt{(\bar{X}-1)^{2\rightleftarrows}+12\bar{X}}}{2\bar{X}} \tag{6}$$

3. Control Limits for Xgamma Distribution

The Individual control charts (I-chart) are used when sub-grouping data is impractical due to infrequent measurements, low production volumes, or destructive testing, allowing monitoring of process mean and stability with single observations over time. When the monitored variable represents lifetime-type data such as repair times, service times, or inter-arrival times, the underlying distribution is usually positive and right-skewed. Positive and skewed lifetime data often exhibit heavy tails. To address this, we construct an individual's chart from the Xgamma distribution through a percentile-based approach. To stabilize variability and reduce skewness, the fourth root transformation is applied.

$$Y = (X^{1/4}) \tag{7}$$

For a given value of θ, the corresponding quantile x_p is determined by solving for it.

$$F(X_p; \hat{\theta}) = p, p \in \{0.00135, 0.50, 0.99865\} \tag{8}$$

The above expression is not in closed form; hence, numerical root-finding algorithms, viz., Newton-Raphson, are adopted to solve $F(x_p; \theta) - p = 0$.

The control limits on the transformed scale are then

$$LCL = {x_L}^{1/4} \tag{9}$$

$$CL = {x_C}^{1/4} \tag{10}$$

$$UCL = {x_U}^{1/4} \tag{11}$$

The performance of control charts is commonly measured using Average Run Length (ARL), which represents the average number of points plotted before an out-of-control signal appears. ARL_0 (in-control ARL) characterizes the stable process, whereas out-of-control ARL (denoted ARL_1) corresponds to processes disrupted by undesired variability. The mathematical expressions of ARL_0 and ARL_1 are as follows

$$p_0 = F(LCL;\theta_0) + 1 - F(UCL;\theta_0) where \theta_0 = \theta \quad (12)$$

$$ARL_0 = \frac{1}{p_0} \quad (13)$$

The probability of signaling an out-of-control condition under the Xgamma individual chart is given as

$$p_1 = F(LCL;\theta_1) + 1 - F(UCL;\theta_1) where \theta_1 = \delta\theta \quad (14)$$

$$ARL_1 = \frac{1}{p_1} \quad (15)$$

4. Simulation Study

A simulation study is carried out to illustrate the construction of the proposed control chart. The control limits for the Xgamma distribution are obtained using a simulated data set for a range of θ. The corresponding results are presented in Table 1.

Table. 1. Control limits using xgamma distribution

θ	LCL	CL	UCL
0.1	0.5997	2.1928	3.1590
0.2	0.5997	2.1928	3.1590
0.3	0.3724	1.6445	2.4295
0.5	0.2949	1.3920	2.1112
0.6	0.2771	1.3253	2.0295
0.8	0.2513	1.2214	1.9036
0.9	0.2398	1.1717	1.8438
1.0	0.2322	1.1378	1.8031
1.1	0.2256	1.1078	1.7671
1.2	0.2154	1.0597	1.7090
1.3	0.2096	1.0320	1.6753
θ	LCL	CL	UCL
1.4	0.2046	1.0074	1.6452
1.5	0.1993	0.9814	1.6131
1.6	0.1958	0.9638	1.5912
1.7	0.1926	0.9476	0.5709
1.8	0.1843	0.9056	1.5175
1.9	0.1816	0.8921	1.5002
2.0	0.1845	0.9069	1.5192
2.1	0.1821	0.8947	1.5036
2.2	0.1795	0.8811	1.4859
2.3	0.1744	0.8549	1.4513
2.4	0.1725	0.8450	1.4382
2.5	0.1701	0.8328	1.4218
2.6	0.1684	0.8240	1.4098
2.7	0.1668	0.8157	1.3984

