International Journal of Mathematics Trends and Technology Volume 72 Issue 4, 83-92, April 2026
ISSN: 2231-5373/ https://doi.org/10.14445/22315373/IJMTT-V72I4P112

Original Article

Use of Known Parameters of Auxiliary Variable for Mean Estimation in Successive Sampling Over Two Occasions

Deepak Majhi[1], Uttam Kumar Prasad[2]

[1,2]PG Department of Mathematics, Veer Kunwar Singh University, Ara, Bihar, India.

Corresponding Author : deepak.ismd@gmail.com

Received: 28 February 2026 Revised: 31 March 2026 Accepted: 22 April 2026 Published: 30 April 2026

Abstract*- This study takes a look at the issue of estimating the population means on this particular occasion using samples taken on two separate dates. The characteristics of three possible estimators are presented and examined, all of which rely on the known coefficients of variation, skewness, and kurtosis. We have spoken about the best practices for replacing the estimators. We have compared the suggested estimators to the optimal estimator, which is a linear combination of the means of the matched and unmatched sections of the sample on this occasion, and to the sample mean estimate when no matching is present.*

Keywords - *Auxiliary information, Bias, Mean Square Error, Optimum replacement strategy, Mathematics subject classification – 62D05.*

1. Introduction

It is impossible to determine the nature or rate of change of a characteristic over different occasions, as well as the average value of the characteristic over all occasions or the most recent one, from a one-time survey (Inquiry) conducted on a single occasion when the characteristic under study of a finite population changes over time. In order to circumvent this problem and get consistent results, sampling is repeated at various intervals.

In his examination of survey data acquired from farms, Jesson [1] initially explored the theory of consecutive sampling, sometimes known as rotation sampling, which involves partially replacing the sample units. He was the first to use all of the data gathered from the earlier studies (occasions). Patterson [2], Rao and Graham [3], Gupta [4], Das [5], Chaturvedi and Tripathi [6], and many more developed the notion of sequential (rotational) sampling.

In order to successfully develop the estimators for the population means on this occasion utilising information on two auxiliary variables available on earlier occasions, Sen [7] employed this idea. Sen expanded on his previous work on many auxiliary variables in [8], [9]. On two occasions, sequential sampling, Singh et al. [10] and Singh and Singh [11] utilised the auxiliary information at the present time and suggested estimators for the current mean of the population. For h-occasions sequential sampling, Singh [12] has expanded their previous work. Biradar and Singh [14] and Feng and Zou [13] both estimated the current mean in a subsequent sample using the auxiliary information.

There are numerous cases where data on an auxiliary variate is easily accessible both times. “For instance, in transportation survey sampling, we know the tonnage (or seat capacity) of every ship and vehicle. In hospital surveys, we know the number of beds available. In environmental surveys, we know the number of polluting industries and vehicles. In demographic surveys, we know the nature of employment, educational status, food availability, and medical aid in the area, so that we can estimate the various demographic parameters. It would be possible to demonstrate the usefulness of the current work by investigating several more scenarios within the biological (life) sciences.

For the purpose of estimating the population means at the current (second) occasion in two-occasion successive sampling, several authors have suggested different estimators using the auxiliary information from both occasions: Majhi [20], Singh and Priyanka [15], Singh and Priyanka [16], Singh and Priyanka [17], and Singh and Karna [19]. Using the known coefficients of variation, skewness, and kurtosis together with auxiliary information, this research proposes three separate estimators for the current population mean in two-occasion consecutive sampling. Empirical results have proven the detailed behaviours of the provided estimators.

2. Formulation of Estimators

Let U = (U_1, U_2, ---, U_N) be the finite population of N units, which has been sampled over two occasions. Let x(y) represent the investigated character on the first and second occurrences, respectively. Presumably, data on an auxiliary variable z, the mean of which is known, is accessible on both occasions and has a strong relationship (positive correlation) with x on the first occasion and y on the second occasion, respectively. In the first instance, let us choose a size n random sample (without replacement). On the second occasion, we draw a new simple random sample (without replacement) of size u = (n-m) = nμ units from the non-sampled units of the population, ensuring that the sample size is n. We retain a matched sub-sample of m = nλ units for this purpose.

The following notations have been used in this work:

$\bar{X}$, $\bar{Y}$, $\bar{Z}$: Population means of x, y, z respectively.

$\bar{x}_n, \bar{x}_m, \bar{y}_u, \bar{y}_m, \bar{z}_u, \bar{z}_m$: Sample mean of the respective variables of the sample sizes shown in suffices.

$\rho_{yx}, \rho_{yz}, \rho_{zx}$: Correlation coefficients between the variables y and x, y and z, and z and x, respectively.

$s_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{X})^2$: Population mean square of the variable x.

S_y^2, S_z^2: population means square of the variables y and z, respectively.

There are two sets of estimators that are developed depending on the method used to estimate the population means on the second occurrence. One set of adjusted ratio-type estimators is provided by, and it is based on a sample size u =nµthat is drawn again for the second time.

$$T_u(\alpha_i) = \bar{y}_u \left[\frac{\alpha_i \bar{Z} + \sigma_z}{\alpha_i \bar{z}_u + \sigma_z}\right] (i=1,2,3) \quad (1)$$

where $\alpha_1 = 1, \alpha_2 = \beta_1(z), \alpha_3 = \beta_2(z)$

The second set estimator is a modified chain- ratio type estimator based on the sample size $m =$ nλ, which is common to both occasions and is defined by

$$T_m(\alpha_i) = \frac{\bar{y}_m}{\bar{x}_m}\bar{x}_n \left[\frac{\alpha_i \bar{Z} + \sigma_z}{\alpha_i \bar{z}_m + \sigma_z}\right] (i=1,2,3) \quad (2)$$

Final estimators are the linear combination of these two sets of estimators and are defined as

$$T(\alpha_i) = \varphi_i T_u(\alpha_i) + (1 - \varphi_i) T_m(\alpha_i); (i=1,2,3) \quad (3)$$

where $\varphi_i (i = 1,2,3)$ the unknown constants are to be determined under certain criteria.

