

Original Article

Antiplane Elastostatic Analysis of a Homogeneous Wedge with a Crack at its Apex

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Abstract - An infinite homogeneous wedge containing an edge crack of length 'a', subjected to concentrated loads at two points, is analyzed for stresses and displacement due to Elastostatic deformation. The series expressions for the stresses $\sigma_{rz}(r, \theta)$, $\sigma_{\theta z}(r, \theta)$ everywhere in the body are then computed for use in estimating other fracture parameters. The wedge contains a crack of length "a" lying along the ray $\theta = 0$, $0 \leq r \leq a$. The problem is formulated by an infinite Mellin transform, and transformed into an integral parameter plane where the transformed problem is solved to get the displacement by the Wiener-Hopf technique. These produced a two-dimensional Neumann boundary value problem in terms of the only non-zero displacement component, $W(r, \theta)$. The presence of the crack motivates the expectation of different transform plane displacement, $W_i(r, 0)$ $i = 1, 2$ for $0 \leq r \leq a$ Mellin transform is next applied, where s is the transform parameter, which introduces four coefficients, $A_1(s), A_2(s), B_1(s)$ and $B_2(s)$ that are evaluated by the Wiener-Hopf technique using given boundary conditions. A series of closed-form solutions was thereafter obtained for displacement and stresses that were used to analyze the fracture parameters. The stress field at the crack tip of the wedge is employed to compute the mode III stress intensity factor, K_{III} . The result is that along the crack region, the displacements $W(r, 0^+) \neq W(r, 0^-)$ for $0 \leq r \leq a$, which implies discontinuity of the displacement field and that the tearing stress $\sigma_{\theta z}(r, 0)$ along the region is zero. Also, at the region ahead of the crack, $a \leq r \leq a$, $W(r, 0^+) = W(r, 0^-)$ implies the continuity of the displacement fields, and that the tearing stress $\sigma_{\theta z}(r, 0^+) = \sigma_{\theta z}(r, 0^-)$ It is also continuous there. We then conclude that, as a result of the crack, there are different fracture parameter responses at every region of the wedge material. It is also found that the stress intensity factor, K_{III} It is independent of the material property, but depends linearly on the concentrated load. Therefore, irrespective of the structural material, there are always certain fracture responses due to the application of load.

Keywords - Antiplane, Crack, Elastostatic, Homogeneous, Wedge.

1. Introduction

The stress and strain distributions in the mechanical mechanism of fractures in elastic bodies of small size are important in determining which materials will fail and in the study of fracture behavior [1]. Especially, there has been an intense focus on the antiplane (mode III) elastostatic problems due to their mathematical convenience and one-dimensionality in applications to torsional loading and out-of-plane shear deformation [2]. The cracks cause singularities on the stress field, particularly on the tip of the crack, where the stress intensity factor is decisive in defining the behavior of fractures.

Wedge-shaped domains are a significant category of geometries in elasticity, which occur in practice in engineering structures in the form of notched components, joints, and geological structures [3]. The theoretical and practical value of studying cracks in wedges is therefore important. As a crack is found at or close to the apex of a wedge, the interaction between geometric singularities and stress concentration at the crack results in a complex set of boundary value problems whose resolution is advanced with the application of powerful analytical methods [4].



A number of mathematical techniques have been developed over the years to solve the crack problems in the elastic media, with some of these being the integral transform techniques, complex variable methods, and numerical methods [5]. The Mellin transform has been found to be useful in the above problems, especially those that are associated with wedge geometries, since it inherently takes into account the radial geometry of the domain. Moreover, the Wiener-Hopf method offers a powerful method of solving mixed boundary value problems that occur in a semi-infinite and wedge-like domain to enable calculation of unknown transform coefficients by factorization processes [6].

With antiplane elasticity, the equations of the problem become a scalar Laplace equation of the out-of-plane displacement field, with suitable boundary conditions. The development of a crack along a radial line of the wedge, however, causes discontinuities in the displacement field, and special conditions of the boundary conditions on each side of the crack are required Cheng et al. [4]. This will give a two-dimensional Neumann boundary value problem, the solution of which will provide information on the stress distribution and fracture properties of the system. By making use of the infinite Mellin transform, a formal solution is obtained for the stress distribution in an infinite wedge under concentrated surface loading (Boundary condition). The results for the case in which each boundary surface is subjected to such a load for a finite distance measured from the vertex of the wedge are reduced to definite values by use of infinite integrals. These can be evaluated exactly when the wedge is a semi-infinite solid and are in a form suitable for numerical computation for other parameters. The surface of the crack is stress-free ($\sigma_{\theta z}(r, 0) = 0; 0 \leq r \leq a$) But it is displaced such that

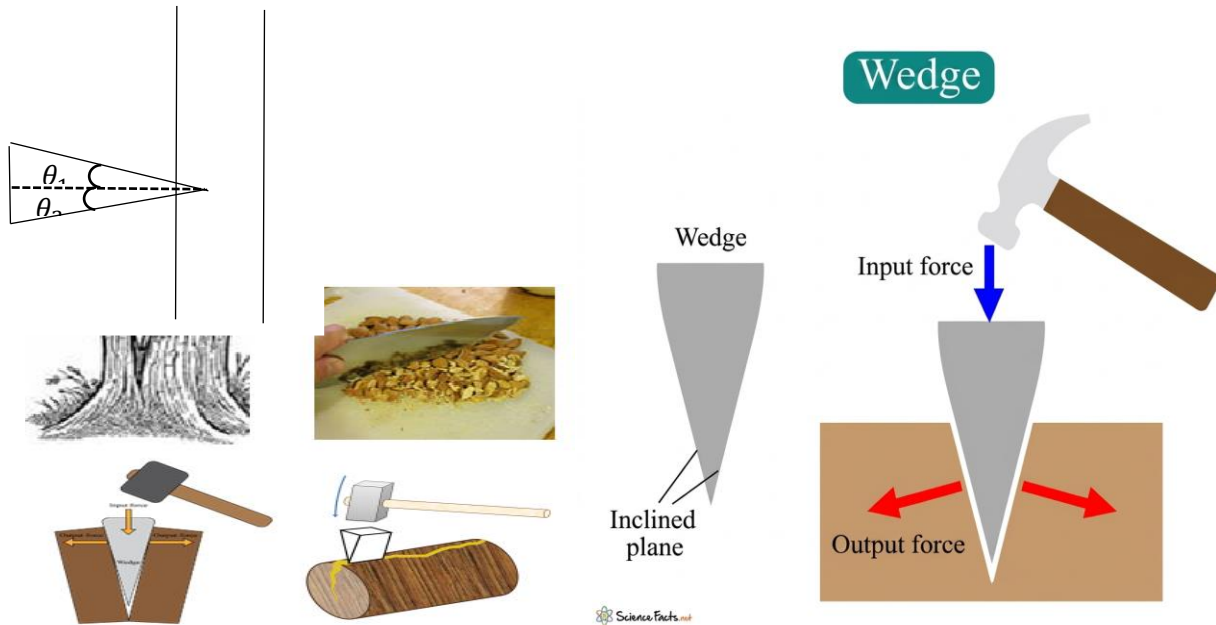
$$w(r, 0^+) = w(r, 0^-); 0 \leq r \leq a$$

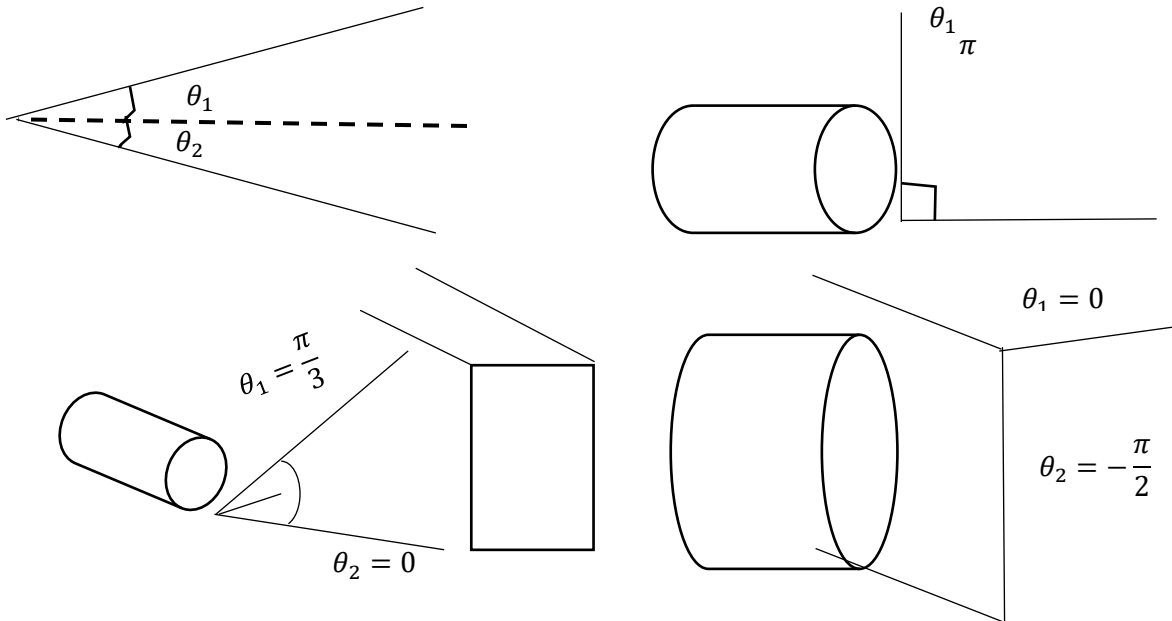
Continuity conditions

$$w(r, 0^+) \neq w(r, 0^-), \quad \sigma_{\theta z}(r, 0^+) = \sigma_{\theta z}(r, 0^-)$$

hold along $\theta = 0, r \geq a$.

The task is to find valid expressions for the polar stresses. $\sigma_{\theta z}(r, \theta)$ and $\sigma_{rz}(r, \theta)$ everywhere in the body. This defines a problem with different boundaries on which derivatives of displacement are prescribed on one part, and displacement is prescribed on the other part. Such problems are tackled by the Wiener-Hopf technique to get the transformed displacement, which is used to analyze the system. The solution of the Wiener-Hopf problems gives the transformed solution in terms of the transformed parameter.





Although there are studies on crack issues in a wedge and a half-plane, an edge crack situated at the apex of a homogeneous infinite wedge under concentrated loading is an unexplored area, especially in the context of exact analytical solutions. The current research fills this gap by using the infinite Mellin transform with the Wiener-Hopf method to come up with the closed-form series solutions to the displacement and stress fields. Key fracture parameters, such as mode III stress intensity factor, are then evaluated using these solutions.

The objective of this work is therefore to provide a rigorous analytical treatment of the antiplane elastostatic behavior of a homogeneous wedge with an apex crack, and to examine the influence of loading conditions on the resulting fracture response. The findings contribute to a deeper understanding of singular stress fields in wedge geometries and offer potential applications in fracture mechanics, structural integrity assessment, and related areas of applied mathematics.

2. Methodology

2.1. Formulation of the Governing Equations

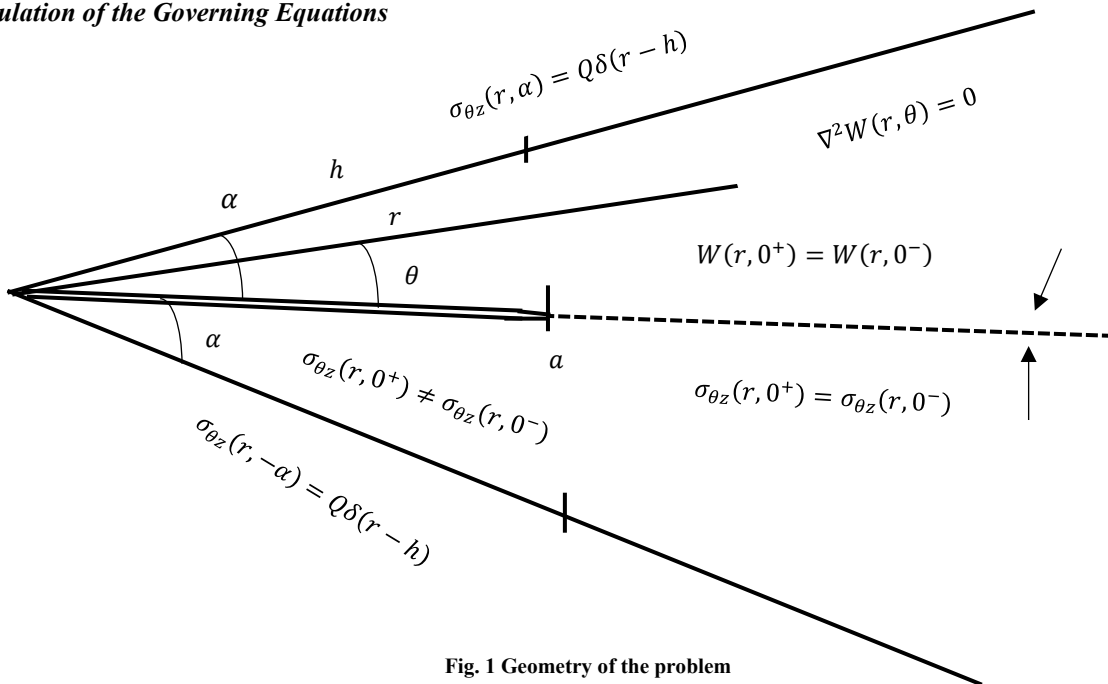


Fig. 1 Geometry of the problem

We modeled the prescribed displacement and stress to suit the analysis by the Wiener-Hopf technique. The Wiener-Hopf technique was used to obtain the displacement in the transformed parameter plane, and the inverse Mellin transform was used to obtain the displacement everywhere in the material. We then used the displacement to get other fracture parameters.

The problem is to find the displacement everywhere in a cracked wedge put in antiplane state by application of two concentrated loads of magnitude Q at the boundaries of the wedge at the points given in polar coordinates as (r, α) and $(r, -\alpha)$ at a distance h from the origin. The problem is formulated in cylindrical polar coordinates.

(r, θ, z) $r \geq 0, -\alpha < \theta < \alpha$ and $-\infty < z < \infty$. The applied concentrated loads are given by $\sigma_{\theta z}(r, \alpha) = Q\delta(r - h)$ and $\sigma_{\theta z}(r, -\alpha) = Q\delta(r - h)$ expressed in terms of the Dirac delta function $\delta(r - h)$. The crack surfaces of length “a” are stress-free. $\sigma_{\theta z}(r, 0) = 0, 0 \leq r \leq a$. Each of the three components of displacement (U, V, W) is independent of the z – variable that is, $U = U(x, y), V = V(x, y)$ and $W = W(x, y)$ Moreover, for the state of antiplane deformation, the only non-vanishing component is the one in the z – direction. When the displacement $w(x, y)$ Once it is found, the other fracture parameters can be obtained using.

$$\sigma_{\theta z}(r, \theta) = \frac{\mu}{r} \frac{\partial W}{\partial \theta}, \sigma_{rz}(r, \theta) = \mu \frac{\partial W}{\partial r}(r, \theta).$$

In other words, $U(x, y) = 0, V(x, y) = 0$ but $W(x, y) \neq 0$ for antiplane strain to exist. In this state, the non-vanishing polar stresses given by Timoshenko & Goodier (1951) are as follows:

$$\sigma_{\theta z}(r, \theta) = \frac{\mu}{r} \frac{\partial W}{\partial \theta}, \sigma_{rz}(r, \theta) = \mu \frac{\partial W}{\partial r}(r, \theta) \tag{3.1}$$

In the absence of body forces, the equilibrium equation leads to the Laplace equation for the displacement, $W(r, \theta)$ given by

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) W(r, \theta) = 0, r \geq 0, -\alpha \leq \theta \leq \alpha \tag{3.2}$$

The stresses along the radial boundary are

$$\sigma_{\theta z}(r, \alpha) = Q\delta(r - h) \tag{3.3}$$

$$\sigma_{\theta z}(r, -\alpha) = Q\delta(r - h) \tag{3.4}$$

The surface of the crack is stress-free; that is

$$\sigma_{\theta z}(r, 0) = 0, \quad 0 \leq r \leq a \tag{3.5}$$

The surface of the crack is displaced, hence

$$w(r, 0^+) \neq w(r, 0^-) \quad 0 \leq r \leq a \tag{3.6}$$

Notation

Let $\epsilon < \rho < \beta$ and $\Omega(r, \rho)$ be a function then

$\Omega(r, \epsilon^+) =$ the limit of $\Omega(r, \rho)$ as ρ approaches ϵ from the right.