2.8	0.1653	0.8077	1.3875
2.9	0.1638	0.8001	1.3770
3.0	0.1624	0.7929	1.3669
3.1	0.1611	0.7860	1.3573
3.2	0.1598	0.7794	1.3479
3.3	0.1585	0.7730	1.3389
3.4	0.1574	0.7669	1.3302
3.5	0.1562	0.7611	1.3218
3.5	0.1562	0.7611	1.3218
3.6	0.1551	0.7554	1.3137
3.7	0.1541	0.7500	1.3059
3.8	0.1519	0.7385	1.2890
3.9	0.1509	0.7335	1.2817
4.0	0.1499	0.7287	1.2745
4.1	0.1490	0.7240	1.2676
4.2	0.1482	0.7195	1.2608
4.3	0.1473	0.7151	1.2543
4.4	0.1465	0.7108	1.2479
4.6	0.1449	0.7027	1.2356
4.7	0.1441	0.6989	1.2297
4.8	0.1434	0.6951	1.2239
4.9	0.1427	0.6914	1.2183
5.0	0.1420	0.6878	1.2128

The result shows that the control limit decreases as θ increases, reflecting reduced variability in the fourth-root transformed process data. Figure 1 displays the individual's control chart constructed using a simulated data set for the Xgamma distribution for θ = 1.2. All observations lie within the upper and lower control limits and fluctuate randomly around the central line, indicating that the process is in control; thus, there is no statistical evidence of assignable causes in the process over the observed period.

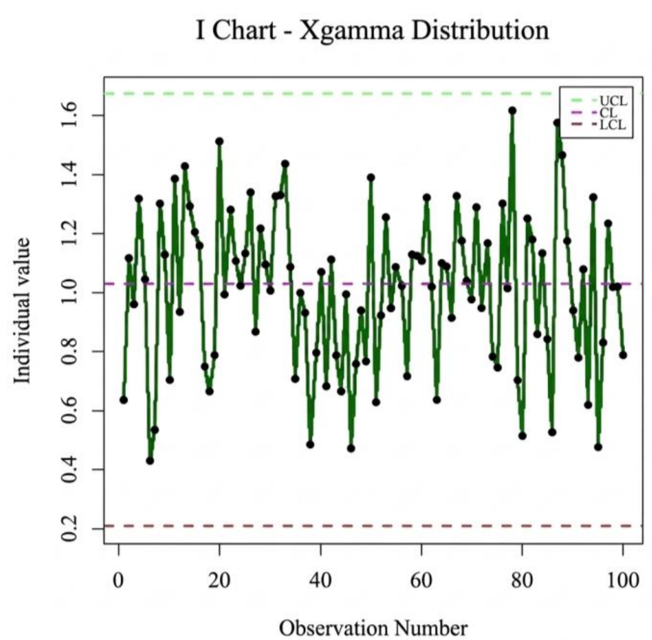

Fig. 1 Individual chart of xgamma distribution for simulated data

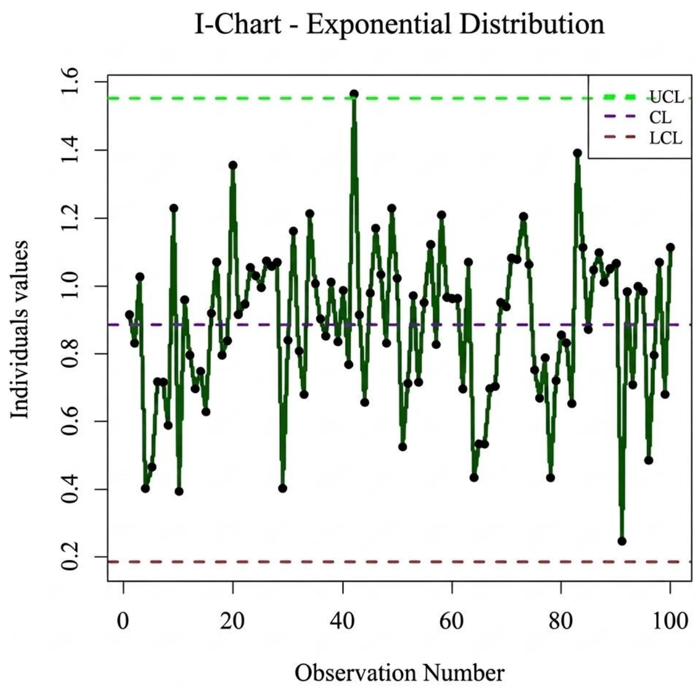

Fig. 2 Individual chart of the exponential distribution for simulated data

Figure 2 presents an individual's control chart of exponential distribution (Kittlitz's (1999)) for a simulated data set of θ = 1.2, where all points fall randomly around the center line, and the 42nd signal indicates an out-of-control condition. The corresponding control limits are UCL=1.4792 and LCL= 0.17688. In contrast, Figure 1 shows the proposed Xgamma-based individual control chart with all observations in an in-control process. This suggests that the Xgamma-based control chart may reduce false alarms and provide a more reliable monitoring tool for skewed process data.