2.1. Bias and mean square errors of $T(\alpha_i)(i = 1,2,3)$

The estimators $T_u(\alpha_i)(i = 1,2,3)$ and $T_m(\alpha_i)(i = 1,2,3)$ are the modified ratio-type estimator and ratio chain-type estimator, respectively; they are biased estimators for the population mean $\bar{Y}$. Therefore, the final estimator $T(\alpha_i)(i = 1,2,3)$ suggested in equation (3) is also a biased estimator of the population mean $\bar{Y}$. The bias B(.) and mean square errors M(.) up to the first order approximations are derived using the large sample approximations given below:

$\bar{y}_u = \bar{Y}(1 + e_1), \bar{y}_m = \bar{Y}(1 + e_2), \bar{x}_m = \bar{X}(1+e_3),, \bar{z}_u = \bar{Z}(1+e_4), \bar{z}_m = \bar{Z}(1+e_5), \ \bar{x}_n = \bar{X}(1+e_6)$

Such that $E(e_j) = 0, \forall j=1,2,3,4,5,6$ and $|e_j| < 1, \forall j=1,2,3,4,5,6$.

Under the above transformation $T_u(\alpha_i)$(i=1,2,3) and $T_m(\alpha_i)(i=1,2,3)$ take the following forms:

$$T_u(\alpha_i) = \bar{Y}\left[1+ e_1 - \delta_i e_4 - \delta_i e_1 e_4 + \delta_i^2 e_4^2\right] \tag{4}$$

$$T_m(\alpha_i) = \bar{Y}\left[(1+ e_6 + e_2 + e_2 e_6 - e_3 - e_3 e_6 - e_2 e_3 + e_3^2) - \delta_i(e_5 + e_5 e_6 + e_2 e_5 - e_3 e_5) + \delta_i^2 e_5^2\right] \tag{5}$$

where $\delta_i = \frac{\alpha_i \bar{Z}}{\alpha_i \bar{Z} + \sigma_z}$

Thus, we have the following theorems:

Theorem 2.2. Bias of the estimators $T(\alpha_i)$(i=1,2,3) defined in equation (3) to the first order approximation is obtained as

$$B\{T(\alpha_i)\} = \varphi_i B\{T_u(\alpha_i)\} + (1-\varphi_i) B\{T_m(\alpha_i)\};\ (i=1,2,3) \tag{6}$$

where $$B\{T_u(\alpha_i)\} = \bar{Y}\left[\left(\frac{1}{u} - \frac{1}{N}\right)\delta_i\left(\delta_i C_z^2 - \rho_{yz} C_y C_z\right)\right];\ (i=1,2,3) \tag{7}$$

$$B\{T_m(\alpha_i)\} = \bar{Y}\left[\left(\frac{1}{m} - \frac{1}{n}\right)\left(C_x^2 - \rho_{yx} C_x C_y\right) - \delta_i\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\rho_{yx} C_x C_y - \left(\frac{1}{m} - \frac{1}{n}\right)\rho_{xz} C_x C_z\right\} + \delta_i^2\left(\frac{1}{m} - \frac{1}{N}\right) C_z^2\right]\ (i=1,2,3) \tag{8}$$

$$B\{T_u(\alpha_1)\} = \bar{Y}\left[\left(\frac{1}{n} - \frac{1}{N}\right)\delta_1\left(\delta_1 C_z^2 - \rho_{yz} C_y C_z\right)\right] \tag{9}$$

$$B\{T_u(\alpha_2)\} = \bar{Y}\left[\left(\frac{1}{n} - \frac{1}{N}\right)\delta_2\left(\delta_2 C_z^2 - \rho_{yz} C_y C_z\right)\right] \tag{10}$$

$$B\{T_u(\alpha_3)\} = \bar{Y}\left[\left(\frac{1}{n} - \frac{1}{N}\right)\delta_3\left(\delta_3 C_z^2 - \rho_{yz} C_y C_z\right)\right] \tag{11}$$

$$B\{T_m(\alpha_1)\} = \bar{Y}\left[\left(\frac{1}{m} - \frac{1}{n}\right)\left(C_x^2 - \rho_{yx} C_x C_y\right) - \delta_1\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\rho_{yx} C_x C_y - \left(\frac{1}{m} - \frac{1}{n}\right)\rho_{xz} C_x C_z\right\} + \delta_1^2\left(\frac{1}{m} - \frac{1}{N}\right) C_z^2\right] \tag{12}$$

$$B\{T_m(\alpha_2)\} = \bar{Y}\left[\left(\frac{1}{m} - \frac{1}{n}\right)\left(C_x^2 - \rho_{yx} C_x C_y\right) - \delta_2\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\rho_{yx} C_x C_y - \left(\frac{1}{m} - \frac{1}{n}\right)\rho_{xz} C_x C_z\right\} + \delta_2^2\left(\frac{1}{m} - \frac{1}{N}\right) C_z^2\right] \tag{13}$$

$$B\{T_m(\alpha_3)\} = \bar{Y}\left[\left(\frac{1}{m} - \frac{1}{n}\right)\left(C_x^2 - \rho_{yx} C_x C_y\right) - \delta_3\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\rho_{yx} C_x C_y - \left(\frac{1}{m} - \frac{1}{n}\right)\rho_{xz} C_x C_z\right\} + \delta_3^2\left(\frac{1}{m} - \frac{1}{N}\right) C_z^2\right] \tag{14}$$