$\Omega(r, \beta^-)$ means the limit of $\Omega(r, \rho)$ as ρ approaches β from the left

Both displacement and stress are continuous across the line, $\theta = 0$. That is

$$W(r, 0^+) = W(r, 0^-), \quad r > a \tag{3.7}$$

$$\sigma_{\theta z}(r, 0^+) = \sigma_{\theta z}(r, 0^-), \quad r > a \tag{3.8}$$

Equations (3.7) and (3.8) are known as continuity conditions.

Use of the Infinite Mellin Transform

The task is to solve equation (3.2) subject to (3.3) - (3.8) by the method of infinite Mellin transform.

The problem equation (3.2) can be written as

$$r \frac{\partial}{\partial r} \left(r \frac{\partial W}{\partial r}(r, \theta) \right) + \frac{\partial^2 W}{\partial \theta^2}(r, \theta) = 0$$

or

$$-\frac{\partial^2 W}{\partial \theta^2}(r, \theta) = r \frac{\partial}{\partial r} \left(r \frac{\partial W}{\partial r}(r, \theta) \right) \tag{3.9}$$

The infinite Mellin transform of $W(r, \theta)$ is defined as

$$\bar{W}(s, \theta) = \int_0^\infty W(r, \theta)r^{s-1}dr \tag{3.10}$$

If $\bar{W}(s, \theta)$ is obtained, the displacement $W(r, \theta)$ sought for is readily recovered by use of the inversion formula for the infinite Mellin transform defined as:

$$W(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{W}(s, \theta)r^{-s} ds \tag{3.11}$$

Application of (3.10) to (3.9) yields

$$\begin{aligned} - \int_0^\infty r^{s-1} \frac{\partial^2 W}{\partial \theta^2}(r, \theta) dr &= \int_0^\infty r^{s-1} r \frac{\partial}{\partial r} \left(r \frac{\partial W}{\partial r}(r, \theta) \right) dr \\ &= \int_0^\infty r^s \frac{\partial}{\partial r} \left(r \frac{\partial W}{\partial r}(r, \theta) \right) dr \end{aligned}$$

That is

$$-\frac{\partial^2}{\partial \theta^2} \left\{ \int_0^\infty W(r, \theta)r^{s-1} dr \right\} = r^2 \left(r \frac{\partial W}{\partial r}(r, \theta) \right) \Big|_0^\infty - s \int_0^\infty r^{s-1} - r \frac{\partial W}{\partial r}(r, \theta) dr$$

or

$$\begin{aligned} -\frac{\partial^2 \bar{W}}{\partial \theta^2}(s, \theta) &= r^{s+1} \frac{\partial W}{\partial r}(r, \theta) \Big|_0^\infty - s \left[r^s W(r, \theta) - s \int_0^W r^{s-1} W - W(r, \theta) dr \right] \\ -\frac{\partial^2 \bar{W}}{\partial \theta^2}(s, \theta) &= r^{s+1} \frac{\partial W}{\partial r}(r, \theta) \Big|_0^\infty - sr^s W(r, \theta) + s^2 \bar{W}(s, \theta) \end{aligned}$$

Hence

$$\frac{\partial^2 \bar{W}}{\partial \theta^2}(s, \theta) + s^2 \bar{W}(s, \theta) = 0 \tag{3.12}$$

Provided

$$r^{s+1} \frac{\partial W}{\partial r}(r, \theta) \Big|_0^\infty - sr^{-s} W(r, \theta) \Big|_0^\infty = 0 \tag{3.13}$$

We have used the integration by parts formula, denoted by

$$\int_a^b f(x) \frac{dg(x)}{dx} dx = (f(x)g(x)) \Big|_a^b - \int_a^b g(x) \frac{df(x)}{dx} dx \tag{3.14}$$

Bounds For $\bar{W}(s, \theta)$

The asymptotic behaviors of the stresses and their satisfaction of (3.14) determine bounds for $\bar{W}(s, \theta)$. Because stresses are concentrated at sharp corners, we expect the behavior.

$$\sigma_{rz}(r, \theta) = \sigma_{\theta z}(r, \theta) = 0(r^{-\lambda}) \quad 0 < \lambda < 1 \quad \text{as } r \rightarrow 0$$

That is

$$W(r, \theta) = 0(r^{1-\lambda}), \quad \text{as } r \rightarrow 0 \tag{3.15}$$

The stresses are expected to vanish at infinity, as given in Erdogan & Gupta (1975), the expected behavior is

$$\sigma_{rz}(r, \theta) = \sigma_{\theta z}(r, \theta) = 0(r^{-1-\epsilon}) \quad \text{as } r \rightarrow \infty \tag{3.16}$$

then

$$W(r, \theta) = 0(r^{-\epsilon}) \quad \text{as } r \rightarrow \infty$$

From (3.14)

$$\begin{aligned} \lim_{r \rightarrow 0} (r^{s+1} r^{-\lambda} - sr^s r^{1-\lambda}) &= 0 \\ \lim_{r \rightarrow 0} r^{s+1-\lambda} (1-s) &= 0 \end{aligned}$$

Then

$$Re \ s > \lambda - 1$$

From (3.14) also

$$\begin{aligned} \lim_{r \rightarrow \infty} (r^{s+1} r^{-1-\epsilon} - sr^s r^{-\epsilon}) &= 0 \\ \lim_{r \rightarrow \infty} r^{s-\epsilon} (1-s) &= 0 \end{aligned}$$

Hence,

$$Re(s - \epsilon) < 0$$

or

Consequently

$$Re s < \epsilon$$

$$\lambda - 1 < Re s < \epsilon$$

We may choose $\epsilon = \frac{1}{2}$ so that

$$\lambda - 1 < Re s < \frac{1}{2} \tag{3.17}$$

Transformation of the Boundary Conditions

Application of (3.10) to (3.3) to (3.7) and (3.8) gives

$$\int_0^\infty \sigma_{\theta z}(r, \alpha)r^{s-1}dr = \int_0^\infty Q\delta(r-h)r^{s-1}dr$$

That is by (3.1) and (3.3),

$$\begin{aligned} \int_0^\infty r\sigma_{\theta z}(r, \alpha)r^{s-1}dr &= \int_0^\infty \mu \frac{\partial W}{\partial \theta}(r, \alpha)r^{s-1}dr \\ &= \int_0^\infty rQ\delta(r-h)r^{s-1}dr \end{aligned}$$

or

$$\int_0^\infty \frac{\partial W}{\partial \theta}(r, \alpha)r^{s-1}dr = \frac{Q}{\mu}h^s$$

or

$$\frac{d\bar{W}}{d\theta}(s, \alpha) = \frac{Q}{\mu}h^s$$

Similarly, by (3.1), (3.4), and (3.8)

$$\begin{aligned} \int_0^\infty r\sigma_{\theta z}(r, -\alpha)r^{s-1}dr &= \int_0^\infty \mu \frac{dW}{d\theta}(r, -\alpha)r^{s-1}dr \\ &= \int_0^\infty rQ\delta(r-h)r^{s-1}dr \\ &= \int_0^\infty r^sQ\delta(r-h)dr \\ &= Qh^s \end{aligned}$$

Hence,

$$\frac{d\bar{W}}{d\theta}(r, -\alpha) = \frac{Qh^s}{\mu}$$

On the line, $\theta = 0$, we have

$$\begin{aligned} \bar{W}(s, 0^+) &= \int_0^\infty W(r, 0^+)r^{s-1}dr \\ &= \int_0^a W(r, 0^+)r^{s-1}dr + \int_a^\infty W(r, 0^+)r^{s-1}dr \end{aligned}$$

and

$$\begin{aligned} \bar{W}(s, 0^-) &= \int_0^\infty W(r, 0^-)r^{s-1}dr \\ &= \int_0^a W(r, 0^-)r^{s-1}dr + \int_a^\infty W(r, 0^-)r^{s-1}dr \end{aligned}$$

Hence

$$\bar{W}(s, 0^+) - \bar{W}(s, 0^-) = \int_0^a [W(r, 0^+) - W(r, 0^-)]r^{s-1}dr + \int_a^\infty [W(r, 0^+) - W(r, 0^-)]r^{s-1}dr$$

By (3.7) and (3.8), we have the half-known function.

$$\bar{W}(s, 0^+) - \bar{W}(s, 0^-) = \int_0^a [W(r, 0^+) - W(r, 0^-)]r^{s-1}dr$$

The transform of (3.5) and its complement (3.8) gives

$$\begin{aligned} r\sigma_{\theta z}(r, 0) &= \int_0^{\infty} r\sigma_{\theta z}(r, 0)r^{s-1}dr \\ &= \int_0^a r\sigma_{\theta z}(r, 0)r^{s-1}dr + \int_a^{\infty} r\sigma_{\theta z}(r, 0)r^{s-1}dr \end{aligned}$$

Thus, we derive another half-known function given by

$$\mu \frac{d\bar{W}}{d\theta}(r, 0) = \mu \int_a^{\infty} \frac{dW}{d\theta}(r, 0)r^{s-1}dr$$

Leads to

$$\frac{d\bar{W}}{d\theta}(r, 0^+) = \int_a^{\infty} \frac{dW}{d\theta}(r, 0^+)r^{s-1}ds$$

and

$$\frac{d\bar{W}}{d\theta}(r, 0^-) = \int_a^{\infty} \frac{dW}{d\theta}(r, 0^-)r^{s-1}ds$$

Let $r = a\rho$ in the half-known function related to displacement. Then $dr = ad\rho$
 $r = a$ implies $\rho = 1$ and $r = 0$ implies $\rho = 0$

Thus

$$\begin{aligned} \bar{W}(s, 0^+) - \bar{W}(s, 0^-) &= \int_0^1 [W(a\rho, 0^+) - W(a\rho, 0^-)](a\rho)^{s-1}ad\rho \\ &= a^s \int_0^1 [W(a\rho, 0^+) - W(a\rho, 0^-)]\rho^{s-1}d\rho \\ &= a^s H(s) \end{aligned}$$

where

$$H(s) = \int_0^1 [W(a\rho, 0^+) - W(a\rho, 0^-)]\rho^{s-1}d\rho$$

Similarly, using $r = a\rho$, in the other half-known function gives
 $dr = ad\rho$, $r = a$ implies $\rho = 1$ and $r = \infty$ implies $\rho = \infty$.

Hence

$$\begin{aligned} \frac{d\bar{W}}{d\theta}(s, 0^+) &= \frac{1}{\mu} \int_a^{\infty} \mu \frac{dW}{d\theta}(r, 0^+)r^{s-1}ds \\ &= \frac{1}{\mu} \int_1^{\infty} \mu \frac{dW}{d\theta}(a\rho, 0^+)(a\rho)^{s-1}ad\rho \\ &= \frac{a^s}{\mu} \int_1^{\infty} \mu \frac{dW}{d\theta}(a\rho, 0^+)\rho^{s-1}d\rho \\ &= \frac{a^s}{\mu} \int_1^{\infty} a\rho\sigma_{\theta z}(a\rho, 0^+)\rho^{s-1}d\rho = \frac{a^s}{\mu} G(s) \\ \frac{d\bar{W}}{d\theta}(s, 0^-) &= \frac{a^s}{\mu} \int_1^{\infty} \mu \frac{dW}{d\theta}(a\rho, 0^-)\rho^{s-1}d\rho \\ &= \frac{a^s}{\mu} \int_1^{\infty} \mu \frac{dW}{d\theta}(a\rho, 0^-)\rho^{s-1}d\rho = a^s G(s) \end{aligned}$$

where

$$G(s) = \int_1^{\infty} a\rho\sigma_{\theta z}(a\rho, 0^+)\rho^{s-1}d\rho$$

Theorem 3.1 A Wiener-Hopf by Noble [7]

The transform plane problem to be solved is therefore:

$$\frac{d^2\bar{W}}{d\theta^2}(s, \theta) + s^2\bar{W}(s, \theta) = 0, \quad \lambda - 1 < Res < \frac{1}{2} \tag{3.18}$$

$$\frac{d\bar{W}}{d\theta}(s, \alpha) = \frac{Q}{\mu} h^s \tag{3.19}$$

$$\frac{d\bar{W}}{d\theta}(s, -\alpha) = \frac{Q}{\mu} h^s \tag{3.20}$$

$$\bar{W}(s, 0^+) - \bar{W}(s, 0^-) = a^s H(s) \tag{3.21}$$

$$\frac{d\bar{W}}{d\theta}(s, 0^+) = \frac{d\bar{W}}{d\theta}(s, 0^-) = \frac{a^s}{\mu} G(s) \quad (3.22)$$

Writing the solution of (3.15) as

$$\bar{W}(s, \theta) = A_1(s) \cos \theta s + B_1(s) \sin \theta s, \quad 0 \leq \theta \leq \alpha$$

$$A_2(s) \cos \theta s + B_2(s) \sin \theta s, \quad -\alpha \leq \theta \leq 0 \quad (3.23)$$

Leads to

$$\frac{d\bar{W}}{d\theta}(s, \theta) = -sA_1(s) \sin \theta s + sB_1(s) \cos \theta s, \quad 0 \leq \theta \leq \alpha \quad (3.24)$$

$$= -sA_2(s) \sin \theta s + sB_2(s) \cos \theta s, \quad -\alpha \leq \theta \leq 0$$

The coefficients, $A_i(s)$ and $B_i(s)$, $i = 1, 2$ will be deduced from the boundary conditions
From (3.19) and (3.20), we get

$$-A_1(s) \sin \alpha s + B_1(s) \cos \alpha s = \frac{Q}{\mu s} h^s, \quad 0 \leq \theta \leq \alpha \quad (3.25)$$

$$A_2(s) \sin \alpha s + B_2(s) \cos \alpha s = \frac{Q}{\mu s} h^s \quad -\alpha \leq \theta \leq 0$$

From (3.21) and (3.22), wedge

$$\bar{W}(s, 0^+) = A_1(s)$$

$$\bar{W}(s, 0^-) = A_2(s)$$

$$\bar{W}(s, 0^+) - \bar{W}(s, 0^-) = a^s H(s)$$

Implies

$$A_1(s) - A_2(s) = a^s H(s) \quad (3.26)$$

From (3.24) and (3.22), we get

$$B_1(s) = B_2(s) = \frac{a^s}{\mu s} G(s) \quad (3.27)$$

Adding (3.24) and (3.25) gives

$$[B_1(s) + B_2(s)] \cos \alpha s - [A_1(s) - A_2(s)] \sin \alpha s = 2 \frac{Q}{\mu s} h^s \quad (3.28)$$

Incorporating (3.26) and (3.27) into (3.28) gives

$$2 \frac{a^s}{\mu s} G(s) \cos \alpha s - a^s H(s) \sin \alpha s = \frac{2Q}{\mu s} h^s$$

or

$$\frac{2}{\mu s} G(s) \cos \alpha s = H(s) \sin \alpha s + \frac{2Q}{\mu s} \left(\frac{h}{a}\right)^s$$

That is

$$G(s) = \frac{\mu s \sin \alpha s}{2 \cos \alpha s} \left[H(s) + \frac{2}{\mu s \sin \alpha s} \left(\frac{h}{a}\right)^s \right]$$

or

$$G(s) = \frac{\mu s \sin \alpha s}{\cos \alpha s} \left[\frac{1}{2} H(s) + \frac{Q}{\mu s \sin \alpha s} \left(\frac{h}{a}\right)^s \right] \quad (3.29)$$

Let the subscript + be attached to a function that is analytic in the right half plane, $Res > \lambda - 1$, while the subscript - is attached to a function that is analytic in the left half plane $Res < \frac{1}{2}$. Such functions overlap on the strip $\lambda - 1 < Res < \frac{1}{2}$. From the behavior (3.15), it follows that

$$H(s) = \int_0^1 [W(a\rho, 0^+) - W(a\rho, 0^-)] \rho^{s-1} d\rho$$

Then

$$W(r, \theta) = 0(r^{1-\lambda}), \quad r \rightarrow 0$$

$W(a\rho, \theta) = 0(\rho^{1-\lambda})$ leads to

$$H(s) \sim c \int_0^1 \rho^{1-\lambda} \rho^{s-1} d\rho$$

$$\sim c \int_0^1 \rho^{s-\lambda} d\rho$$

$$\sim \frac{c}{s - \lambda + 1} \rho^{s-\lambda+1}, \rho \rightarrow 0$$

That is $Res > 1 - \lambda, \rho \rightarrow 0$

Therefore, $H(s)$ is a right half plane function that is $H(s)$ should be replaced by $H_+(s)$

On the other hand

$$G(s) = \int_1^\infty a\rho\sigma_{\theta z}(a\rho, 0^+)\rho^{s-1}d\rho$$

The behavior (3.16) with $\epsilon = \frac{1}{2}$ yields

$$\begin{aligned} G(s) &\sim B \int_1^\infty \rho^{-1-\epsilon} \rho^s d\rho \\ &\sim B \int_1^\infty \rho^{s-\frac{1}{2}-1} d\rho, \rho \rightarrow \infty \\ &\sim \frac{B}{s - \frac{1}{2}} \rho^{s-\frac{1}{2}}, \rho \rightarrow \infty \end{aligned}$$

That is $s - \frac{1}{2} < 0$ or $s < \frac{1}{2}$

Hence $G(s)$ is analytic in the left half plane $Re s < \frac{1}{2}$ and should be written as $G_-(s)$.