Table. 2. ARL of proposed control chart for out-of-control process with $arl_0 \approx 370$

δ \ θ	0.5	1.0	1.5	2.0	2.5
1	370.27	370.22	370.14	370.08	369.99
0.8	115.18	117.31	119.81	121.56	123.83
0.6	24.29	25.27	26.42	27.23	28.31
0.4	5.48	5.74	6.05	6.27	6.57
0.2	1.65	1.71	1.77	1.82	1.88

Table 3. Comparison of arl of the existing exponential-based control chart with the proposed xgamma based control chart for an out-of-control process (θ = 0.5)

δ	Existing	Proposed
1	370.37	370.27
0.8	162.83	115.18
0.6	50.54	24.28
0.4	13.95	5.48
0.2	3.75	1.65

Table 2 indicates that the proposed control chart maintains the in-control ARL of approximately 370 for various values of θ. For out-of-control conditions (δ < 1), the ARL decreases substantially as the shift decreases. Table 3 presents a comparison to the existing exponential chart; the proposed Xgamma-based chart consistently yields smaller ARL values, indicating faster detection of process shifts. The improvement is particularly significant for moderate and large shifts, demonstrating the superior sensitivity of the proposed design.

5. Conclusion

This paper presents an individual's control chart based on the Xgamma distribution for monitoring positively skewed and strictly positive process data. By applying the fourth-root transformation, percentile-based control limits are derived for the Xgamma distribution, and a simulation study is conducted to illustrate the performance and practical utility of the proposed chart. The individual's chart for the transformed Xgamma data shows that the process is stable and operating under statistical control, with only common-cause variation present. A comparative analysis with the existing individual control chart based on the exponential distribution further demonstrates the effectiveness of the control chart based on the Xgamma distribution. The proposed Xgamma-based chart indicates that all observations remain within control limits, reflecting improved stability and reliability in monitoring the process. In addition, the proposed method consistently produces smaller ARL values, enabling faster detection of process shifts. The improvement is particularly notable for moderate and large shifts, highlighting the superior sensitivity and monitoring performance of the proposed Xgamma-based control chart. Overall, the proposed approach provides an appropriate and effective tool for monitoring positively skewed process data.

References

[1] W.A. Shewhart, "Quality Control Charts," *Bell System Technical Journal*, vol. 5, no. 4, pp. 593-603, 1926. [CrossRef] [Google Scholar] [Publisher Link]

[2] Rudolf G. Kittlitz, "Transforming the Exponential for SPC Applications," *Journal of Quality Technology*, vol. 31, no. 3, pp. 301-308, 1999. [CrossRef] [Google Scholar] [Publisher Link]

[3] R. Zheng, and S. Chakraborti, "A Study of the Quantile Control Chart," *Quality and Productivity Section*, pp. 2239-2246, 2014. [Publisher Link]

[4] Kayoung Park, Dongmin Jung, and Jong-Min Kim, "Control Charts based on Randomized Quantile Residuals," *Applied Stochastic Models in Business and Industry*, vol. 36, no. 4, pp. 716-729, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[5] Taswar Iqbal et al., "Advancing Quality Assurance in Manufacturing: A Weighted Exponential Distribution Control Chart for Enhanced Production Monitoring," *Migration Letters*, vol. 21, no. S5, pp. 1344-1362, 2024. [Google Scholar]