Proof: The Bias of estimators $T(\alpha_i)$; (i=1,2,3) is defined as

$$B\{T(\alpha_i)\} = E[T(\alpha_i) - \bar{Y}] = \varphi_i E[T_u(\alpha_i) - \bar{Y}] + (1-\varphi_i) E[T_m(\alpha_i) - \bar{Y}]$$

$$= B\{T_u(\alpha_i)\} + (1-\varphi_i) B\{T_m(\alpha_i)\}$$

Where $$B\{T_u(\alpha_i)\} = E[T_u(\alpha_i) - \bar{Y}] = \bar{Y} E\left[e_1 - \delta_i e_4 - \delta_i e_1 e_4 - \delta_i^2 e_4^2\right] \tag{15}$$

Taking expectations up to first-order approximations, we have the expression for the bias of the estimators $T_u(\alpha_i)$; (i=1,2,3) as presented in equation (7), and by substituting (i=1, 2, 3)respectively in equation (7), we found that the bias of the estimators $T_u(\alpha_1), T_u(\alpha_2)andT_u(\alpha_3)$ is described in equations (9)-(11). In a similar manner, the bias of the estimator $T_m(\alpha_i)$; (i=1,2,3)is

$$B\{T_m(\alpha_i)\} = E[T_m(\alpha_i) - \bar{Y}]$$

$$= \bar{Y} E\left[(e_2 - e_3 + e_6 + e_2 e_6 - e_3 e_6 - e_2 e_3 + e_3^2) - \delta_i(e_5 + e_5 e_6 + e_2 e_5 - e_3 e_5) + \delta_i^2 e_5^2\right] \tag{16}$$

Taking expectations up to first order approximations, we have the expression for the bias of the estimators $T_m(\alpha_i)$; (i=1,2,3) as defined in equation (8), and by putting the values of (i=1,2,3)respectively in equation (8), the bias of the estimators $T_m(\alpha_1), T_m(\alpha_2)andT_m(\alpha_3)$ is calculated in equations (12)-(14).

Theorem 2.3. Mean square error of the estimators $T(\alpha_i)$; (i=1,2,3) formulated in equation (3) to the first order approximations is derived as

$$M\{T(\alpha_i)\} = \varphi_i^2 M\{T_u(\alpha_i)\} + (1-\varphi_i^2) M\{T_m(\alpha_i)\} + 2\varphi_i(1-\varphi_i) C\{T_u(\alpha_i), T_m(\alpha_i)\};\ (i=1,2,3) \tag{17}$$

$$\text{where } M\{T_u(\alpha_i)\} = \left(\frac{1}{u} - \frac{1}{N}\right)\left(1 + \delta_i^2 - 2\delta_i\rho_{yz}\right)S_y^2 \tag{18}$$

$$M\{T_m(\alpha_i)\} = \left[\left(\frac{1}{m} - \frac{1}{n}\right)\left(1 + 2\delta_i\rho_{xz} - 2\rho_{yx}\right) + \left(\frac{1}{m} - \frac{1}{N}\right)\left(1 + \delta_i^2 - 2\delta_i\rho_{yz}\right)\right]S_y^2 \tag{19}$$

$$M\{T_u(\alpha_1)\} = \left(\frac{1}{u} - \frac{1}{N}\right)\left(1 + \delta_1^2 - 2\delta_1\rho_{yz}\right)S_y^2 \tag{20}$$

$$M\{T_u(\alpha_2)\} = \left(\frac{1}{u} - \frac{1}{N}\right)\left(1 + \delta_2^2 - 2\delta_2\rho_{yz}\right)S_y^2 \tag{21}$$

$$M\{T_u(\alpha_3)\} = \left(\frac{1}{u} - \frac{1}{N}\right)\left(1 + \delta_3^2 - 2\delta_3\rho_{yz}\right)S_y^2 \tag{22}$$

$$M\{T_m(\alpha_1)\} = \left[\left(\frac{1}{m} - \frac{1}{n}\right)\left(1 + 2\delta_1\rho_{xz} - 2\rho_{yx}\right) + \left(\frac{1}{m} - \frac{1}{N}\right)\left(1 + \delta_1^2 - 2\delta_1\rho_{yz}\right)\right]S_y^2 \tag{23}$$

$$M\{T_m(\alpha_2)\} = \left[\left(\frac{1}{m} - \frac{1}{n}\right)\left(1 + 2\delta_2\rho_{xz} - 2\rho_{yx}\right) + \left(\frac{1}{m} - \frac{1}{N}\right)\left(1 + \delta_2^2 - 2\delta_2\rho_{yz}\right)\right]S_y^2 \tag{24}$$

$$M\{T_m(\alpha_3)\} = \left[\left(\frac{1}{m} - \frac{1}{n}\right)\left(1 + 2\delta_3\rho_{xz} - 2\rho_{yx}\right) + \left(\frac{1}{m} - \frac{1}{N}\right)\left(1 + \delta_3^2 - 2\delta_3\rho_{yz}\right)\right]S_y^2 \tag{25}$$

$$C\{T_u(\alpha_i), T_m(\alpha_i)\} = -\frac{1}{N}\left[1 + \delta_i^2 - 2\delta_i\rho_{yz}\right]S_y^2 \tag{26}$$

$$C\{T_u(\alpha_1), T_m(\alpha_1)\} = -\frac{1}{N}\left[1 + \delta_1^2 - 2\delta_1\rho_{yz}\right]S_y^2 \tag{27}$$

$$C\{T_u(\alpha_2), T_m(\alpha_2)\} = -\frac{1}{N}\left[1 + \delta_2^2 - 2\delta_2\rho_{yz}\right]S_y^2 \tag{28}$$

$$C\{T_u(\alpha_3), T_m(\alpha_3)\} = -\frac{1}{N}\left[1 + \delta_3^2 - 2\delta_3\rho_{yz}\right]S_y^2 \tag{29}$$