Equation (3.29) can be expressed in terms of the two overlapping functions. $H_+(s)$ and $G_-(s)$ as

$$G_-(s) = \frac{\mu s \sin \alpha s}{\cos \alpha s} \left[\frac{1}{2} H_+(s) + \frac{Q}{\mu s \sin \alpha s} \left(\frac{h}{a} \right)^s \right] \quad (3.30)$$

Equation (3.30) is of the type known as the Wiener-Hopf equation.

An equation equivalent to (3.30) is

$$H_+(s) = \frac{2 \cos \alpha s}{s \sin \alpha s} \left[\frac{1}{\mu} G_-(s) - \frac{Q}{\mu \cos \alpha s} \left(\frac{h}{a} \right)^s \right] \quad (3.31)$$

2.2. Solution of the Wiener-Hopf Equation

Let equation (3.31) be written as

$$H_+(s) = 2\Phi(s) \left[G_-(s) - \frac{\Omega(s)}{\cos \alpha s} \right] \quad (3.32)$$

where

$$\Phi(s) = \frac{\cos \alpha s}{\mu s \sin \alpha s} \quad (3.33)$$

and

$$\Omega(s) = Q \left(\frac{h}{a} \right)^s \quad (3.34)$$

We need to decompose $\Phi(s)$ into a ratio of two functions in the form

$$\Phi(s) = \frac{\Phi_-(s)}{\Phi_+(s)} \quad (3.35)$$

When (3.34) is substituted into (3.32), the equation becomes

$$\begin{aligned} \frac{\mu}{2} H_+(s) \Phi_+(s) &= \Phi_-(s) \left[\frac{1}{\mu} G_-(s) - \frac{\Omega(s)}{\cos \alpha s} \right] \\ &\left[\frac{1}{\mu} \Phi_-(s) G_-(s) - \frac{\Phi_-(s) \Omega(s)}{\cos \alpha s} \right] \end{aligned} \quad (3.36)$$

Theorem 3.2 Weierstrass' Theorem for Infinite Products by Murray [8]

Let $f(z)$ be analytic for all z [i. e. $f(z)$] is an entire function, and suppose that it has simple zeros at a_1, a_2, a_3, \dots where $0 < |a_1| < |a_2| < |a_3| < \dots$ and $\lim_{n \rightarrow \infty} |a_n| = \infty$.

Then

$$i. \quad f(z) = f(0) e^{\frac{f'(0)z}{f(0)}} \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{z}{a_k} \right) e^{\frac{z}{a_k}} \right\}$$

A generalization of this states that if $f(z)$ has zeros at $a_k \neq 0, 1, 2, 3, \dots$, of respective multiplicities or orders μ_k , and if for some integer N , $\sum_{k=1}^{\infty} \frac{1}{a_k^N}$ is absolutely convergent, then

$$ii. \quad f(z) = f(0)e^{G(z)} \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2} \frac{z^2}{a_k^2} + \dots + \frac{1}{N-1} \frac{z^{N-1}}{a_k^{N-1}}} \right\}^{\mu_k}$$

Where $G(z)$ is an entire function. The result is also true if some of the a_k 's are poles, in which case their multiplicities are negative. The results (i) and (ii) are sometimes called Weierstrass' factor theorems.

The functions $\Phi_+(s)$ and $\Phi_-(s)$ can be derived by use of the infinite product definitions of trigonometric functions given by Korn & Korn (1999)

$$\begin{aligned} \cos \alpha s &= \prod_{k=1}^{\infty} \left(1 - \frac{4\alpha^2 s^2}{(2k-1)^2 \pi^2}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{2\alpha s}{(2k-1)\pi}\right) \left(1 + \frac{2\alpha s}{(2k-1)\pi}\right) \\ &= \prod_{k=1}^{\infty} \left(1 - \frac{2\alpha s}{(2k-1)\pi}\right) \prod_{k=1}^{\infty} \left(1 + \frac{2\alpha s}{(2k-1)\pi}\right) \\ S \sin \alpha s &= \alpha s^2 \prod_{k=1}^{\infty} \left(1 - \frac{\alpha^2 s^2}{k^2 \pi^2}\right) = \alpha s^2 \prod_{k=1}^{\infty} \left(1 - \frac{\alpha s}{k\pi}\right) \left(1 + \frac{\alpha s}{k\pi}\right) \\ &= \alpha s^2 \prod_{k=1}^{\infty} \left(1 - \frac{\alpha s}{k\pi}\right) \prod_{k=1}^{\infty} \left(1 + \frac{\alpha s}{k\pi}\right) \end{aligned}$$

Therefore,

$$\frac{\cos \alpha s}{S \sin \alpha s} = \frac{\prod_{k=1}^{\infty} \left(1 - \frac{2\alpha s}{(2k-1)\pi}\right) \prod_{k=1}^{\infty} \left(1 + \frac{2\alpha s}{(2k-1)\pi}\right)}{\alpha s^2 \prod_{k=1}^{\infty} \left(1 - \frac{\alpha s}{k\pi}\right) \prod_{k=1}^{\infty} \left(1 + \frac{\alpha s}{k\pi}\right)}$$

To achieve the decomposition $\frac{\Phi_-(s)}{\Phi_+(s)}$ we write

$$\Phi(s) = \frac{\cos \alpha s}{s \sin \alpha s} = \frac{2 \sin \alpha s \cos \alpha s}{2s \sin \alpha s \sin \alpha s} = \frac{\sin 2\alpha s}{2s \sin \alpha s \sin \alpha s}$$

Consequently,

$$\begin{aligned} \Phi(s) &= \frac{2\alpha s \prod_{k=1}^{\infty} \left(1 - \frac{2\alpha s}{k\pi}\right) \prod_{k=1}^{\infty} \left(1 + \frac{2\alpha s}{k\pi}\right) e^{\Psi_s}}{2s \left[\alpha s \prod_{k=1}^{\infty} \left(1 - \frac{\alpha s}{k\pi}\right) \prod_{k=1}^{\infty} \left(1 + \frac{\alpha s}{k\pi}\right)\right]^2 e^{\Psi_s}} \\ &= \frac{\alpha \prod_{k=1}^{\infty} \left(1 - \frac{2\alpha s}{k\pi}\right) e^{\Psi_s}}{\alpha^2 s^2 \left[\prod_{k=1}^{\infty} \left(1 - \frac{\alpha s}{k\pi}\right)\right]^2} \\ &= \frac{1}{\left[\prod_{k=1}^{\infty} \left(1 + \frac{\alpha s}{k\pi}\right)\right]^2 e^{\Psi_s}} \\ &= \frac{1}{\prod_{k=1}^{\infty} \left(1 + \frac{2\alpha s}{k\pi}\right)} \end{aligned}$$

where e^{Ψ_s} is introduced to render algebraic behaviours to the infinite products as $|S| \rightarrow \infty$
Let,

$$\Phi_+(s) = \frac{\alpha s^2 \left[\prod_{k=1}^{\infty} \left(1 + \frac{\alpha s}{k\pi}\right)\right]^2}{\prod_{k=1}^{\infty} \left(1 + \frac{2\alpha s}{k\pi}\right)} e^{\Psi_s} \tag{3.37}$$

and

$$\Phi_-(s) = \frac{\prod_{k=1}^{\infty} \left(1 - \frac{\alpha s}{k\pi}\right)}{\left[\prod_{k=1}^{\infty} \left(1 - \frac{\alpha s}{k\pi}\right)\right]^2} e^{\Psi_s} \tag{3.38}$$

Then,

$$\Phi(s) = \frac{\Phi_-(s)}{\Phi_+(s)}$$

To understand the behaviors of $\Phi_-(s)$ and $\Phi_+(s)$ at infinity, that is, as $|s| \rightarrow \infty$ We connect the infinite products to their gamma function equivalents through the formula

Theorem 3.3 The Gamma Function by Murray [8]

For $Re(z) > 0$, we define the gamma function by

i. $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$

Then, we have the recursion formula

ii. $\Gamma(z + 1) = z \Gamma(z)$ where $\Gamma(1) = 1$

If z is a positive integer n , we see from (ii) that

iii. $\Gamma(n + 1) = n!$

So that the gamma function is a generalization of the factorial. For this reason, the gamma function is also called the factorial function and is written as $z!$ Rather than $\Gamma(z + 1)$.

$$\frac{1}{\Gamma(z)} = ze^{rz} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \tag{3.39}$$

or

$$\prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) = \frac{1}{ze^{rz}\Gamma(z)e^{-\frac{z}{k}}} = \frac{1}{e^{rz}\Gamma(z+1)e^{-\frac{z}{k}}}$$

and use the asymptotic relation

$$\Gamma(z + 1) = \sqrt{2\pi z} z^z e^{-z} \text{ as } |z| \rightarrow \infty, \quad -\pi < argz < \pi \tag{3.40}$$

It is noteworthy that in (3.40) the factor Z^z may lead to a non-algebraic behavior of $\Gamma(z + 1)$. Thus, if λ is a real number, then

$$\Gamma(\lambda z + 1) = \sqrt{2\pi\lambda z} (\lambda z)^{\lambda z} e^{-\lambda z}, \quad |z| \rightarrow \infty \tag{3.41}$$

But, $\lambda^{\lambda z}$ may manifest non-algebraic behavior at $|z| = \infty$ and so must be eliminated.

Since

$$\lambda^{\lambda z} = e^{\ln \lambda^{\lambda z}} = e^{(\lambda \ln \lambda)z}$$

We must take valid mathematical steps to eliminate $e^{(\lambda \ln \lambda)z}$ at the appropriate stage.

Now, by use of (3.37), we get

$$\prod_{k=1}^{\infty} \left(1 + \frac{w}{k}\right) = \frac{1}{we^{rw} \Gamma(w) e^{-\frac{w}{\alpha}}}, \quad w = \frac{\alpha s}{\pi}$$

Hence,

$$\alpha \left[s \prod_{k=1}^{\infty} \left(1 + \frac{\alpha s}{\pi k}\right) \right]^2 = \alpha \left[\frac{s}{\frac{\alpha s}{\pi k} e^{\frac{\alpha}{\pi} s \gamma} \Gamma\left(\frac{\alpha}{\pi} s\right) e^{-\frac{\alpha s}{\pi k}}} \right]^2 = \frac{\alpha s^2}{e^{2\frac{\alpha}{\pi} s \gamma} \left[\Gamma\left(\frac{\alpha}{\pi} s\right) + 1\right]^2 e^{-2\frac{\alpha s}{\pi k}}}$$

Similarly,

$$\prod_{k=1}^{\infty} \left(1 + \frac{2\alpha s}{\pi k}\right) = \frac{1}{\frac{2\alpha}{\pi} s e^{\frac{2\alpha}{\pi} s \gamma} \Gamma\left(2\frac{\alpha}{\pi} s\right) e^{-\frac{2\alpha s}{\pi k}}}$$

Hence, by (3.37)

$$\phi_t(s) = \frac{\alpha s^2 e^{\frac{2\alpha}{\pi} s \gamma} \Gamma\left(2\frac{\alpha}{\pi} s + 1\right) e^{-\frac{2\alpha s}{\pi k}}}{e^{\frac{2\alpha}{\pi} s \gamma} \left[\Gamma\left(\frac{\alpha}{\pi} s\right) + 1\right]^2 e^{-\frac{2\alpha s}{\pi k}}} e^{\psi s} = \frac{\alpha s^2 \Gamma\left(\frac{2\alpha s}{\pi} + 1\right)}{\left[\Gamma\left(\frac{\alpha}{\pi} s\right) + 1\right]^2} e^{\psi s}$$

As $|S| \rightarrow \infty$

$$\begin{aligned}
 \alpha s^2 \Gamma\left(\frac{2\alpha}{\pi}s + 1\right) &= \alpha s^2 \sqrt{2\pi\left(\frac{2\alpha}{\pi}\right)s} \left(\frac{2\alpha}{\pi}\right)^{\frac{2\alpha}{\pi}s} (S)^{\frac{2\alpha}{\pi}s} e^{-\frac{2\alpha}{\pi}s} \\
 &= \alpha s^2 2\sqrt{\alpha s} \left(\frac{2\alpha}{\pi}\right)^{\frac{2\alpha}{\pi}s} (S)^{\frac{2\alpha}{\pi}s} e^{-\frac{2\alpha}{\pi}s} \\
 \Gamma\left(\frac{\alpha}{\pi}s + 1\right) &= \sqrt{2\pi\frac{\alpha}{\pi}s} \left(\frac{\alpha}{\pi}\right)^{\frac{\alpha}{\pi}s} (S)^{\frac{\alpha}{\pi}s} e^{-\frac{\alpha}{\pi}s} \\
 &= 2^{\frac{1}{2}} \alpha^{\frac{1}{2}} S^{\frac{1}{2}} \left(\frac{\alpha}{\pi}\right)^{\frac{\alpha}{\pi}s} (S)^{\frac{\alpha}{\pi}s} e^{-\frac{\alpha}{\pi}s} \\
 \frac{\alpha s^2 \Gamma\left(\frac{2\alpha}{\pi}s + 1\right)}{\left[\Gamma\left(\frac{\alpha}{\pi}s + 1\right)\right]^2} e^{\psi s} &= \frac{2\alpha s^2 (\alpha)^{\frac{1}{2}} S^{\frac{1}{2}} \left(\frac{\alpha}{\pi}\right)^{\frac{2\alpha}{\pi}s} 2^{\frac{2\alpha}{\pi}s} e^{-\frac{2\alpha}{\pi}s} (S)^{\frac{2\alpha}{\pi}s}}{2\alpha s \left(\frac{\alpha}{\pi}\right)^{\frac{2\alpha}{\pi}s} (S)^{\frac{2\alpha}{\pi}s} e^{-\frac{2\alpha}{\pi}s}} e^{\psi s} \\
 &= \alpha^{\frac{1}{2}} S^{\frac{3}{2}} 2^{\frac{2\alpha}{\pi}s} e^{\psi s}, \quad \text{as } |Z| \rightarrow \infty \\
 &= \alpha^{\frac{1}{2}} S^{\frac{3}{2}} e^{\left(\frac{2\alpha}{\pi} \ln 2\right)s} e^{\psi s}
 \end{aligned}$$

Let $e^{\frac{2\alpha}{\pi} \ln 2s} e^{\psi s} = 1$
Hence,

$$\psi = -\frac{2\alpha}{\pi} \ln 2$$

Consequently,

$$\phi_+(s) = \frac{\alpha s^2 \Gamma\left(\frac{2\alpha}{\pi}s + 1\right)}{\left[\Gamma\left(\frac{\alpha}{\pi}s + 1\right)\right]^2} e^{-\left(\frac{2\alpha}{\pi} \ln 2\right)s} \tag{3.42}$$

and

$$\phi_+(+s) = \alpha^{\frac{1}{2}} S^{\frac{3}{2}}, \quad |S| \rightarrow \infty \tag{3.43}$$

Observe that

$$\phi_+(-s) = \frac{\alpha s^2 \Gamma\left(1 - \frac{2\alpha}{\pi}s\right)}{\left[\Gamma\left(1 - \frac{\alpha}{\pi}s\right)\right]^2} e^{\left(\frac{2\alpha}{\pi} \ln 2\right)s}$$

Implies

$$\frac{\left[\Gamma\left(1 - \frac{\alpha}{\pi}s\right)\right]^2}{\Gamma\left(1 - \frac{2\alpha}{\pi}s\right)} e^{-\left(\frac{2\alpha}{\pi} \ln 2\right)s} = \frac{\alpha s^2}{\phi_+(-s)}$$