[6] Guney Ozsan, Murat Caner Testik, and Christian H. Weiß, "Properties of the Exponential EWMA Chart with Parameter Estimation," *Quality and Reliability Engineering International*, vol. 26, no. 6, pp. 555-569, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[7] Aamir Saghir, Liaquat Ahmad, and Muhammad Aslam, "Modified EWMA Control Chart for Transformed Gamma Data," *Communications in Statistics – Simulation and Computation*, vol. 50, no. 10, pp. 3046-3059, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[8] C.W. Zhang et al., "A Control Chart for the Gamma Distribution as a Model of Time between Events," *International Journal of Production Research,* vol. 45, no. 23, pp. 5649-5666. 2007. [CrossRef] [Google Scholar] [Publisher Link]

[9] Subhradev Sen, Sudhansu S. Maiti, and N. Chandra, "The XGamma Distribution: Statistical Properties and Application," *Journal of Modern Applied Statistical Methods*, vol. 15, no. 1, pp. 774-788, 2016. [CrossRef] [Google Scholar] [Publisher Link]

[10] S. Sen, and N. Chandra, "The Quasi XGamma Distribution with Application in Bladder Cancer Data," *Journal of Data Science*, vol. 15, no. 1, pp. 61-76, 2017. [Google Scholar]

[11] Subhradev Sen, N. Chandra, and Sudhansu S. Maiti, "On Properties and Applications of a Two- Parameter XGamma Distribution," *Journal of Statistical Theory and Applications*, vol. 17, no. 4, pp. 674-685, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[12] Hayrinisa Demirci Biçer, "Properties and Inference for a New Class of XGamma Distributions with an Application," *Mathematical Sciences*, vol. 13, no. 4, pp. 335-346. 2019. [CrossRef] [Google Scholar] [Publisher Link]

[13] Abhimanyu Singh Yadav, Sudhansu S. Maiti, and Mahendra Saha, "The Inverse XGamma Distribution: Statistical Properties and Different Methods of Estimation," *Annals of Data Science*, vol. 8, no. 2, pp. 275-293, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[14] Abhimanyu Singh Yadav et al., "The Exponentiated XGamma Distribution: A New Monotone Failure Rate Model and Its Applications to Lifetime Data," *Statistica*, vol. 81, no. 3, pp. 303-334, 2021. [CrossRef] [Google Scholar] [Publisher Link]

[15] Harsh Tripathi et al., "A New Flexible Extension of XGamma Distribution and Its Application to Covid-19 Data," *Nepal Journal of Mathematical Sciences*, vol. 3, no. 1, pp. 11-30, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[16] Mohammed Abulebda et al., "On a Bivariate XGamma Distribution Derived from Copula," *Statistica*, vol. 82, no. 1, pp. 15-40, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[17] Ashok Kumar Palanisamy, and Muthukumar Madaswamy, "Frailty Models under XGamma Distribution with Application to Survival Data," *Mathematics and Computer Science*, vol. 8, no. 4, pp. 87-93, 2023. [CrossRef] [Publisher Link]

[18] Subhradev Sen et al., “On Truncated Versions of XGamma Distribution: Various Estimation Methods and Statistical Modelling. Statistics,” *Optimization and Information Computing*, vol. 12, no. 4, pp. 943-961, 2024. [CrossRef] [Google Scholar] [Publisher Link]

[19] Therrar Kadri et al., “Sum of Independent XGamma Distribution,” *arXiv Preprint arXiv 2504.15186*, pp. 1-14, 2025. [CrossRef] [Google Scholar] [Publisher Link]

[20] Ghadah Alomani, and Amer I. Al-Omari, “Single Acceptance Sampling Plans Based on Truncated Lifetime Tests for Two-Parameter Xgamma Distribution with Real Data Application,” *Mathematical Biosciences and Engineering*, vol. 19, no.12, pp. 13321-13336, 2022. [CrossRef] [Google Scholar] [Publisher Link]

[21] Subhradev Sen, Mahendra Saha and Harsh Tripathi, “Double and Group Acceptance Sampling Inspection Plans Based on Truncated Life Test for the Quasi-Xgamma Distribution,” *Stochastic and Quality Control*, vol. 40, no.1, pp. 61-70, 2025. [CrossRef] [Google Scholar] [Publisher Link]