Proof: The Mean square error of the estimator $T(\alpha_i)$; (i=1,2,3)defined in (3) is given by

$$M\{T(\alpha_i)\} = E[T(\alpha_i) - \bar{Y}]^2 = E\left[\varphi_i\{T_u(\alpha_i) - \bar{Y}\} + \left(1 - \varphi_i\right)\{T_m(\alpha_i) - \bar{Y}\}\right]^2$$

$$= \varphi_i^2 M\{T_u(\alpha_i)\} + (1 - \varphi_i)^2 M\{T_m(\alpha_i)\} + 2\varphi_i(1 - \varphi_i)C\{T_u(\alpha_i), T_m(\alpha_i)\} \tag{30}$$

Where $M\{T_u(\alpha_i)\} = E[T_u(\alpha_i) - \bar{Y}]^2 = \bar{Y}^2 E[e_1 - \delta_i e_4]^2$

Squaring the quantity in the bracket and taking expectations up to first order approximations, we have the expression for $M\{T_u(\alpha_i)\}$(i=1,2,3) as described in equation (18), and by substituting (i=1,2,3)respectively in equation (18), the mean square of error of estimators $T_u(\alpha_1), T_u(\alpha_2)andT_u(\alpha_3)$ are presented in equations (20)-(22).

Similarly, $M\{T_m(\alpha_i)\} = E[T_m(\alpha_i) - \bar{Y}]^2 = \bar{Y}^2 E[e_2 - e_3 + e_6 - \delta_i e_5]^2$

Squaring the quantity in the bracket and using mathematical expectations up to first order approximations, we have the expression for $M\{T_m(\alpha_i)\}$(i=1,2,3) as described in equation (19), and by putting (i=1,2,3)respectively in equation (19), the mean square of error of estimators $T_m(\alpha_1), T_m(\alpha_2)andT_m(\alpha_3)$ are presented in equations (23)-(25).

Finally, $C[T_u(\alpha_i), T_m(\alpha_i)] = E[\{T_u(\alpha_i) - \bar{Y}\}\{T_m(\alpha_i) - \bar{Y}\}]$

$$= E[\bar{Y}(e_1 - \delta_i e_4)\bar{Y}(e_2 - e_3 + e_6 - \delta_i e_5)]$$

Simplify the expression and apply mathematical expectations up to first-order approximations. We have the expression for $C[T_u(\alpha_i), T_m(\alpha_i)]$; (i=1,2,3) as described in equation (26), and by taking (i=1,2,3)respectively in equation (26), the covariance of estimators $C[T_u(\alpha_1), T_m(\alpha_1)], C[T_u(\alpha_2), T_m(\alpha_2)] and C[T_u(\alpha_3), T_m(\alpha_3)]$ is obtained in equations from (27) to (29).

Remark: The above results are derived under the assumption that the coefficients of variation of the variables x, y, and z are approximately equal.

3. Minimum Mean Squared Error of Estimators $T(\alpha_i)$; (i =1,2, 3).

Since the mean square error expression of estimators $T(\alpha_i)$; (i =1,2, 3) derived in equation (17) is a function of unknown constants φ_i, it is minimized with respect to φ_i , and subsequently, the optimum value of φ_i is obtained as

$$\varphi_{i_{\text{opt}}} = \frac{M\{T_m(\alpha_i)\} - C\{T_u(\alpha_i), T_m(\alpha_i)\}}{M\{T_u(\alpha_i)\} + M\{T_m(\alpha_i)\} - 2C\{T_u(\alpha_i), T_m(\alpha_i)\}}; (i = 1,2,3) \tag{31}$$

Now, substituting the value $\varphi_{i\text{opt}}$ given in equation (31), we have the optimum mean square error of estimators $T(\alpha_i)$; (i =1,2, 3) as

$$M\{T(\alpha_i)\}_{\text{opt}} = \frac{M\{T_u(\alpha_i)\}.M\{T_m(\alpha_i)\} - [C\{T_u(\alpha_i), T_m(\alpha_i)\}]^2}{M\{T_u(\alpha_i)\} + M\{T_m(\alpha_i)\} - 2C\{T_u(\alpha_i), T_m(\alpha_i)\}} \tag{32}$$

Further, substituting the values from equations (20) - (29) in equations (31) and (32), simplified values of $\varphi_{i_{\text{opt}}}$ and $M\{T(\alpha_i)\}_{\text{opt}}$ are shown in Theorem 3.1.