Hence, by (3.32) and the equation above

$$\phi_-(s) = \frac{\alpha s^2}{\phi_+(-s)} \tag{3.44}$$

Similarly, from (3.38) and (3.39)

$$\begin{aligned}
 \prod_{k=1}^{\infty} \left(1 - \frac{2\alpha s}{\pi k}\right) &= \frac{1}{-\frac{2\alpha}{\pi} s e^{-\frac{2\alpha}{\pi} s \gamma} \Gamma\left(-\frac{2\alpha}{\pi} s\right) e^{-\frac{2\alpha}{\pi} \frac{s}{k}}} \\
 &= \frac{1}{e^{-\frac{2\alpha}{\pi} s \gamma} \Gamma\left(-\frac{2\alpha s}{\pi} + 1\right) e^{\frac{2\alpha}{\pi} \frac{s}{k}}} \\
 \left[\prod_{k=1}^{\infty} \left(1 - \frac{\alpha s}{\pi k}\right)\right]^2 &= \frac{1}{e^{-\frac{2\alpha}{\pi} s \gamma} \left[\Gamma\left(-\frac{\alpha s}{\pi} + 1\right)\right]^2 e^{\frac{2\alpha}{\pi} \frac{s}{k}}}
 \end{aligned}$$

Therefore,

$$\phi_-(s) = \frac{e^{-\frac{2\alpha}{\pi}sy} \left[\Gamma\left(-\frac{2\alpha s}{\pi} + 1\right) \right]^2 e^{\frac{2\alpha s}{\pi k}}}{e^{-\frac{2\alpha}{\pi}sy} \Gamma\left(-\frac{2\alpha s}{\pi} + 1\right) e^{\frac{2\alpha s}{\pi k}}} e^{\psi s}$$

That is,

$$\phi_-(s) = \frac{\left[\Gamma\left(-\frac{\alpha s}{\pi} + 1\right) \right]^2}{\Gamma\left(-\frac{2\alpha s}{\pi} + 1\right)} e^{\psi s}$$

To obtain the behavior for large values of |S| We note that

$$\begin{aligned} \left[\Gamma\left(-\frac{\alpha s}{\pi} + 1\right) \right]^2 &= \left[\sqrt{-\frac{\alpha s}{\pi}} \cdot 2\pi, \quad \left(-\frac{\alpha s}{\pi}\right)^{-\frac{\alpha s}{\pi}} e^{-\frac{\alpha s}{\pi}} \right]^2 \\ &= -2\alpha s \left(\frac{\alpha}{\pi}\right)^{-\frac{2\alpha s}{\pi}} (-s)^{-\frac{2\alpha s}{\pi}} e^{-\frac{2\alpha s}{\pi}} \\ \Gamma\left(-\frac{2\alpha s}{\pi} + 1\right) &= \sqrt{2\pi \left(-\frac{2\alpha s}{\pi}\right)} \left(-\frac{2\alpha s}{\pi}\right)^{-\frac{2\alpha s}{\pi}} e^{-\frac{2\alpha s}{\pi}} \\ &= (-4\alpha s)^{\frac{1}{2}} \left(\frac{2\alpha}{\pi}\right)^{-\frac{2\alpha s}{\pi}} (-s)^{-\frac{2\alpha s}{\pi}} e^{-\frac{2\alpha s}{\pi}} \end{aligned}$$

Then,

$$\begin{aligned} \phi_-(s) &= \frac{-2\alpha s \left(\frac{\alpha}{\pi}\right)^{-\frac{2\alpha s}{\pi}} (-s)^{-\frac{2\alpha s}{\pi}} e^{-\frac{2\alpha s}{\pi}}}{2(-\alpha s)^{\frac{1}{2}} (2)^{-\frac{2\alpha s}{\pi}} \left(\frac{\alpha}{\pi}\right)^{-\frac{2\alpha s}{\pi}} (-s)^{-\frac{2\alpha s}{\pi}} e^{-\frac{2\alpha s}{\pi}}} e^{-\left(\frac{2\alpha}{\pi} \ln 2\right)s} \\ &= \alpha s (-\alpha s)^{-\frac{1}{2}} = \alpha \alpha^{-\frac{1}{2}} s (-1)^{-\frac{1}{2}} S^{-\frac{1}{2}} \\ &= \frac{\alpha^{\frac{1}{2}} S^{\frac{1}{2}}}{(-1)^{\frac{1}{2}}}, \quad |S| \rightarrow \infty \end{aligned}$$

To check our results, we can use (3.43) and (3.44) to get

$$\begin{aligned} \phi_-(s) &= \frac{\alpha s^2}{\phi_+(-s)} \quad \text{as } |S| \rightarrow \infty \quad (3.45) \\ &= \frac{\alpha s^2}{\alpha^{-\frac{1}{2}} (-s)^{\frac{3}{2}}} = \frac{\alpha^{\frac{1}{2}} S^2 S^{-\frac{3}{2}}}{(-1)^{\frac{3}{2}}} = \frac{\alpha^{\frac{1}{2}} S^{\frac{1}{2}}}{[(-1)^3]^{\frac{1}{2}}} \\ &= \frac{\alpha^{\frac{1}{2}} S^{\frac{1}{2}}}{[-1]^{\frac{1}{2}}}, \quad |S| \rightarrow \infty \end{aligned}$$

This is in agreement with the result obtained in (3.45) by a different method.

In (3.36), we decompose the term.

$L(S) = \phi_-(s)\Omega(S) \sec \alpha s$ into a sum given by

$$\phi_-(s)\Omega(S) \sec \alpha s = [L_+(S) + L_-(S)] \quad (3.46)$$

Theorem 3.4. Mittag-Leffler’s Expansion Theorem [8]

states that

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{z - a_n} + \frac{1}{a_n} \right\}$$

Mittag-Leffler’s expansion theorem for $\sec z$ is given by

$$\sec z = \pi \left(\frac{1}{\left(\frac{\pi}{2}\right)^2 - z^2} - \frac{3}{\left(\frac{3\pi}{2}\right)^2 - z^2} + \frac{5}{\left(\frac{5\pi}{2}\right)^2 - z^2} - \dots \right)$$

With the aid of Mittag-Leffler’s expansion theorem for $\sec z$ we get

$$\sec w = \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{w - (n - \frac{1}{2})\pi} - \frac{1}{w + (n - \frac{1}{2})\pi} \right]$$

Consequently,

$$\sec \alpha s = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{\frac{\alpha s}{\pi} - \gamma_n} - \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{\frac{\alpha s}{\pi} + \gamma_n}$$

$$\gamma_n = n - \frac{1}{2}$$

We have,

$$\begin{aligned} & \phi_-(s)\Omega(S) \sec \alpha s \\ &= \phi_-(s)\Omega(S) \frac{1}{\pi} \sum_{n=1}^{\infty} - \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\phi_-(s)\Omega(S)}{\frac{\alpha s}{\pi} + \gamma_n} + \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\phi_-(-\gamma_n)\Omega(-\gamma_n)}{\frac{\alpha s}{\pi} + \gamma_n} \\ & \quad - \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\phi_-(-\gamma_n)\Omega(-\gamma_n)}{\frac{\alpha s}{\pi} + \gamma_n} \\ &= L_-(S) - \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \left(\frac{\phi_-(s)\Omega(S) - \phi_-(-\gamma_n)\Omega(-\gamma_n)}{\frac{\alpha s}{\pi} + \gamma_n} \right) + L_+(S) \end{aligned}$$

Hence,

$$L_-(S) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\phi_-(s)\Omega(S)}{\frac{\alpha s}{\pi} - \gamma_n} \tag{3.47}$$

$$L_+(S) = \frac{-1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\phi_-(-\gamma_n)\Omega(-\gamma_n)}{\frac{\alpha s}{\pi} + \gamma_n} \tag{3.48}$$

The decomposition (3.46) makes (3.36) to be

$$\frac{\mu}{2} H_+(S)\phi_+(S) = \phi_-(S)G_-(S) - [L_+(S) + L_-(S)]$$

Therefore,

$$\frac{\mu}{2} H_+(S)\phi_+(S) + L_+(S) = \phi_-(S)G_-(S) - L_-(S) \tag{3.49}$$

The two functions are analytic in different half planes, $Re s > \max\left(-\frac{1}{2}, \lambda - 1\right)$ for the right half plane and $Re s < \frac{1}{2}$ for the left half plane, with both of them being analytic in the strip $\left(-\frac{1}{2}, \lambda - 1\right) < Re s < \frac{1}{2}$. Therefore, one of the functions is the analytic continuation of the other.

Theorem 3.5 Liouville’s Theorem [8]

Suppose that for all z in the entire complex plane,

- (i) $f(z)$ is analytic
- (ii) $f(z)$ is bounded, i.e., $|f(z)| < M$ for some constant M . Then $f(z)$ must be a constant.

By Liouville’s theorem, the functions must be equal to a constant.

Hence

$$\frac{\mu}{2} H_+(S)\phi_+(S) + L_+(S) = \phi_-(S)G_-(S) - L_-(S) = c$$

The constant C can be easily determined by the behavior of $\phi_+(S), H_+(S)$ and $L_+(S)$ near the origin.

Now

$$\lim_{|s| \rightarrow 0} H_+(S) \neq 0, \quad \lim_{|s| \rightarrow 0} L_+(S) \neq 0 \text{ but } \lim_{|s| \rightarrow 0} \phi_+(S) = 0$$

We see that $C = L_+(0)$

Therefore,

$$\begin{aligned} \frac{\mu}{2}H_+(S)\phi_+(S) + L_+(S) &= L_+(0) \\ \phi_-(S)G_-(S) - L_-(S) &= L_+(0) \end{aligned}$$

Leads to

$$H_+(S) = \frac{2}{\mu} \left[\frac{L_+(0) - L_+(S)}{\phi_+(S)} \right] \tag{3.50}$$

and

$$G_-(S) = \frac{L_+(0) + L_-(S)}{\phi_-(S)} \tag{3.51}$$

Recall that

$$\phi_-(s) = \frac{\left[\Gamma\left(-\frac{\alpha s}{\pi} + 1\right) \right]^2}{\Gamma\left(-\frac{2\alpha s}{\pi} + 1\right)} e^{-(\frac{2\alpha}{\pi} \ln 2)s}$$

Is defined for $Re\ s < \frac{1}{2}$; that is $S_n = 0, -1, -2, -3, \dots$ when $S_n = \frac{n\pi}{\alpha}$

Therefore,

$$G_-(S) = \frac{L_+(0) + L_-(S)}{\phi_-(S)}$$

Is defined for $Re\ s < \frac{1}{2}$; that is $S_n = 0, -1, -2, -3, \dots$ when $S_n = \frac{n\pi}{\alpha}$

Recall also that

$\Gamma(1) = 1, \Gamma(n + 1) = n!$ For a positive integer n .

$$\begin{aligned} \Gamma(2n) &= \frac{1}{\sqrt{\pi}} 2^{2n-1} \Gamma(n) \Gamma\left(n + \frac{1}{2}\right) \\ \Gamma(n) &= \Gamma\{(n - 1) + 1\} = (n - 1)! \\ \Gamma\left(n + \frac{1}{2}\right) &= \frac{(2n - 1)!!}{2^n} \sqrt{\pi} \end{aligned}$$

Then,

$$\begin{aligned} \Gamma(2n) &= \frac{1}{\sqrt{\pi}} 2^{2n-1} (n - 1)! \frac{(2n - 1)!!}{2^n} \sqrt{n} \\ &= 2^{2n-1} 2^{-n} (n - 1)! (2n - 1)!! \\ &= 2^{n-1} (n - 1)! (2n - 1)!! \end{aligned}$$

From (3.35) and (3.27), we get

$$\begin{aligned} \mu s A_1(s) \sin \alpha s - a^s G_-(s) \cos \alpha s &= -Qh^s \\ A_1(s) &= \frac{a^s G_-(s) \cos \alpha s - Qh^s}{\mu s \sin \alpha s} \\ \mu s A_2(s) \sin \alpha s &= -a^s G_-(s) + Qh^s \\ A_2(s) &= \frac{-a^s G_-(s) + Qh^s}{\mu s \sin \alpha s} \\ B_1(s) &= \frac{a^s}{\mu s} G_-(s) \\ B_2(s) &= \frac{a^s}{\mu s} G_-(s) \end{aligned}$$

The transformed displacement is therefore obtained from (3.23) as

$$\begin{aligned} \bar{W}(s, \theta) &= \left[\frac{a^s G_-(s) \cos \alpha s - Qh^s}{\mu s \sin \alpha s} \right] \cos \theta s + \frac{a^s}{\mu s} G_-(s) \sin \theta s, \quad 0 \leq \theta \leq \alpha \\ &= \left[\frac{-a^s G_-(s) \cos \alpha s + Qh^s}{\mu s \sin \alpha s} \right] \cos \theta s + \frac{a^s}{\mu s} G_-(s) \sin \theta s \quad -\alpha \leq \theta \leq 0 \end{aligned}$$

For $0 \leq \theta \leq \alpha$

$$\bar{W}(s, \theta) = a^s G_-(s) \frac{\cos \alpha s \cos \theta s}{\mu s \sin \alpha s} - \frac{Qh^s \cos \theta s}{\mu s \sin \alpha s} + a^s G_-(s) \frac{\sin \alpha s \sin \theta s}{\mu s \sin \alpha s}$$

$$\begin{aligned}
 &= a^s G_-(s) \frac{(\cos \alpha s \cos \theta s + \sin \alpha s \sin \theta s)}{\mu s \sin \alpha s} - \frac{Qh^s \cos \theta s}{\mu s \sin \alpha s} \\
 &= a^s G_-(s) \frac{\cos(\alpha - \theta)s}{\mu s \sin \alpha s} - \frac{Qh^s \cos \theta s}{\mu s \sin \alpha s}
 \end{aligned} \tag{3.52}$$

For $-\alpha \leq \theta \leq 0$

$$\begin{aligned}
 \bar{W}(s, \theta) &= \frac{-a^s G_-(s) \cos \alpha s \cos \theta s}{\mu s \sin \alpha s} + \frac{Qh^s \cos \theta s}{\mu s \sin \alpha s} + a^s G_-(s) \frac{\sin \alpha s \sin \theta s}{\mu s \sin \alpha s} \\
 &= -a^s G_-(s) \frac{(\cos \alpha s \cos \theta s - \sin \alpha s \sin \theta s)}{\mu s \sin \alpha s} - \frac{Qh^s \cos \theta s}{\mu s \sin \alpha s} \\
 &= -a^s G_-(s) \frac{\cos(\alpha + \theta)s}{\mu s \sin \alpha s} + \frac{Qh^s \cos \theta s}{\mu s \sin \alpha s}
 \end{aligned} \tag{3.53}$$

Referring to (3.50) and (3.51), we deduce

$$\begin{aligned}
 \frac{\mu}{2} H_+(s) \Phi_+(s) &= L_+(0) - L_+(s) \\
 G_-(s) \Phi_-(s) &= L_+(0) + L_-(s)
 \end{aligned}$$

From which we get

$$\begin{aligned}
 G_-(s) \Phi_-(s) - \frac{\mu}{2} H_+(s) \Phi_+(s) &= L_+(s) + L_-(s) = L(s) \\
 &= \Phi_-(s) \frac{\Omega(s)}{\cos \alpha s} \quad \Phi(s) = \frac{\cos \alpha s}{s \sin \alpha s}
 \end{aligned}$$

Then,

$$\begin{aligned}
 G_-(s) &= \frac{\mu}{2} H_+(s) \frac{\Phi_+(s)}{\Phi(s)} + \frac{\Omega(s)}{\cos \alpha s} \\
 G_-(s) &= \frac{\mu}{2} \frac{H_+(s)}{\Phi(s)} + \frac{\Omega(s)}{\cos \alpha s}
 \end{aligned} \tag{3.54}$$

In view of (3.54), the forms of $\bar{W}(s, \theta)$ For $0 \leq \theta \leq \alpha$, we use (3.52) and (3.54) to get

$$\bar{W}(s, \theta) = \frac{a^s}{2} \frac{\cos(\alpha - \theta)s}{\cos \alpha s} H_+(s) + \frac{Q}{\mu} \frac{\sin \theta s}{s \cos \alpha s} h^s \tag{3.55}$$

For $-\alpha \leq \theta \leq 0$, we use (3.53) and (3.54) to get

$$\bar{W}(s, \theta) = -\frac{a^s}{2} \frac{\cos(\alpha + \theta)s}{\cos \alpha s} H_+(s) + \frac{Qh^s}{\mu s} \frac{\sin \theta s}{\cos \alpha s} \tag{3.56}$$