Theorem 3.1.

$$\varphi_{1_{\text{opt}}} = \frac{\mu_1[A_1 + \mu_1 A_2]}{[A_1 + \mu_1{}^2 A_2]} \tag{33}$$

$$\varphi_{2_{\text{opt}}} = \frac{\mu_2[B_1 + \mu_2 B_2]}{[B_1 + \mu_2{}^2 B_2]} \tag{34}$$

$$\varphi_{3_{\text{opt}}} = \frac{\mu_3[C_1 + \mu_3 C_2]}{[C_1 + \mu_3{}^2 C_2]} \tag{35}$$

$$M\{T(\alpha_1)\}_{\text{opt}} = \frac{[A_3 + \mu_1 A_4 - \mu_1^2 f A_4]}{[A_1 + \mu_1^2 A_2]} \frac{S_Y^2}{n} \tag{36}$$

$$M\{T(\alpha_2)\}_{\text{opt}} = \frac{[B_3 + \mu_2 B_4 - \mu_2^2 f B_4]}{[B_1 + \mu_2^2 B_2]} \frac{S_Y^2}{n} \tag{37}$$

$$M\{T(\alpha_3)\}_{\text{opt}} = \frac{[C_3 + \mu_3 C_4 - \mu_3^2 f C_4]}{[C_1 + \mu_3^2 C_2]} \frac{S_Y^2}{n} \tag{38}$$

Where $A_1 = 1+\delta_1^2-2\delta_1\rho_{yz}$, $A_2 = 1 - 2\rho_{yx} + 2\delta_1\rho_{xz}$, $A_3 = (1-f)A_1^2$, $A_4 = A_1 A_2$, $B_1 = 1+\delta_2^2-2\delta_2\rho_{yz}$, $B_2 = 1 - 2\rho_{yx} + 2\delta_2\rho_{xz}$, $B_3 = (1-f)B_1^2$, $B_4 = B_1 B_2$, $C_1 = 1+\delta_3^2-2\delta_3\rho_{yz}$, $C_2 = 1 - 2\rho_{yx} + 2\delta_3\rho_{xz}$, $C_3 = (1-f)C_1^2$, $C_4 = C_1 C_2$, $f = \frac{n}{N}$ (sampling fraction) and $\boldsymbol{\mu_i} = \frac{\boldsymbol{u}}{\boldsymbol{n}}$ **(i=1,2, 3)** is the fraction of fresh sample drawn at the second (current) occasion.

4. Optimum Replacement Strategy for $T(\alpha_i)$; (i =1,2, 3).

To determine the optimum values of μ_i(i=1,2, 3) (fraction of samples to be taken afresh at the second occasion) so that population means $\bar{Y}$ may be estimated with maximum precision, we minimize mean square errors $T(\alpha_i)$; (i =1,2, 3) given in equations (36) to (38) respectively with respect to μ_i. For this, differentiating equations (36), (37), and (38) with respect to μ_i and equating these to zero, which result in quadratic equations in μ_i and respective solutions of $\hat{\mu}_i$(i=1,2, 3) are given below:

$$Q_1\mu_1^2 + 2Q_2\mu_1 + Q_3 = 0 \tag{39}$$

$$\hat{\mu}_1 = \frac{-Q_2 \pm \sqrt{Q_2^2 - Q_1 Q_3}}{Q_1} \tag{40}$$

$$P_1\mu_2^2 + 2P_2\mu_2 + P_3 = 0 \tag{41}$$

$$\hat{\mu}_2 = \frac{-P_2 \pm \sqrt{P_2^2 - P_1 P_3}}{P_1} \tag{42}$$

$$R_1\mu_2^2+2R_2\mu_2+R_3=0 \tag{43}$$

$$\hat{\mu}_3=\frac{-R_2\pm\sqrt{R_2^2-R_1R_3}}{R_1} \tag{44}$$

where $Q_1=A_2A_4$, $Q_2=fA_4A_1+A_2A_3$, $Q_3=-A_1A_4$, $P_1=B_2B_4$, $P_2=fB_4B_1+B_2B_3$, $P_3=-B_1B_4$,

$R_1=C_2C_4$, $R_2=fC_4C_1+C_2C_3$, $R_3=-C_1C_4$.

From equations (40), (42), and (44), it is clear that the real values of $\hat{\mu}_i$(i=1,2,3)exits if and only if the quantities under square roots are greater than or equal to zero. For any combinations of correlation coefficients ρ_{yx}, ρ_{yz}andρ_{xz}that satisfy the condition of real solutions. Two real values of $\hat{\mu}_i$(i=1,2,3) are possible. Hence, while choosing the values of $\hat{\mu}_i$(i=1,2,3), it should be mentioned that $0\leq\hat{\mu}_i\leq 1$and all other values of $\hat{\mu}_i$are inadmissible. If both the values of $\hat{\mu}_i$ are admissible, the least one is the best choice as it will reduce the cost of the survey. Substituting the admissible value of $\hat{\mu}_i$say $\mu_i^{(0)}$(i=1,2,3)from equations (40), (42), and (44) into equations (36), (37), and (38), respectively, we have the optimum values of mean square errors $M\{T(\alpha_i)\}^*{}_{\text{opt}}$, which are shown below

$$M\{T(\alpha_1)\}^*{}_{\text{opt}}=\frac{[A_3+\mu_1^{(0)}A_4-\mu_1^{(0)2}fA_4]}{[A_1+\mu_1^{(0)2}A_2]}\frac{S_Y^2}{n} \tag{45}$$

$$M\{T(\alpha_2)\}^*{}_{\text{opt}}=\frac{[B_3+\mu_2^{(0)}B_4-\mu_2^{(0)2}fB_4]}{[B_1+\mu_2^{(0)2}B_2]}\frac{S_Y^2}{n} \tag{46}$$

$$M\{T(\alpha_3)\}^*{}_{\text{opt}}=\frac{[C_3+\mu_3^{(0)}C_4-\mu_3^{(0)2}fC_4]}{[C_1+\mu_3^{(0)2}C_2]}\frac{S_Y^2}{n} \tag{47}$$