3. Results

3.1. Derivation of the Displacement $W(R, \theta)$

The displacement sought for, $W(r, \theta)$ is obtained by application of the inversion formula for the infinite Mellin transform given in (3.11), which is repeated as

$$W(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{W}(s, \theta) r^{-s} ds, \quad \max\left(-\frac{1}{2}, \lambda - 1\right) < c < \frac{1}{2}$$

$\bar{W}(s, \theta)$ has the form, $G(s)$ obtained in (3.52) and (3.53)

For $0 \leq \theta \leq \alpha$, use is made of (3.52) to get

$$\begin{aligned}
 W(r, \theta) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(a^s G_-(s) \frac{\cos(\alpha - \theta)s}{\mu s \sin \alpha s} - \frac{Qh^s \cos \theta s}{\mu s \sin \alpha s} \right) r^{-s} ds \\
 &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G_-(s) \frac{\cos(\alpha - \theta)s}{\mu s \sin \alpha s} \left(\frac{r}{a}\right)^{-s} ds - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{Q}{\mu} \frac{\cos \theta s}{s \sin \alpha s} \left(\frac{r}{h}\right)^{-s} ds
 \end{aligned} \tag{4.1}$$

$$W(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{-a^s G_-(s) \cos(\alpha + \theta)s + Qh^s \cos \theta s}{\mu s \sin \alpha s} \right) r^{-s} ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G_-(s) \cos(\alpha + \theta)s}{\mu s \sin \alpha s} \left(\frac{r}{a}\right)^{-s} ds + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{Q \cos \theta s}{\mu s \sin \alpha s} \left(\frac{r}{h}\right)^{-s} ds \quad (4.2)$$

Alternatively, we can use the equivalent forms in terms of $H_+(s)$ given by (3.55) and (3.56) which produce: for $0 \leq \theta \leq \alpha$

$$\begin{aligned} W(r, \theta) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \frac{\alpha^s \cos(\alpha - \theta)s}{2 \cos \alpha s} H_+(s) + \frac{Q \cos \theta s h^s}{\mu s \cos \alpha s} \right\} r^{-s} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1 \cos(\alpha - \theta)s}{2 \cos \alpha s} H_+(s) \left(\frac{r}{a}\right)^{-s} ds + \frac{1}{2\pi i} \frac{Q}{\mu} \int_{c-i\infty}^{c+i\infty} \frac{\sin \theta s h^s}{s \cos \alpha s} \left(\frac{r}{h}\right)^{-s} ds \end{aligned} \quad (4.3)$$

$$\begin{aligned} W(r, \theta) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left\{ \frac{-a^s \cos(\alpha + \theta)s}{2 \cos \alpha s} H_+(s) + \frac{Q h^s \sin \theta s}{\mu s \cos \alpha s} \right\} r^{-s} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{1 \cos(\alpha + \theta)s}{2 \cos \alpha s} H_+(s) \left(\frac{r}{a}\right)^{-s} ds + \frac{1}{2\pi i} \frac{Q}{\mu} \int_{c-i\infty}^{c+i\infty} \frac{\sin \theta s}{s \cos \alpha s} \left(\frac{r}{h}\right)^{-s} ds \end{aligned} \quad (4.4)$$

3.2. Displacement of The Crack Region, $0 \leq \theta \leq \alpha, -\alpha \leq \theta \leq 0$

Because we are interested in the displacements and stresses at the wedge tip, we consider the evaluation of (4.1) and (4.2) as $r \rightarrow 0$. To use the residue method, we use the simple poles of $\sin \alpha s$, which are located at $s_n = \frac{n\pi}{\alpha}$, $n = 1, 2, 3, \dots$ the double pole at $s = 0$ leads to constants for $r \geq 0$, which do not affect the solutions of a Neumann problem. Therefore, the poles associated with $s = 0$ are ignored. The results are expected for $0 < r < a$, $0 < r < h$ and $0 \leq \alpha \leq \pi$

Using (4.1), the residues are obtained as follows:

For the first integral, set $S_n = -\frac{n\pi}{\alpha}$, $n = 1, 2, 3, \dots$ and

$$R_1(S) = \frac{G_-(s) \cos(\alpha - \theta)s}{s \sin \alpha s} \left(\frac{r}{a}\right)^{-s}$$

Then,

$$\begin{aligned} \lim_{s \rightarrow S_n} R_1(S) &= \lim_{s \rightarrow S_n} \left\{ \frac{S - S_n}{\sin \alpha s} \frac{G_-(s) \cos(\alpha - \theta)s}{s} \left(\frac{r}{a}\right)^{-s} \right\} \\ &= \lim_{s \rightarrow S_n} \frac{1}{\alpha \cos \alpha s} \left\{ \frac{G_-(-s_n) \cos(\alpha - \theta)s_n}{s_n} \left(\frac{r}{a}\right)^{-s_n} \right\} \\ &= \frac{1}{\alpha} \frac{1}{\cos n\pi} \left\{ \frac{G_-(-s_n) \cos(\alpha - \theta)s_n}{s_n} \left(\frac{r}{a}\right)^{s_n} \right\} \\ &= \frac{(-1)^n}{\alpha} \frac{G_-(s_n) \cos(\alpha - \theta)s_n}{\frac{-n\pi}{\alpha}} \left(\frac{r}{a}\right)^{-s_n} \\ &= \frac{-(-1)^n}{\pi n} G_- \left(\frac{-n\pi}{\alpha}\right) \cos(\alpha - \theta)s_n \left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}} \end{aligned}$$

For the second integral set $S_n = -\frac{n\pi}{\alpha}$, $n = 1, 2, 3, \dots$ and

$$R_2(S) = \frac{\cos \theta s}{s \sin \alpha s} \left(\frac{r}{h}\right)^{-s}$$

Then,

$$\begin{aligned} \lim_{s \rightarrow S_n} R_2(S) &= \lim_{s \rightarrow S_n} \left\{ \frac{S - S_n}{\sin \alpha s} \frac{\cos \theta s}{s} \left(\frac{r}{h}\right)^{-s} \right\} \\ &= \lim_{s \rightarrow S_n} \frac{1}{\alpha \cos \alpha s} \frac{\cos \theta s_n}{\frac{-n\pi}{\alpha}} \left(\frac{r}{h}\right)^{-s_n} \\ &= -\frac{1}{n\pi \cos n\pi} \cos \frac{n\pi}{\alpha} \theta \left(\frac{r}{h}\right)^{\frac{n\pi}{\alpha}} \end{aligned}$$

$$= -\frac{1}{n} \frac{(-1)^n}{n} \cos \frac{n\pi}{\alpha} \theta \left(\frac{r}{h}\right)^{\frac{n\pi}{\alpha}}$$

Theorem 4.1. The Residue Theorem [8]

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities a, b, c, \dots inside C , which have residues given by $a_{-1}, b_{-1}, c_{-1}, \dots$ then

$$\oint_C f(z) dz = 2\pi i (a_{-1} + b_{-1} + c_{-1} + \dots)$$

i.e., the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C .

By the residue theorem, these produce:

For $0 \leq \theta \leq \alpha, 0 < r < a, r < h, 0 \leq \alpha \leq \pi$

$$W(r, \theta) = \frac{-1}{\pi\mu} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} G_- \left(\frac{-n\pi}{\alpha}\right) \cos(\alpha - \theta) \frac{n\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}} + \frac{Q}{\pi\mu} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos \theta \frac{n\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{n\pi}{\alpha}}$$

As $r \rightarrow 0$, the dominant term occurs when $n = 1$. Hence,

$$W(r, \theta) = \frac{1}{\pi\mu} G_- \left(\frac{-\pi}{\alpha}\right) \cos(\alpha - \theta) \frac{\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}} - \frac{Q}{\pi\mu} \cos \theta \frac{\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{\pi}{\alpha}} \quad r \rightarrow 0 \quad (4.5)$$

Utilizing (3.1) and (4.5) leads to the stresses

$$\begin{aligned} \sigma_{\theta z}(r, \theta) &= \frac{\mu}{r} \frac{\partial W}{\partial \theta}(r, \theta) \\ &= \frac{a^{-1}}{\pi} \left[\frac{\pi}{\alpha} G_- \left(\frac{-\pi}{\alpha}\right) \sin(\alpha - \theta) \frac{\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}-1} \right] + \frac{Q}{\pi h \alpha} \sin \theta \frac{\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{\pi}{\alpha}-1} \\ &= \frac{1}{a \alpha} G_- \left(\frac{-\pi}{\alpha}\right) \sin(\alpha - \theta) \frac{\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}-1} + \frac{Q}{h \alpha} \sin \theta \frac{\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{\pi}{\alpha}-1} \end{aligned} \quad (4.6)$$

$$\begin{aligned} \sigma_{rz}(r, \theta) &= \mu \frac{\partial W(r, \theta)}{\partial r} \\ &= \frac{1}{\pi\mu} \left[\frac{\mu \pi}{a \alpha} G_- \left(\frac{-\pi}{\alpha}\right) \cos(\alpha - \theta) \frac{\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}-1} \right] - \frac{Q}{\pi\mu} \left[\frac{\mu \pi}{h \alpha} \cos \theta \frac{\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{\pi}{\alpha}-1} \right] \\ &= \frac{1}{a \alpha} G_- \left(\frac{-\pi}{\alpha}\right) \cos(\alpha - \theta) \frac{\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}-1} - \frac{Q}{h \alpha} \cos \theta \frac{\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{\pi}{\alpha}-1} \end{aligned} \quad (4.7)$$

For $-\alpha \leq \theta \leq 0$ use is made of (4.2) because $0 \leq \alpha \leq \pi, 0 \leq r \leq a, r \leq h$ and we are investigating the fields as $r \rightarrow 0$, we need powers of $\frac{r}{a}$ to be positive. The residues are obtained as follows:

For the first integral, set $S_n = -\frac{n\pi}{\alpha}, n = 1, 2, 3, \dots$ and

$$R_3(s) = \frac{-G_-(s) \cos(\alpha + \theta) s}{S \sin \alpha s} \left(\frac{r}{a}\right)^{-s}$$

Then,

$$\begin{aligned} \lim_{s \rightarrow s_n} R_3(s) &= -\lim_{s \rightarrow s_n} \frac{(S - s_n) G_-(s)}{\sin \alpha s} \frac{G_-(s)}{S} \cos(\alpha + \theta) s \left(\frac{r}{a}\right)^{-s} \\ &= -\lim_{s \rightarrow s_n} \frac{1}{\alpha \cos \alpha s} \frac{G_-(s_n)}{S_n} \cos(\alpha + \theta) s_n \left(\frac{r}{a}\right)^{-s_n} \\ &= -\frac{1}{\alpha \cos n\pi} \frac{G_- \left(\frac{-n\pi}{\alpha}\right)}{-\frac{n\pi}{\alpha}} \cos(\alpha + \theta) \frac{n\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}} \\ &= \frac{(-1)^n}{\pi n} G_- \left(\frac{-n\pi}{\alpha}\right) \cos(\alpha + \theta) \frac{n\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}} \end{aligned}$$

For the second integral, positive powers of $\frac{r}{a}$ are also needed, so we set

$$S_n = -\frac{n\pi}{\alpha}, n = 1, 2, 3, \dots \text{ and } R_4(s) = \frac{\cos \theta s}{S \sin \alpha s} \left(\frac{r}{h}\right)^{-s}$$

$$\begin{aligned} \operatorname{Re}_{S \rightarrow s} R_4(s) &= - \lim_{s \rightarrow s_n} \frac{(S - s_n) \cos \theta s}{\sin \alpha s} \frac{s}{s_n} \left(\frac{r}{h}\right)^{-s} \\ &= \lim_{s \rightarrow s_n} \frac{1}{\alpha \cos \alpha s} \frac{\cos \theta s_n}{s_n} \left(\frac{r}{h}\right)^{-s_n} \\ &= \frac{1}{\alpha \cos n\pi} \frac{-\cos \theta \frac{n\pi}{\alpha}}{\frac{n\pi}{\alpha}} \left(\frac{r}{h}\right)^{\frac{n\pi}{\alpha}} \\ &= -\frac{(-1)^n}{\pi n} \cos \theta \frac{n\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{n\pi}{\alpha}} \end{aligned}$$

By the residue theorem, these produce

$$\begin{aligned} W(r, \theta) &= \frac{1}{\mu\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} G_- \left(-\frac{n\pi}{\alpha}\right) \cos(\alpha + \theta) \frac{n\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}} \\ &\quad - \frac{Q}{\mu\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos \theta \frac{n\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{n\pi}{\alpha}}, \quad r < a \end{aligned}$$

The dominant term, as $r \rightarrow 0$, occurs when $n = 1$. Hence,

$$W(r, \theta) = -\frac{1}{\mu\pi} G_- \left(-\frac{\pi}{\alpha}\right) \cos(\alpha + \theta) \frac{\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}} + \frac{Q}{\mu\pi} \cos \theta \frac{\pi}{h} \left(\frac{r}{h}\right)^{\frac{\pi}{\alpha}}, \quad r \rightarrow 0 \quad (4.8)$$

Use of (3.01) and (4.8) gives

$$\begin{aligned} \sigma_{\theta z}(r, \theta) &= \frac{\mu}{r} \frac{\partial W}{\partial \theta}(r, \theta) \\ &= \frac{1}{\alpha\pi} \left[-G_- \left(-\frac{\pi}{\alpha}\right) \frac{\pi}{\alpha} \sin(\alpha + \theta) \frac{\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}-1} \right] - \frac{Q}{\pi\mu h} \left[\frac{\pi}{\alpha} \sin \theta \frac{\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{\pi}{\alpha}-1} \right] \\ &= -\frac{1}{\alpha\pi} \left[G_- \left(-\frac{\pi}{\alpha}\right) \sin \frac{\pi}{\alpha} (\alpha + \theta) \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}} \right] + \frac{Q}{\pi h} \left[\sin \theta \frac{\pi}{\alpha} \left(\frac{r}{h}\right)^{\frac{\pi}{\alpha}-1} \right], \quad r \rightarrow 0 \quad (4.9) \end{aligned}$$

$$\begin{aligned} \sigma_{rz}(r, \theta) &= \mu \frac{\partial W}{\partial \theta}(r, \theta) \\ &= -\frac{1}{\alpha\alpha} \left[G_- \left(-\frac{\pi}{\alpha}\right) \cos(\alpha + \theta) \frac{\pi}{\alpha} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}-1} \right] + \frac{Q}{h\alpha} \left[\cos \theta \frac{r}{\alpha} \left(\frac{r}{h}\right)^{\frac{\pi}{\alpha}} \right], \quad r \rightarrow 0 \quad (4.10) \end{aligned}$$

3.3. Displacement of the Region beyond the Crack Tip With $r \geq a$

The region beyond the crack tip is considered when $r \geq a, -\alpha \leq \theta \leq \alpha$.