5. Efficiency Comparison

The percent relative efficiencies $T(\alpha_i)$; (i =1,2, 3) with respect to (i) the sample mean estimator $\bar{y}_n$, when there is no matching, and (ii) $\hat{\bar{Y}}=\varphi^*\bar{y}_u+(1\text{-}\varphi^*)\bar{y}'_m$when there is no auxiliary information is used on any occasion, where, since $\bar{y}_n$ and $\hat{\bar{Y}}$ are unbiased estimators of $\bar{Y}$, therefore, following Sukhatma et al. [20], the variance of $\bar{y}_n$ and optimum variance of $\hat{\bar{Y}}$are given by

$$V(\bar{y}_n)=\left(\frac{1}{n}-\frac{1}{N}\right)S_y^2 \tag{48}$$

$$V(\hat{\bar{Y}})_{\text{opt}}=\left[1+\sqrt{1\text{-}\rho_{yx}^2}\right]\frac{S_y^2}{2n}-\frac{S_y^2}{N} \tag{49}$$

For different choices of ρ_{yx}, ρ_{yz} and ρ_{xz}, Table 1 -3 shows the Optimum values of μ_i(i=1,2,3) i.e., $\mu_i{}^{(0)}$(i=1,2, 3) and percent relative efficiencies $E_i^{(1)}$ and $E_i^{(2)}$of the estimators $T(\alpha_i)$; (i =1,2, 3) (under optimal condition) with respect to $\bar{y}_n$ and $\hat{\bar{Y}}$, respectively, where

$$E_i^{(1)}=\frac{V(\bar{y}_n)}{M\{T(\alpha_i)\}^*_{opt}}\text{X }100 \text{ and } E_i^{(2)}=\frac{V(\hat{\bar{Y}})_{\text{opt}}}{M\{T(\alpha_i)\}^*_{opt}}\text{X}100;\ (\text{i }=1,2,3)$$

Table 1. Optimum values of $\mu_1^{(0)}$ and percent relative efficiencies of the estimator $T(\alpha_1)$ with respect to $\bar{y}_n$ and $\hat{\hat{Y}}$ for $f = 0.1$ and $\delta_1 = 0.20$.

ρ_{yz} ↓	ρ_{xz} ↓	ρ_{yx} →	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.3	0.3	$\mu_1^{(0)}$ $E_1^{(1)}$ $E_1^{(2)}$	* - -	* - -	* - -	0.9034 110.67 **	0.8150 115.14 **	0.7862 123.32 **	0.8031 137.82 **
	0.4	$\mu_1^{(0)}$ $E_1^{(1)}$ $E_1^{(2)}$	* - -	* - -	* - -	0.9457 119.22 105.97	0.8051 123.79 104.13	0.7648 133.18 103.58	0.7823 150.05 103.02
	0.5	$\mu_1^{(0)}$ $E_1^{(1)}$ $E_1^{(2)}$	0.11 * *	* - -	* - -	* - -	0.8111 136.64 114.94	0.7431 147.53 114.75	0.7583 167.72 115.16
0.5	0.3	$\mu_1^{(0)}$ $E_1^{(1)}$ $E_1^{(2)}$	0.2639 115.92 112.96	0.2220 120.65 115.05	**0.0741** 124.74 115.46	* - -	0.8931 156.85 131.94	0.7241 169.04 131.47	0.7298 194.14 133.30
	0.4	$\mu_1^{(0)}$ $E_1^{(1)}$ $E_1^{(2)}$	0.3139 138.56 135.01	0.3141 145.40 138.66	0.3011 153.49 142.07	0.2285 162.77 144.68	* - -	0.7295 203.71 158.44	0.6947 236.60 162.45
	0.5	$\mu_1^{(0)}$ $E_1^{(1)}$ $E_1^{(2)}$	0.3127 177.00 172.47	0.3250 186.63 177.98	0.3377 198.35 183.59	0.3482 213.09 189.41	0.3370 232.41 195.50	* - -	0.6514 315.03 216.30

* Indicate $\mu_1^{(0)}$ do not exit, ** indicate no gain.

Table 2. Optimum values of $\mu_2^{(0)}$ and percent relative efficiencies of $T(\alpha_2)$ with respect to $\bar{y}_n$ and $\hat{\hat{Y}}$ for $f = 0.2$ and $\delta_2 = 0.40$

ρ_{yz}	ρ_{xz}	ρ_{yx} →	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.3	0.3	$\mu_2^{(0)}$ $E_2^{(1)}$ $E_2^{(2)}$	* - -	* - -	* - -	0.8825 111.12 **	0.7873 117.78 **	0.7616 129.95 **	0.7848 152.63 **
	0.4	$\mu_2^{(0)}$ $E_2^{(1)}$ $E_2^{(2)}$	* - -	* - -	* - -	0.9323 119.33 102.28	0.7766 126.13 100.37	0.7404 139.92 **	0.7647 166.00 **

	0.5	$\mu_2^{(0)}$ $E_2^{(1)}$ $E_2^{(2)}$	0.2099 ** **	0.0861 ** **	* - -	* - -	0.7821 138.28 110.05	0.7186 154.04 110.03	0.7417 184.74 110.30
0.5	0.3	$\mu_2^{(0)}$ $E_2^{(1)}$ $E_2^{(2)}$	0.3003 110.14 106.51	0.2776 116.11 109.18	**0.1935** 122.57 110.84	* - -	0.8705 157.19 125.10	0.6990 174.68 124.77	0.7142 211.96 126.55
	0.4	$\mu_2^{(0)}$ $E_2^{(1)}$ $E_2^{(2)}$	0.3289 129.18 124.93	0.3338 136.99 128.82	0.3305 146.53 132.51	0.2896 158.29 135.67	* - -	0.7014 207.06 147.90	0.6800 254.35 151.86
	0.5	$\mu_2^{(0)}$ $E_2^{(1)}$ $E_2^{(2)}$	0.3181 161.19 155.89	0.3316 171.57 161.34	0.3464 184.48 166.83	0.3608 201.18 172.44	0.3621 224.02 178.28	* - -	0.6366 329.82 196.92

* Indicate $\mu_2^{(0)}$ do not exit, ** indicate no gain.