In this case, we need the powers of $\frac{a}{r}$ to be positive (or powers of $\left(\frac{r}{a}\right)$ to be negative) so that the corresponding series will converge. To obtain negative powers of $\left(\frac{r}{a}\right)^{-s}$ (4.3) and (4.4) will be used, and the integrals will be evaluated by the method of residues. The poles are simple and located at

$$S_n = \left(n - \frac{1}{2}\right) \frac{\pi}{\alpha}, \quad n = 1, 2, 3, \dots$$

For $0 \leq \theta \leq \alpha, r \geq a, r \geq h, -\pi \leq \alpha \leq \pi$

The residues for the first integral in (3.59) are

$$\begin{aligned} \operatorname{Resi} S_n &= \lim_{s \rightarrow s_n} \left\{ \frac{S - s_n}{\cos \alpha s} \cos(\alpha - \theta) s H_+(s) \left(\frac{r}{a}\right)^{-s} \right\} \\ &= -\lim_{s \rightarrow s_n} \frac{1}{\alpha \sin \alpha s} \cos(\alpha - \theta) s_n H_+(s_n) \left(\frac{r}{a}\right)^{-s_n} \\ &= -\frac{1}{\alpha \sin \left(n - \frac{1}{2}\right) \pi} \cos(\alpha - \theta) s_n H_+(s_n) \left(\frac{r}{a}\right)^{-s_n} \\ &= -\frac{(-1)^n}{\alpha} \cos(\alpha - \theta) s_n H_+(s_n) \left(\frac{r}{a}\right)^{-s_n}, \quad n = 1, 2, 3, \dots \end{aligned}$$

The residues for the second integral in (4.3) are

$$\begin{aligned} \text{Resi } S_n &= \lim_{s \rightarrow s_n} \frac{(S - s_n) \sin \theta s}{\cos \alpha s} \frac{1}{S} \left(\frac{r}{h}\right)^{-s} \\ &= -\frac{(-1)^n \sin \theta s_n}{\alpha (2n-1) \frac{\pi}{2\alpha}} \left(\frac{r}{h}\right)^{-s_n} \\ &= \frac{1}{\pi} \frac{-2(-1)^n}{(2n-1)} \sin \theta s_n \left(\frac{r}{h}\right)^{-s_n} \end{aligned}$$

The residue theorem gives

$$\begin{aligned} W(r, \theta) &= \frac{-1}{2\alpha} \sum_{n=1}^{\infty} (-1)^n \cos(\alpha - \theta) s_n H_+(s_n) \left(\frac{r}{a}\right)^{-s_n} \\ &= -\frac{2Q}{\mu\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sin \theta s_n \left(\frac{r}{h}\right)^{-s_n} \quad r \geq a, \quad r \geq h \end{aligned} \quad (4.11)$$

or

$$\begin{aligned} W(r, \theta) &= -\frac{1}{2\alpha} \sum_{n=1}^{\infty} (-1)^n \cos(\alpha - \theta) s_n H_+(S) \left(\frac{a}{r}\right)^{s_n} \\ &\quad - \frac{2Q}{\mu\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sin \theta s_n \left(\frac{h}{r}\right)^{s_n} \quad a \leq r, \quad h \leq r \end{aligned} \quad (4.12)$$

When the asymptotic nature of (3.68) is considered, the dominant term is that for which $n = 1$. That is

$$W(r, \theta) = \frac{1}{2\alpha} \cos(\alpha - \theta) s_1 H_+(s_1) \left(\frac{a}{r}\right)^{s_1} + \frac{1}{\mu\pi} \sin \theta S_1 \left(\frac{h}{r}\right)^{s_1} \quad a \leq r, \quad h \leq r$$

or equivalently,

$$W(r, \theta) = \frac{1}{2\alpha} \cos \frac{\pi}{2\alpha} (\alpha - \theta) H_+ \left(\frac{\pi}{2\alpha}\right) \left(\frac{a}{r}\right)^{\frac{\pi}{2\alpha}} + \frac{1}{\mu\pi} \sin \theta \frac{\pi}{2\alpha} \left(\frac{h}{r}\right)^{\frac{\pi}{2\alpha}} \quad r \geq a, \quad r \geq h \quad (4.13)$$

Using (3.1) and (4.13) gives the stresses

$$\begin{aligned} \sigma_{\theta z}(r, \theta) &= \frac{\mu}{r} \frac{\partial W}{\partial \theta}(r, \theta) \\ &= \frac{\mu}{r} \left[\frac{1}{2\alpha} \left\{ \frac{\pi}{2\alpha} \sin \frac{2}{2\alpha} (\alpha - \theta) H_+ \left(\frac{\pi}{2\alpha}\right) \left(\frac{a}{r}\right)^{\frac{\pi}{2\alpha}} \right\} + \frac{2}{\mu\pi} \frac{\pi}{2\alpha} \cos \frac{\pi\theta}{2\alpha} \left(\frac{h}{r}\right)^{\frac{h}{2\alpha}} \right] \\ \sigma_{\theta z}(r, \theta) &= \frac{\pi}{2\alpha} \left\{ \frac{\mu}{2\alpha a} \sin \frac{\pi}{2\alpha} (\alpha - \theta) H_+ \left(\frac{\pi}{2\alpha}\right) \left(\frac{a}{r}\right)^{\frac{\pi}{2\alpha}+1} + \frac{2}{\pi h} \cos \frac{\pi\theta}{2\alpha} \left(\frac{h}{r}\right)^{\frac{h}{2\alpha}+1} \right\} \quad r \geq a, \quad r \geq h \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} \sigma_{rz}(r, \theta) &= \mu \frac{\partial W}{\partial r}(r, \theta) \\ &= \mu \left\{ \frac{1}{2\alpha a} \cos \frac{\pi}{2\alpha} (\alpha - \theta) H_+ \left(\frac{\pi}{2\alpha}\right) \left(-\frac{\pi}{2\alpha}\right) \left(\frac{a}{r}\right)^{\frac{\pi}{2\alpha}+1} + \frac{2}{\mu\pi h} \sin \frac{\pi\theta}{2\alpha} \left(-\frac{\pi}{2\alpha}\right) \left(\frac{h}{r}\right)^{\frac{\pi}{2\alpha}+1} \right\} \\ &= -\frac{\pi}{2\alpha} \left\{ \frac{\mu}{2\alpha a} \cos \frac{\pi}{2\alpha} (\alpha - \theta) H_+ \left(\frac{\pi}{2\alpha}\right) \left(\frac{a}{r}\right)^{\frac{\pi}{2\alpha}+1} + \frac{2}{\pi h} \sin \frac{\pi\theta}{2\alpha} \left(\frac{h}{r}\right)^{\frac{\pi}{2\alpha}+1} \right\} \quad r \geq a, \quad r \geq h \end{aligned} \quad (4.15)$$

For $-\alpha \leq \theta \leq 0$

We use (4.4) to indicate that the poles are simple and located at

$$S_n = \left(n - \frac{1}{2}\right) \frac{\pi}{\alpha}, \quad n = 1, 2, 3, \dots$$

The residues for the first integral are

$$\begin{aligned}
 \text{Resi } S_n &= \lim_{s \rightarrow s_n} \frac{S - s_n}{\cos \alpha s} \cos(\alpha + \theta) s H_+(s) \left(\frac{r}{a}\right)^{-s} \\
 &= -\frac{1}{\alpha \sin \alpha s} \left| \cos(\alpha + \theta) s_n H_+(s_n) \left(\frac{r}{a}\right)^{-s_n} \right. \\
 &\quad \left. S = s_n \right. \\
 &= -\frac{1}{\alpha \sin \left(n - \frac{1}{2}\right) n} \cos(\alpha + \theta) s_n H_+(s_n) \left(\frac{r}{a}\right)^{-s_n} \\
 &= -\frac{(-1)^{n+1}}{\alpha} \cos(\alpha + \theta) s_n H_+(s_n) \left(\frac{r}{a}\right)^{-s_n} \\
 &= \frac{(-1)^n}{\alpha} \cos(\alpha + \theta) s_n H_+(s_n) \left(\frac{r}{a}\right)^{-s_n}
 \end{aligned}$$

Residues of the second integral are

$$\begin{aligned}
 (\text{Residue at } S = s_n) &= \lim_{s \rightarrow s_n} \frac{S - s_n}{\cos \alpha s} \frac{\sin \theta s}{S} \left(\frac{r}{h}\right)^{-s} \\
 &= -\frac{1}{\alpha \sin \alpha s} \left| \frac{\sin \theta s_n}{S_n} \left(\frac{r}{h}\right)^{-s_n} \right. \\
 &\quad \left. S = s_n \right. \\
 &= \frac{-2(-1)^{n+1}}{\alpha(2n - 1)} \frac{\sin \theta s_n}{S_n} \left(\frac{r}{h}\right)^{-s_n}
 \end{aligned}$$

The residue theorem then leads to

$$\begin{aligned}
 W(r, \theta) &= -\frac{1}{2\alpha} \sum_{n=1}^{\infty} (-1)^n \cos(\alpha + \theta) s_n H_+(s_n) \left(\frac{r}{a}\right)^{-s_n} \\
 &\quad \frac{2Q}{\alpha\mu} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n - 1} \sin \theta s_n \left(\frac{r}{h}\right)^{-s_n}, \quad r \geq a, \quad r \geq h \quad (4.16)
 \end{aligned}$$

or equivalently

$$\begin{aligned}
 W(r, \theta) &= -\frac{1}{2\alpha} \sum_{n=1}^{\infty} (-1)^n \cos(\alpha + \theta) s_n H_+(s_n) \left(\frac{a}{r}\right)^{s_n} \\
 &\quad -\frac{2}{\alpha} \frac{Q}{\mu} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n - 1} \sin \theta s_n \left(\frac{r}{h}\right)^{s_n}, \quad r \geq a, \quad r \geq h \quad (4.17)
 \end{aligned}$$

The dominant term in (4.16) and (4.17) occurs when $n = 1$.

Therefore

$$W(r, \theta) = \frac{1}{2\alpha} \cos(\alpha + \theta) s_1 H_+(s_1) \left(\frac{a}{r}\right)^{s_1} + \frac{2Q}{\alpha\mu} \sin \theta S_1 \left(\frac{h}{r}\right)^{s_1}, \quad r \geq a, \quad r \geq h$$

That is

$$W(r, \theta) = \frac{1}{2\alpha} \cos(\alpha + \theta) \frac{\pi}{2\alpha} H_+\left(\frac{\pi}{2\alpha}\right) \left(\frac{a}{r}\right)^{\frac{\pi}{2\alpha}} + \frac{2Q}{\alpha\mu} \sin \theta \frac{\pi}{2\alpha} \left(\frac{h}{r}\right)^{\frac{\pi}{2\alpha}} \quad (4.18)$$

The corresponding stresses are obtained from (3.1) and (3.74)

$$\begin{aligned}
 \sigma_{\theta z}(r, \theta) &= \frac{\mu}{r} \frac{\partial W}{\partial \theta}(r, \theta) \\
 &= \frac{\mu}{r} \left[-\frac{1}{2\alpha} \frac{\pi}{2\alpha} \sin \frac{\pi}{2\alpha} (\alpha + \theta) H_+\left(\frac{\pi}{2\alpha}\right) \left(\frac{a}{r}\right)^{\frac{\pi}{2\alpha}} + \frac{2Q}{\alpha\mu} \frac{\pi}{2\alpha} \cos \frac{\pi\theta}{2\alpha} \left(\frac{h}{r}\right)^{\frac{\pi}{2\alpha}} \right] \\
 &= \frac{\pi}{2\alpha} \left\{ -\frac{\mu}{2\alpha a} \sin \frac{\pi}{2\alpha} (\alpha + \theta) H_+\left(\frac{\pi}{2\alpha}\right) \left(\frac{a}{r}\right)^{\frac{\pi}{2\alpha}+1} + \frac{2}{\alpha h} \cos \frac{\pi\theta}{2\alpha} \left(\frac{h}{r}\right)^{\frac{\pi}{2\alpha}+1} \right\} \\
 &\quad , r \geq a, \quad r \geq h \\
 \sigma_{rz}(r, \theta) &= \mu \frac{\partial W}{\partial r}(r, \theta)
 \end{aligned} \quad (4.19)$$

$$\begin{aligned}
 &= \mu \left\{ \frac{1}{2\alpha} \cos \frac{\pi}{2\alpha} (\alpha + \theta) H_+ \left(\frac{\pi}{2\alpha} \right) \left(-\frac{\pi}{2\alpha} \right) \frac{1}{a} \left(\frac{a}{r} \right)^{\frac{\pi}{2\alpha} + 1} + \frac{2}{\alpha} \frac{Q}{\mu} \left(-\frac{\pi}{2\alpha} \right) \sin \frac{\pi\theta}{2\alpha} \frac{1}{h} \left(\frac{h}{r} \right)^{\frac{\pi}{2\alpha} + 1} \right\} \\
 &= -\frac{\pi}{2\alpha} \left\{ \frac{\mu}{2\alpha} \cos \frac{\pi}{2\alpha} (\alpha + \theta) H_+ \left(\frac{\pi}{2\alpha} \right) \left(\frac{a}{r} \right)^{\frac{\pi}{2\alpha} + 1} + \frac{2Q}{\alpha h} \sin \frac{\pi\theta}{2\alpha} \left(\frac{h}{r} \right)^{\frac{\pi}{2\alpha} + 1} \right\} \quad (4.20)
 \end{aligned}$$

$r \geq a, \quad r \geq h$

3.4. Fields at the Crack Tip

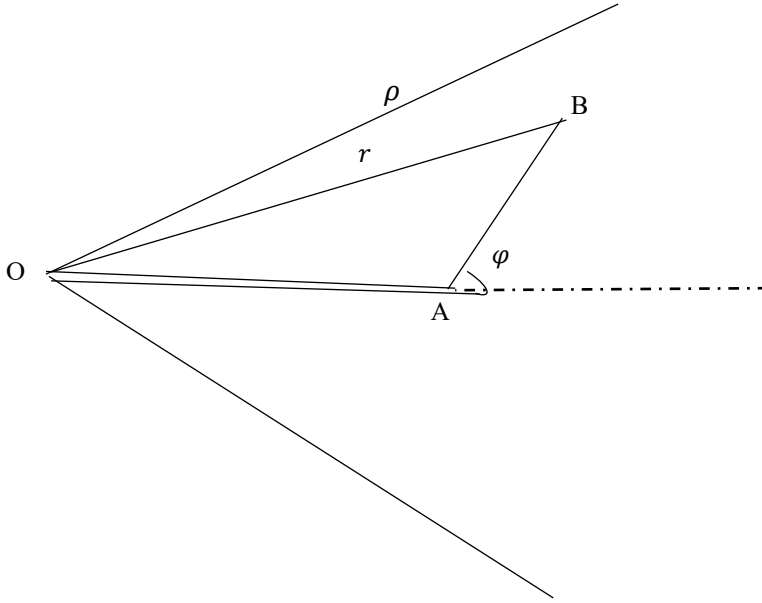


Fig. 2 Local Polar Coordinate (ρ, φ) at the crack tip

To investigate the fields at the crack tip, we introduce a local polar coordinate system. (ρ, φ) With origin at the tip there. Then the cosine rule gives

$$r^2 = a^2 + \rho^2 - 2a\rho \cos(\pi - \varphi)$$

The crack tip is approached if the following conditions are satisfied.

$$\theta \approx 0, r \approx a, \varphi \approx 0 \quad \text{and} \quad \rho \rightarrow 0 \quad (4.21)$$

The cosine rule then yields

$$r^2 = a^2 + 2a\rho \cos \varphi, \quad \rho \rightarrow 0$$

That is,

$$\left(\frac{r}{a} \right)^2 = 1 + 2 \frac{\rho}{a} \cos \varphi, \quad \rho \rightarrow 0$$

Or

$$\begin{aligned}
 \frac{r}{a} &= \left(1 + 2 \frac{\rho}{a} \cos \varphi \right)^{\frac{1}{2}}, \quad \rho \rightarrow 0 \\
 &= 1 + \frac{\rho}{a} \cos \varphi + 0 \left(\frac{\rho}{a} \right)^2, \quad \rho \rightarrow 0 \quad (4.22)
 \end{aligned}$$

Hence, conditions (4.21) are equivalent to

$$\theta \approx 0, \frac{r}{a} \rightarrow 1, \quad \varphi \approx 0 \quad \text{and} \quad \rho \rightarrow 0 \quad (4.23)$$

The vector triangle rule applied in the triangle OAB produces $\overline{OA} + \overline{AB} = \overline{OB}$. That is

$$r e^{i\theta} = a e^{i0} + \rho e^{i\varphi}$$

Hence,

$$r \cos \theta + i r \sin \theta = a \cos 0 + i \sin 0 + \rho \cos \varphi + i \rho \sin \varphi$$

Or

$$r \cos \theta + ir \sin \theta = a + \rho \cos \varphi + i\rho \sin \varphi$$

Then,

$$\begin{aligned} r \cos \theta &= a + \rho \cos \varphi \quad (\text{condition (4.21) or (4.22) holds}) \\ r \sin \theta &= \rho \sin \varphi \end{aligned} \tag{4.24}$$

The Maclaurin series expansion for $\sin \theta$ and $\sin^{-1} \theta$ are

$$\begin{aligned} \sin \theta &= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \\ &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \\ &= \theta + \theta^2 \left(-\frac{\theta}{3!} + \frac{\theta^3}{5!} - \dots \right) \\ &= \theta + 0(\theta^2), \quad \theta \rightarrow 0 \end{aligned} \tag{4.25}$$

$$\begin{aligned} \sin^{-1} \theta &= \sum_{n=0}^{\infty} a_n \frac{\theta^{2n+1}}{(2n+1)!}, \quad a_n = \frac{(2n)!}{2^{2n} (n!)^2} \\ &= \theta + \frac{1}{2} \frac{\theta^3}{3} + \frac{1.3}{2.4} \frac{\theta^5}{5} + \dots \\ &= \theta + \theta^2 \left(\frac{1}{2} \frac{\theta}{3} + \frac{1.3}{2.4} \frac{\theta^3}{5} + \dots \right) \\ &= \theta + 0(\theta^2), \quad \theta \rightarrow 0 \end{aligned} \tag{4.26}$$

From (4.25) and $r \simeq a$, we get

$$\sin \theta = \frac{\rho}{a} \sin \varphi$$

Thus, referring to (4.26), we get

$$\begin{aligned} \theta &= \sin^{-1} \left(\frac{\rho}{a} \sin \varphi \right) \\ &= \frac{\rho}{a} \sin \varphi + 0 \left[\left(\frac{\rho}{a} \right)^2 \right], \quad \rho \rightarrow 0 \end{aligned} \tag{4.27}$$

The Brownwich integral will be evaluated by the method introduced by Choi & Earmme (2001). The method requires taking $c = 0$ and integrating from $-i\infty$ to $i\infty$. It also requires dividing the interval of integration into three. That is

$$(-i\infty, i\infty) = \left(-i\infty, -i \left(\frac{\rho}{a} \right)^\beta \right) U \left(-i \left(\frac{\rho}{a} \right)^\beta, i \left(\frac{\rho}{a} \right)^\beta \right) U \left(i \left(\frac{\rho}{a} \right)^\beta, i\infty \right), \quad -1 < \beta < 0 \tag{4.28}$$

Advantage is then taken of the integration properties of integrals of continuous functions on a symmetric interval in the two symmetric integrals associated with $(-i\infty, i\infty)$ and $\left(-i \left(\frac{\rho}{a} \right)^\beta, i \left(\frac{\rho}{a} \right)^\beta \right)$. The local crack tip condition, $r \simeq a, \theta \simeq \phi$ as $\rho \rightarrow 0$ ie $\frac{r}{a} \rightarrow 1$ is associated with the interval $\left(-i \left(\frac{\rho}{a} \right)^\beta, i \left(\frac{\rho}{a} \right)^\beta \right)$, and subsequently with the other two.