Table 3. Optimum values of $\mu_3^{(0)}$ and percent relative efficiencies of the estimator $T(\alpha_3)$ with respect to $\bar{y}_n$ and $\hat{\hat{Y}}$ for $f = 0.3 and \delta_3 = 0.60$.

ρ_{yz}	ρ_{xz}	ρ_{yx}	0.5	0.6	0.7	0.8	0.9
0.3	0.3	$\mu_3^{(0)}$ $E_3^{(1)}$ $E_3^{(2)}$	0.3704 157.63 145.90	0.3652 168.59 149.85	0.1328 181.44 152.62	0.6461 206.08 160.28	0.6550 241.99 166.15
	0.4	$\mu_3^{(0)}$ $E_3^{(1)}$ $E_3^{(2)}$	0.2844 192.15 177.85	0.2665 205.19 182.39	0.1494 219.83 184.92	* - -	0.7165 307.99 211.47
	0.5	$\mu_3^{(0)}$ $E_3^{(1)}$ $E_3^{(2)}$	0.1530 260.98 241.55	0.1199 275.05 244.49	**0.0137** 285.63 240.27	* - -	* - -
0.5	*0.3*	$\mu_3^{(0)}$ $E_3^{(1)}$ $E_3^{(2)}$	0.4166 160.40 148.46	0.4440 171.96 152.85	0.4797 187.27 157.53	0.5303 209.38 162.85	0.6149 247.46 169.91
	0.4	$\mu_3^{(0)}$ $E_3^{(1)}$ $E_3^{(2)}$	0.3661 201.66 186.65	0.3890 217.05 192.93	0.4145 237.63 199.89	0.3378 267.32 207.92	0.5957 321.07 220.45
	0.5	$\mu_3^{(0)}$ $E_3^{(1)}$ $E_3^{(2)}$	0.2712 296.38 274.32	0.2855 320.48 284.87	0.2985 352.84 296.81	0.2855 399.58 310.78	* - -

* Indicate $\mu_3^{(0)}$ do not exit, ** indicate no gain.

6. Interpretation of Empirical Results

(1) From Table-1, it is clear that

(a) For fixed value of $\rho_{xz}$and ρ_{yz}, the values of $E_1^{(1)}$ are increasing uniformly while $E_1^{(2)}$and $\mu_1^{(0)}$do not follow any pattern with increasing values of ρ_{yx}.

(b) For fixed values of ρ_{xz} and ρ_{yx}, the values of $E_1^{(1)}$ and $E_1^{(2)}$ are increasing, while the values of $\mu_1^{(0)}$do not follow any pattern with increasing values of ρ_{yz}.

(c) For fixed values of ρ_{yz} and ρ_{yx}, the values of $E_1^{(1)}$ and $E_1^{(2)}$ are increasing while the values of $\mu_1^{(0)}$do not follow any pattern as the values of ρ_{xz}. This is highly desirable because as the information of auxiliary increases, the efficiencies of the estimators increase.

(d) Minimum value of $\mu_1^{(0)}$ is 0.0741, which indicates that the fraction to be replaced at the current occasion is about 7 percent of the total sample size, leading to an appreciable reduction in cost.

(2) From Table 2, it is found that

(a) For fixed value of $\rho_{xz}$and ρ_{yz}, the values of $E_2^{(1)}$are increasing uniformly while $E_2^{(2)}$and $\mu_2^{(0)}$do not follow any pattern with increasing values of ρ_{yx}.

(b) For fixed values of ρ_{xz} and ρ_{yx}, the values of $E_2^{(1)}$ and $E_2^{(2)}$ are increasing, while the values of$\mu_2^{(0)}$ do not follow any pattern with increasing values of ρ_{yz}.

(c) For fixed values of ρ_{yz} and ρ_{yx}, the values of $\mu_2^{(0)}$, $E_2^{(1)}$ and $E_2^{(2)}$ do not follow any pattern while increasing values of ρ_{xz}.

(d) Minimum value of $\mu_2^{(0)}$is 0.1935, which indicates that the fraction to be replaced at the current occasion is about nineteen percent of the total sample size, leading to an appreciable reduction in cost.

(3) From Table-3, it is visible that

(a) For fixed value of $\rho_{xz}$and ρ_{yz}, the values of $E_3^{(1)}$ and $E_3^{(2)}$ are increasing while the values of $\mu_3^{(0)}$ do not follow any pattern with increasing values of ρ_{yx}.

(b) For fixed values of ρ_{xz} and ρ_{yx}, the values of $E_3^{(1)}$and $E_3^{(2)}$ are increasing while the values of $\mu_3^{(0)}$are decreasing with increasing values of ρ_{yz} and for some largest values of ρ_{xz}, the values of $\mu_3^{(0)}$ do not follow any pattern with increasing values of ρ_{yz}.

(c) For fixed values of ρ_{yz} and ρ_{yx}, the values of $\mu_3^{(0)}$, $E_3^{(1)}$ and $E_3^{(2)}$ do not follow any pattern while increasing values of ρ_{xz}.

(d) Minimum value of $\mu_3^{(0)}$ is 0.0137, which indicates that the fraction to be replaced at the current occasion is about 1 percent of the total sample size, leading to an appreciable reduction in cost.