Then, for $0 \leq \theta \leq \alpha$, as $\theta \rightarrow \varphi, \theta \rightarrow 0, \varphi \rightarrow 0, \rho \rightarrow 0$
(4.27) becomes $0 \leq \varphi \leq \alpha$

$$\begin{aligned} W(\rho, \varphi) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{2} H_+(s) \left(\frac{r}{a} \right)^{-s} ds \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{-i \left(\frac{\rho}{a} \right)^\beta} \frac{1}{2} H_+(s) \left(\frac{r}{a} \right)^{-s} ds + \frac{1}{2\pi i} \int_{-i \left(\frac{\rho}{a} \right)^\beta}^{i \left(\frac{\rho}{a} \right)^\beta} \frac{1}{2} H_+(s) \left(\frac{r}{a} \right)^{-s} ds + \frac{1}{2\pi i} \int_{i \left(\frac{\rho}{a} \right)^\beta}^{i\infty} \frac{1}{2} H_+(s) \left(\frac{r}{a} \right)^{-s} ds \end{aligned} \tag{4.29}$$

Let

$$I_1 = \frac{1}{2\pi i} \int_{-i\infty}^{-i \left(\frac{\rho}{a} \right)^\beta} \frac{1}{2} H_+(s) \left(\frac{r}{a} \right)^{-s} ds \tag{4.30}$$

$$I_2 = \frac{1}{2\pi i} \int_{-i\left(\frac{\rho}{a}\right)^\beta}^{i\left(\frac{\rho}{a}\right)^\beta} \frac{1}{2} H_+(s) \left(\frac{r}{a}\right)^{-s} ds \tag{4.31}$$

$$I_3 = \frac{1}{2\pi i} \int_{i\left(\frac{\rho}{a}\right)^\beta}^{i\infty} \frac{1}{2} H_+(s) \left(\frac{r}{a}\right)^{-s} ds \tag{4.32}$$

For I_2 we use $\left(\frac{r}{a}\right)^{-s} = 1$ because $r \simeq a$ and $\theta \rightarrow \varphi$ implies $\frac{r}{a} \rightarrow 1$; this relation is called into play whenever I_2 is involved. For I_1 and I_3 we use

$$\left(\frac{r}{a}\right)^{-s} = e^{\ln\left(\frac{r}{a}\right)^{-s}} = e^{-s \ln\left(\frac{r}{a}\right)} = e^{-s \ln\left(1 + \frac{\rho}{a} \cos \varphi\right)} = e^{-s \frac{\rho}{a} \cos \varphi} \tag{4.33}$$

We have used

$$\ln(1 + s) = s + O(s^2), \quad s \rightarrow 0$$

Let

$$I_0 = \int_{-i\infty}^{i\infty} \frac{1}{2} H_+(s) \left(\frac{r}{a}\right)^{-s} ds \tag{4.34}$$

Then

$$W(\rho, \varphi) = I_0 = I_1 + I_2 + I_3$$

Since the definite symmetric integral of an analytic function, with constant limits of integration, is a constant, it follows that in (4.33)

$$I_0 = c_0$$

With $\left(\frac{r}{a}\right)^{-s} = 1$ in all the integrands involving I_2 , we have

$$\begin{aligned} I_2 &= \frac{1}{2\pi i} \int_{-i\left(\frac{\rho}{a}\right)^\beta}^{i\left(\frac{\rho}{a}\right)^\beta} \frac{1}{2} H_+(s) ds \\ &= c_0 - \frac{1}{2\pi i} \int_{-i\infty}^{-i\left(\frac{\rho}{a}\right)^\beta} \frac{1}{2} H_+(s) ds - \frac{1}{2\pi i} \int_{i\left(\frac{\rho}{a}\right)^\beta}^{i\infty} \frac{1}{2} H_+(s) ds \end{aligned}$$

To convert the integration limits to real quantities, we make a change of variables through $S = -i\tau$ in the first integral. Then $ds = -id\tau, s = -i\infty$

$$\text{implies } \tau = \infty, s = -i\left(\frac{\rho}{a}\right)^\beta \text{ implies } \tau = i\left(\frac{\rho}{a}\right)^\beta$$

In the second integral, we let $s = i\tau$. Then,

$$ds = id\tau, s = i\infty \text{ implies } \tau = \infty, s = i\left(\frac{\rho}{a}\right)^\beta \text{ implies } \tau = \left(\frac{\rho}{a}\right)^\beta$$

Then,

$$\begin{aligned} I_2 &= C_0 - \frac{1}{4\pi i} \int_{\infty}^{\left(\frac{\rho}{a}\right)^\beta} H_+(-i\tau)(-id\tau) - \frac{1}{4\pi i} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} H_+(i\tau)(id\tau) \\ &= C_0 - \frac{1}{4\pi i} \left\{ \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} [H_+(-i\tau) + H_+(i\tau)] d\tau \right\} \end{aligned}$$

$$= C_0 - \frac{1}{4\pi} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} [H_+(-i\tau) + H_+(i\tau)] d\tau \tag{4.35}$$

It is pertinent to understand the asymptotic behavior of $H_+(s)$ as $|s| \rightarrow \infty$ In order to evaluate the improper integral. From (3.50)

$$H_+(s) = \frac{2Q}{\mu} \left[\frac{L_+(0) - L_-(s)}{\Phi_+(s)} \right]$$

From (3.43)

$$\Phi_+(s) = \alpha^{\frac{1}{2}} S^{\frac{3}{2}}, \quad |s| \rightarrow \infty$$

Hence

$$H_+(s) = \frac{2Q}{\mu} \left[\frac{L_+(0)}{\Phi_+(s)} - \frac{L_-(s)}{\Phi_+(s)} \right] \tag{4.36}$$

Because

$$L_+(s) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \Phi_-\left(-\frac{\pi}{\alpha} \gamma_n\right)}{\frac{\alpha}{\pi} S - \gamma_n} \left(\frac{h}{a}\right)^{\frac{\pi}{\alpha} \gamma_n}$$

The numerators in the series are constants, $k_n, n = 1, 2, 3, \dots$ $|s| \rightarrow \infty$

$$\frac{L_+(s)}{\Phi_+(s)} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n K_n}{\left(\frac{\alpha}{\pi} S - \gamma_n\right) \alpha^{\frac{1}{2}} S^{\frac{3}{2}}} \rightarrow 0 \quad \text{faster than} \quad \frac{L_+(0)}{\Phi_+(s)}$$

Therefore, the dominant term in (4.36) is

$$H_+(s) = \frac{2Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}} S^{\frac{3}{2}}} \quad |S| \rightarrow \infty \tag{4.37}$$

From (4.37) and using $i = e^{\frac{\pi i}{2}}$, we have

$$\begin{aligned} H_+(i\tau) &= \frac{2Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}} i^{\frac{3}{2}} \tau^{\frac{3}{2}}} \\ &= \frac{2Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} \tau^{-\frac{3}{2}} e^{-\frac{3i\pi}{4}}, \quad |s| \rightarrow \infty \\ H_+(-i\tau) &= \frac{2Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}} (-i\tau)^{\frac{3}{2}}} \\ &= \frac{2Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} \tau^{-\frac{3}{2}} e^{+\frac{3i\pi}{4}} \quad |s| \rightarrow \infty \end{aligned}$$

Thus,

$$\begin{aligned} H_+(i\tau) + H_+(-i\tau) &= \frac{2Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} \tau^{-\frac{3}{2}} \left(e^{\frac{3i\pi}{4}} + e^{-\frac{3i\pi}{4}} \right) \\ &= \frac{4Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} \cos \frac{3\pi}{4} \tau^{-\frac{3}{2}} \\ &= \frac{-4Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} \cos \frac{\pi}{4} \tau^{-\frac{3}{2}} \end{aligned}$$

Substituting the above result into (4.35) leads to

$$\begin{aligned} I_2 &= C_0 + \frac{4Q}{4\pi} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} \cos \frac{\pi}{4} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \tau^{-\frac{3}{2}} d\tau \\ &= C_0 - \frac{2Q}{3\pi} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} \cos \frac{\pi}{4} \tau^{-\frac{1}{2}} \Big|_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \end{aligned}$$

$$\begin{aligned}
 &= C_0 - \frac{2Q}{3\pi} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} \cos \frac{\pi}{4} \left(-\left(\frac{\rho}{a}\right)^{-\frac{\beta}{2}} \right) \quad \left(-\frac{1}{2} < \frac{\beta}{2} < 0 \right) \\
 &= C_0 + \frac{2^{\frac{1}{2}} Q}{3\pi\alpha^{\frac{1}{2}}} L_+(0) \left(\frac{\rho}{a}\right)^v, \quad v = -\frac{\beta}{2} > 0
 \end{aligned}$$

Therefore

$$I_2 = C_0, \quad \rho \rightarrow 0 \tag{4.37}$$

Next, we recall from (4.30) and (4.33) that

$$I_1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\left(\frac{\rho}{a}\right)^a} \frac{1}{2} H_+(S) e^{-s\frac{\rho}{a} \cos \varphi} ds$$

I_1 can be converted to an improper integral with real value limits of integration by making the change of variable from s to τ through $s = -i\tau$, then

$$\begin{aligned}
 ds &= -i d\tau, \quad s = -i\infty \text{ gives } \tau = \infty, \quad s = -i\left(\frac{\rho}{a}\right)^a \text{ gives } \tau = \left(\frac{\rho}{a}\right)^\beta \\
 I_1 &= -\frac{1}{2\pi} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \frac{1}{2} H_+(-i\tau) e^{+i\tau\frac{\rho}{a} \cos \varphi} d\tau \\
 &= \frac{1}{4\pi} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} H_+(-i\tau) e^{+i\tau\frac{\rho}{a} \cos \varphi} d\tau
 \end{aligned}$$

From (3.88) and (3.89)

$$I_3 = \frac{1}{2\pi i} \int_{i\left(\frac{\rho}{a}\right)^\beta}^{-i\infty} \frac{1}{2} H_+(s) e^{-s\frac{\rho}{a} \cos \varphi} ds$$

The change of variable from s to τ is effected to make I_3 an improper integral with real value limits by setting $s = i\tau$, then

$$ds = i d\tau, \quad s = i\infty \text{ implies } \tau = \infty, \quad s = i\left(\frac{\rho}{a}\right)^\beta \text{ implies } \tau = \left(\frac{\rho}{a}\right)^\beta$$

Then

$$I_3 = \frac{1}{2\pi} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \frac{1}{2} H_+(i\tau) e^{-i\tau\frac{\rho}{a} \cos \varphi} d\tau$$

Consequently,

$$I_1 + I_3 = \frac{1}{4\pi} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \left[H_+(i\tau) e^{-i\tau\frac{\rho}{a} \cos \varphi} + H_+(-i\tau) e^{i\tau\frac{\rho}{a} \cos \varphi} \right] d\tau \tag{4.39}$$

Behaviors ∞ dominate the consideration of the functions in the integrands. To make computations tractable, we use the fact that at infinity and at 0, we have

$$e^{-i\tau\frac{\rho}{a} \sin \varphi} = \begin{cases} 0 & \text{if } \tau \rightarrow \infty \\ 1 & \text{if } \tau \rightarrow 0 \quad (ie \rho \rightarrow 0) \end{cases} \tag{4.40}$$

We then write (4.39) as

$$\begin{aligned}
 I_1 + I_3 &= \frac{1}{4\pi} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} e^{-\tau\frac{\rho}{a} \sin \varphi} \left[H_+(i\tau) e^{-i\tau\frac{\rho}{a} \cos \varphi} + H_+(-i\tau) e^{i\tau\frac{\rho}{a} \cos \varphi} \right] d\tau \\
 &= \frac{1}{4\pi} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \left[H_+(i\tau) e^{-i\tau\frac{\rho}{a} (\cos \varphi - i \sin \varphi)} + H_+(-i\tau) e^{i\tau\frac{\rho}{a} (\cos \varphi + i \sin \varphi)} \right] d\tau
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4\pi} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \left[\frac{2L_+(0)}{\mu} \frac{1}{\alpha^2} \tau^{-\frac{3}{2}} e^{-\frac{3\pi i}{4}} e^{-i\tau \frac{\rho}{a}} e^{-i\varphi} + \frac{2L_+(0)}{\mu} \frac{1}{\alpha^2} \tau^{-\frac{3}{2}} e^{-\frac{3\pi i}{4}} e^{i\tau \frac{\rho}{a}} e^{i\varphi} \right] d\tau \\
 &= \frac{1}{4\pi} \frac{2L_+(0)}{\mu} \frac{1}{\alpha^2} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \left[e^{-\frac{3\pi i}{4}} e^{-i\tau \frac{\rho}{a}} e^{-i\varphi} + e^{-\frac{3\pi i}{4}} e^{i\tau \frac{\rho}{a}} e^{i\varphi} \right] \tau^{-\frac{3}{2}} d\tau \tag{4.41}
 \end{aligned}$$

Using the entry of Gradshyeyn & Ryzik (1990) given as

$$\int_u^{\infty} \sigma^{v-1} e^{-\eta\sigma} d\sigma = \eta^{-v} \Gamma(v, \eta u), u > 0, \text{Re } \eta > 0 \tag{4.42}$$

and

$$\Gamma(\alpha, \mu u) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\mu u)^{\alpha+n}}{\alpha+n} \tag{4.43}$$

$$\alpha \neq 0, -1, -2, -3, \dots$$

Accordingly, with $\sigma = \tau, v = \frac{-1}{2}, \eta = i \frac{\rho}{a} e^{-i\varphi}$ and (3.98)

$$\begin{aligned}
 e^{\frac{-3\pi i}{4}} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \tau^{-\frac{3}{2}} e^{-i\tau \frac{\rho}{a}} e^{-i\varphi} d\tau &= e^{\frac{-3\pi i}{4}} \left[i \left(\frac{\rho}{a}\right) e^{-\varphi \frac{1}{2}} \Gamma\left(\frac{-1}{2}, i e^{-i\varphi} \left(\frac{\rho}{a}\right)^{\beta+1}\right) \right] \\
 &= e^{\frac{-3\pi i}{4}} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \left\{ \Gamma\left(\frac{-1}{2}\right) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{i \left(\frac{\rho}{a}\right)^{\beta+1} e^{-i\varphi} n^{-\frac{1}{2}}}{n^{-\frac{1}{2}}} \right\} \\
 &= e^{\frac{-3\pi i}{4}} (i)^{\frac{1}{2}} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \Gamma\left(\frac{-1}{2}\right) e^{-\frac{i\varphi}{2}} \quad \rho \rightarrow 0 \\
 &= e^{\frac{-3\pi i}{4}} e^{\frac{\pi i}{4}} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \Gamma\left(\frac{-1}{2}\right) e^{-\frac{i\varphi}{2}} \quad \rho \rightarrow 0 \\
 &= e^{\frac{\pi i}{2}} \Gamma\left(\frac{-1}{2}\right) \left(\frac{\rho}{a}\right)^{\frac{1}{2}} e^{-\frac{i\varphi}{2}} \quad \rho \rightarrow 0 \\
 &= -2 \left(\frac{-1}{2} \Gamma\left(\frac{-1}{2}\right) \left(\frac{\rho}{a}\right)^{\frac{1}{2}} e^{-i\left(\frac{\varphi}{2} + \frac{\pi}{2}\right)} = -2\sqrt{\pi} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} e^{-i\left(\frac{\varphi}{2} + \frac{\pi}{2}\right)} \right) \tag{4.44}
 \end{aligned}$$