7. Conclusion

From the above interpretations shown in Tables 1-3, we conclude that the use of information on an auxiliary variable in the estimation of the population mean of the study variable in two-occasion successive sampling is useful in terms of percent relative efficiencies and reduction of the cost of the survey. Hence, looking at the nice behaviors of the suggested estimators, it may be recommended to the survey statisticians for their practical applications.

Acknowledgement

The authors are thankful to the editor-in- chief and reviewers for their valuable feedback and suggestions to improve the quality of the research paper.

References

[1] Raymond James Jessen, "*Statistical Investigation of a Sample Survey for Obtaining Farm Facts*," Iowa Agricultural Experiment Station Research Bulletin No. 304, Ames, Iowa, 1943. [Google Scholar] [Publisher Link]

[2] H.D. Patterson, "Sampling on Successive Occasions with Partial Replacement of Units," *Journal of the Royal Statistical Society,* vol. 12, no. 2, pp. 241-255, 1950. [Google Scholar] [Publisher Link]

[3] J.N.K. Rao, and Jack E. Graham, "Rotation Design for Sampling on Repeated Occasions," *Journal of the American Statistical Association,* vol. 59, no. 306, pp. 492-509, 1964. [Google Scholar] [Publisher Link]

[4] P.C. Gupta, "Sampling on Two Successive Occasions," *Journal of Statistical Research*, vol. 13, pp. 7-16, 1979. [Google Scholar]

[5] A.K. Das, "Estimation of Population Ratio on Two Occasions," *Journal of Indian Society of Agricultural Statistics*, vol. 34, pp. 1-9, 1982. [Google Scholar]

[6] D.K. Chaturvedi, and T.P. Tripathi, "Estimation of Population Ratio on Two Occasions using Multivariate Auxiliary Information," *Journal of Indian Statistical Asso*ciation, vol. 21, pp. 113-120, 1983. [Google Scholar]

[7] A.R. Sen, "Successive Sampling with Two Auxiliary Variables," *Sankhya: The Indian Journal of Statistics, Series B*, vol. 33, no. 3/4, pp. 371-378, 1971. [Google Scholar] [Publisher Link]

[8] A.R. Sen, "Successive Sampling with p $(p \geq 1)$ Auxiliary Variables," *The Annals of Mathematical Statistics*, vol. 43, no. 6, pp. 2031-2034, 1972. [Google Scholar] [Publisher Link]

[9] A.R. Sen, "Theory and Application of Sampling on Repeated Occasions with Several Auxiliary Variables," *Biometrics*, vol. 29, no. 2, pp. 381-385, 1973. [CrossRef] [Google Scholar] [Publisher Link]

[10] V.K. Singh, G.N. Singh, and D. Shukla, "An Efficient Family of Ratio-Cum-Difference Type Estimators in Successive Sampling Over Two Occasions," *Journal of Science Research*, vol. 41, pp. 149-159, 1991. [Google Scholar]

[11] Garib Nath Singh, and Vijay Kumar Singh, "On the use of Auxiliary Information in Successive Sampling," *Journal Indian Society Agricultural Statistics*, vol. 54, no. 1, pp. 1-12, 2001. [Google Scholar]

[12] G.N. Singh, "Estimation of Population Mean using Auxiliary Information on Recent Occasion in H-occasion Successive Sampling," *Statistics in Transition,* vol. 6, no. 4, pp. 523-532, 2003. [Google Scholar]

[13] Shiyong Feng, and Guohua Zou, "Sample Rotation Method with Auxiliary Variable," *Communications in Statistics-Theory and Methods*, vol. 26, no. 6, pp. 1497-1509, 1997. [CrossRef] [Google Scholar] [Publisher Link]

[14] R.S. Biradar, and H.P. Singh, "Successive Sampling using Auxiliary Information on Both Occasions," *Calcutta Statistical Association Bulletin,* vol. 51, no. 3-4, pp. 243-252, 2001. [CrossRef] [Google Scholar] [Publisher Link]

[15] G.N. Singh, "On the use of Chain-type Ratio Estimator in Successive Sampling," *Statistics in Transition*," vol. 7, no. 1, pp. 21-26, 2005. [Google Scholar]

[16] G.N. Singh, and Kumari Priyanka, "On the use of Chain-type Ratio to Difference Estimator in Successive Sampling," *International Journal of Applied Mathematics and Statistics*, vol. 5, no. S06, 2006. [Google Scholar] [Publisher Link]

[17] G.N. Singh, and Kumari Priyanka, "On the use of Auxiliary Information in Search of Good Rotation Patterns on Successive Occasions,' *International Journal of Statistics & Economics (Bulletin of Statistics and Economics),* vol. 1, no. A07, pp. 42-60, 2007. [Google Scholar]

[18] G.N. Singh, and Kumari Priyanka, "Search of Good Rotation Patterns to Improve the Precision of Estimates at Current Occasion," *Communications in Statistics- Theory and Methods,* vol. 37, no. 3, pp. 337-348, 2008. [CrossRef] [Google Scholar] [Publisher Link]

[19] Garib Nath Singh, and Jaishree Prabha Karna, "Estimation of Population Mean on the Current in Two- Occasion Successive Sampling," *Metron-International Journal of Statistics*, vol. 67, no. 1, pp. 87-103, 2009. [Google Scholar]

[20] Deepak Majhi, "A New Rotation Patterns in Successive Sampling over Two Occasions," *International Journal of Mathematics and Computation*, vol. 29, no. 3, 2018. [Publisher Link]

[21] Pandurang V. Sukhatme et al., *Sampling Theory of Surveys with Applications*, Iowa State University Press, Ames, Iowa (USA) and Indian Society of Agricultural Statistics, New Delhi (India), 1970. [Google Scholar] [Publisher Link]