Use has been made of (4.44)

Next, with $\sigma = \tau, v = -\frac{1}{2}, \eta = -i \frac{\rho}{a} e^{i\varphi}$

$$\begin{aligned}
 &= e^{\frac{3\pi i}{4}} \int_{\left(\frac{\rho}{a}\right)^\beta}^{\infty} \tau^{-\frac{3}{2}} e^{i\tau \frac{\rho}{a}} e^{i\varphi} d\tau = e^{\frac{3\pi i}{4}} \left[-i \left(\frac{\rho}{a}\right) e^{i\varphi \frac{1}{2}} \Gamma\left(\frac{-1}{2}, \left(\frac{\rho}{a}\right)^\beta (-i \frac{\rho}{a} e^{i\varphi}) \right) \right] \\
 &= e^{\frac{3\pi i}{4}} (-i)^{\frac{1}{2}} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} e^{i\frac{\varphi}{2}} \Gamma\left(\frac{-1}{2}\right) \quad \rho \rightarrow 0 \\
 &= e^{\frac{3\pi i}{4}} e^{-\frac{\pi i}{4}} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \Gamma\left(\frac{-1}{2}\right) e^{i\frac{\varphi}{2}} \quad \rho \rightarrow 0 \\
 &= e^{\frac{\pi i}{2}} \Gamma\left(\frac{-1}{2}\right) e^{i\frac{\varphi}{2}} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \quad \rho \rightarrow 0 \\
 &= (-2) \left(\frac{-1}{2} \Gamma\left(\frac{-1}{2}\right) \left(\frac{\rho}{a}\right)^{\frac{1}{2}} e^{i\left(\frac{\varphi}{2} + \frac{\pi}{2}\right)} \right) \\
 &= -2 \Gamma\left(\frac{-1}{2} + 1\right) e^{i\left(\frac{\varphi}{2} + \frac{\pi}{2}\right)}
 \end{aligned}$$

$$= -2 \Gamma\left(\frac{1}{2}\right) e^{i\left(\frac{\varphi}{2} + \frac{\pi}{2}\right)} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} = -2\sqrt{\pi} \cdot \left(\frac{\rho}{a}\right)^{\frac{1}{2}} e^{i\left(\frac{\varphi}{2} + \frac{\pi}{2}\right)} \quad \rho \rightarrow 0$$

Hence, from (4.41)

$$\begin{aligned} I_1 + I_3 &= \frac{1}{4\pi} \frac{2Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} = -2\sqrt{\pi} \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \left(e^{i\left(\frac{\varphi}{2} + \frac{\pi}{2}\right)} e^{-i\left(\frac{\varphi}{2} + \frac{\pi}{2}\right)} \right) \quad \rho \rightarrow 0 \\ &= \frac{1}{4\pi} \frac{-4\sqrt{\pi} Q}{\mu} \frac{L_+(0)}{\alpha^{\frac{1}{2}}} 2 \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \cos\left(\frac{\varphi}{2} + \frac{\pi}{2}\right) \quad \rho \rightarrow 0 \\ &= \frac{-2 Q}{\mu\pi^{\frac{1}{2}} \alpha^{\frac{1}{2}}} L_+(0) \left(-\left(\frac{\rho}{a}\right)^{\frac{1}{2}} \left(\sin \frac{\varphi}{2}\right) \right) \\ &= \frac{2 Q}{\mu\pi^{\frac{1}{2}} \alpha^{\frac{1}{2}}} L_+(0) \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \sin \frac{\varphi}{2} \quad \rho \rightarrow 0 \end{aligned} \quad (4.45)$$

Consequently, for $0 \leq \varphi \leq \alpha$

$$\begin{aligned} W(\rho, \varphi) &= I_1 + I_2 + I_3 \\ &= C_0 + I_1 + I_3 \\ &= C_0 + \frac{2 Q}{\mu\pi^{\frac{1}{2}} \alpha^{\frac{1}{2}}} L_+(0) \left(\frac{\rho}{a}\right)^{\frac{1}{2}} \sin \frac{\varphi}{2} \quad \rho \rightarrow 0 \end{aligned} \quad (4.46)$$

From (4.4), $\theta \rightarrow \varphi, \theta = 0$ implies $\varphi = 0, r \rightarrow a$ implies $\rho \rightarrow 0$

We have

$$W(\rho, \varphi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{1}{2} H_+(s) \left(\frac{r}{a}\right)^{-s} ds$$

A repeat of the computations leads to the same result given in (4.46)

3.5. Analysis

We have derived the displacement and stress fields in closed form.

In the neighborhood of the crack tip given by (4.5), (4.6) and (4.7) for $0 \leq \theta \leq \alpha, 0 < r < a$ and (4.8), (4.9) and (4.10).

For $-\alpha \leq \theta \leq 0, 0 < r < a$, we see that

$$W(r, 0^+) = -\frac{Q}{\mu\pi} \left\{ \frac{1}{Q} G_-\left(\frac{\pi}{\alpha}\right) + \left(\frac{a}{h}\right)^{\frac{\pi}{\alpha}} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}} \right\} \quad r \rightarrow 0, r < a \quad (4.47)$$

$$W(r, 0^-) = \frac{Q}{\mu\pi} \left\{ \frac{1}{Q} G_-\left(\frac{\pi}{\alpha}\right) + \left(\frac{a}{h}\right)^{\frac{\pi}{\alpha}} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}} \right\} \quad r \rightarrow 0, r < a$$

Hence $W(r, 0^+) = -W(r, 0^-)$

That is

$$W(r, 0^+) - W(r, 0^-) = -2W(r, 0^-) \text{ as } r \rightarrow 0, \quad r < a \quad (4.48)$$

$$\sigma_{\theta z}(r, 0^+) = 0 \quad \text{and} \quad \sigma_{\theta z}(r, 0^-) = 0 \quad r < a \quad (4.49)$$

Since,

$$\sigma_{rz}(r, 0^+) = -\frac{Q}{\alpha} \left\{ \frac{1}{aQ} G_-\left(-\frac{\pi}{a}\right) + \frac{1}{h} \left(\frac{a}{h}\right)^{\frac{\pi}{\alpha}-1} \right\} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}-1} \quad r < a \quad (4.50)$$

$$\sigma_{rz}(r, 0^-) = -\frac{Q}{\alpha} \left\{ \frac{1}{aQ} G_-\left(-\frac{\pi}{a}\right) + \frac{1}{h} \left(\frac{a}{h}\right)^{\frac{\pi}{\alpha}-1} \right\} \left(\frac{r}{a}\right)^{\frac{\pi}{\alpha}-1} \quad r < a \quad (4.51)$$

$$\sigma_{rz}(r, 0^+) = -\sigma_{rz}(r, 0^-)$$

In which case, we notice the jump given by

$$\sigma_{rz}(r, 0^+) - \sigma_{rz}(r, 0^-) = -2\sigma_{rz}(r, 0^-)$$

Because (3.51) gives

$$G_-(s) = Q \left[\frac{L_+(0) + L_-(s)}{\Phi_-(s)} \right]$$

It follows that

$$G_{-}\left(-\frac{\pi}{\alpha}\right)=Q\left[\frac{L_{+}(0)-L_{-}\left(-\frac{\pi}{\alpha}\right)}{\Phi\left(-\frac{\pi}{\alpha}\right)}\right] \quad (4.52)$$

Substituting (4.52) into (4.47) – (4.50) gives the fields in closed form

The fields ahead of the crack tip are given by (4.13), (4.14), and (4.15) as $\theta \rightarrow 0^{+}$, $r > a$ and by (4.18), (4.19) and (4.20) as $\theta \rightarrow 0^{-}$, $r > a$ for this case

$$W(r, 0^{+})=0 \quad (4.53)$$

$$\sigma_{\theta z}(r, 0^{+})=\frac{\pi}{2 \alpha}\left\{\frac{\mu}{2 a \alpha} H_{+}\left(\frac{\pi}{2 \alpha}\right)+\frac{2}{\pi h}\left(\frac{h}{a}\right)^{\frac{\pi}{2 \alpha}+1}\right\}\left(\frac{a}{r}\right)^{\frac{\pi}{2 \alpha}+1} \quad r > a \quad (4.54) \quad \sigma_{\theta z}(r, 0^{-})=\frac{\pi}{2 \alpha}\left\{\frac{\mu}{2 a \alpha} H_{+}\left(\frac{\pi}{2 \alpha}\right)+\frac{2}{\pi h}\left(\frac{h}{a}\right)^{\frac{\pi}{2 \alpha}+1}\right\}\left(\frac{a}{r}\right)^{\frac{\pi}{2 \alpha}+1} \quad r > a \quad (4.55)$$

$$\sigma_{r z}(r, 0^{+})=0 \quad (4.56)$$

$$\sigma_{r z}(r, 0^{-})=0$$

Hence, the continuity conditions are satisfied.

Substituting $H_{+}\left(\frac{\pi}{2 \alpha}\right)=\frac{2 Q}{\mu \Phi\left(\frac{\pi}{2 \alpha}\right)}\left[L_{+}(0)-L_{+}\left(\frac{\pi}{2 \alpha}\right)\right]$ into the fields yields closed-form expressions. The fields at the crack tip $r = a$ are assessed by use of (4.46) for the Mode III stress intensity factor, which is defined by Tada et al. (1993)

$$W(r, \theta)=\frac{K_{III}}{\mu}\left(\frac{2 r}{\pi}\right)^{\frac{1}{2}} \cos \frac{\theta}{2}$$

or

$$\sigma_{\theta z}(r, \theta)=-K_{III} \frac{1}{\sqrt{2 \pi r}} \sin \frac{\theta}{2}$$

Ignoring the constant term in (4.46), which signifies rigid displacement, we get

$$W(\rho, \varphi)=\frac{2 Q}{\mu \pi^{\frac{1}{2}} \alpha^{\frac{1}{2}}} L_{+}(0)\left(\frac{\rho}{\alpha}\right)^{\frac{1}{2}} \sin \frac{\varphi}{2} \quad \rho \rightarrow 0$$

$$=\frac{\sqrt{2} Q}{\sqrt{a} \alpha} \frac{L_{+}(0)}{\mu}\left(\frac{2 \rho}{\pi}\right)^{\frac{1}{2}} \sin \frac{\varphi}{2}$$

Hence, (SIF)

$$K_{III}=\sqrt{\frac{2}{a \alpha}} Q L_{+}(0) \quad (4.57)$$

From the results

- $W(r, 0^{+}) \neq -W(r, 0^{-})$ as $r \rightarrow 0, r < a$ in (4.47) implies that the displacement field is discontinuous for $r < a, \theta = 0$
- $W(r, 0^{+}) = W(r, 0^{-}) = 0 \quad r > a$ in (4.53) implies that the line of symmetry $\theta = 0, r > 0$ is not deformed, though displacement across it is continuous.
- $\sigma_{\theta z}(r, 0^{+}) = \sigma_{\theta z}(r, 0^{-}) = 0, r < a$ in (4.49) is an indication that the crack surface is traction-free.
- $\sigma_{\theta z}(r, 0^{+}) = \sigma_{\theta z}(r, 0^{-}), r > a$ in (4.54) and (4.55) shows that the tearing stress ahead of the crack is continuous across the line of symmetry $\theta = 0$.
- $\sigma_{r z}(r, 0^{+}) = \sigma_{r z}(r, 0^{-}) = 0 \quad r > a$ in (4.56) shows that the stress in the radial direction is continuous.
- $K_{III} = \sqrt{\frac{2}{a \alpha}} Q L_{+}(0)$ of (4.57) indicates that the mode III stress intensity factor at the crack tip is independent of the material constant μ and linearly dependent on the applied traction Q

4. Discussion

A cracked infinite wedge under antiplane shear has been investigated for the nature of its displacement and stresses at the wedge apex and crack tip. The stresses were found, and the Mode III stress intensity factor was found to be linearly dependent on the prescribed traction Q . The findings of Shahani [9] noted the derivation of closed-form relations for the stress distribution

in the wedge. The task was to solve equation (3.2) subject to (3.3) - (3.8) by the method of infinite Mellin transform equation (3.10), by the use of integration by parts formula (3.14).

The transform plane displacement $\bar{W}(S, \theta)$ is obtained by the use of the inversion formula equation (3.11) for the infinite Mellin transform. Also, the asymptotic behavior of the stresses and their satisfaction of (3.13) determines bounds for $\bar{W}(S, 0)$ expected behavior of the stresses concentrated at sharp corners, yield the expected behavior as the stresses are expected to vanish to infinity, is (3.16). The transform plane problem solved is therefore (3.18) - (3.22), writing the solution of (3.15) as (3.23) leads to (3.24). The coefficient, $A_i(s)$ and $B_i(s)$ $i = 1, 2$ will be deduced from the boundary conditions from (3.19) and (3.20), we get (3.25). Addition of (3.24) and (3.25) gives (3.28). Incorporating (3.26) and (3.27) into (3.28) gives (3.29). From the behavior of (3.15), it follows that $H(s)$ is a right half plane function, that is, $H(s)$ should be replaced by $H_+(s)$, on the other hand, the behavior of (3.16) with $\epsilon = \frac{1}{2}$ yields that $G(s)$ is analytic in the left half plane $Re\ s < \frac{1}{2}$ and should be written as $G_-(s)$, that brought about equation (3.29) as the two overlapping functions $H_+(s)$ and $G_-(s)$ that yields equation (3.30). Equation (3.30) and its equivalent (3.31) are known as Wiener-Hopf equation application of Wiener-Hopf technique to (3.31) yields (3.32) – (3.35) when (3.34) is substituted to (3.32) it becomes (3.36) with the help of infinite product and their gamma function equivalents we were able to arrive at (3.46) and the decomposition of (3.46) with the aid of the Mittag-Leffler's expansion theorem for $\sec \alpha s$ makes (3.36) to be (3.49). By Liouville's theorem, that makes the function (3.49) equal to a constant $C = L_+(0)$ leads to (3.50) and (3.51). The transformed displacement obtained from (3.23) $\bar{W}(S, \theta)$ (A Wiener – Hopf problem) gave birth to (3.52) and (3.54). In view of (3.54), the forms of $\bar{W}(S, \theta)$ For $0 \leq \theta \leq \alpha$, we use (3.52) and (3.54) to get (3.55). For $-\alpha \leq \theta \leq 0$, we use (3.53) and (3.54) to get (3.56). $\bar{W}(S, \theta)$ has the form $\max\left(-\frac{1}{2}, \lambda - 1\right) < c < \frac{1}{2}$ In terms of $G(s)$ obtained in (3.52) and (3.53), for $0 \leq \theta \leq \alpha$, (3.53) is used to get (4.1) for $-\alpha \leq \theta \leq 0$, and (3.53) is used to get (4.2). Alternatively, equivalent forms in terms of $H_+(s)$ given by (3.56) for $0 \leq \theta \leq \alpha$ produces (4.3) and $-\alpha \leq \theta \leq 0$ produces (4.4).

4.1. Displacement at the Crack Region

Because we are interested in the displacement and stresses at the wedge tip, we consider the evaluation of (4.1) and (4.2) as $r \rightarrow 0$. The results are expected for $0 < r < a$, $0 < r < h$ and $0 \leq \alpha \leq \pi$. Utilizing (3.1) and (4.5) $\bar{W}(r, \theta)$ leads to the stress equation (4.6) $\sigma_{rz}(r, \theta)$ and (4.7) $\sigma_{\theta z}(r, \theta)$. Again, by the residue theorem, these produce $w(r, 0)$, the dominant term, as $r \rightarrow 0$ occurs when $n = 1$, hence equation (4.8). Use of (3.1) and (4.8) gives $\sigma_{\theta z}(r, \theta)$ (4.9) and $\sigma_{rz}(r, \theta)$ (4.10).

4.2. Displacement of the Region beyond the Crack Tip $r \geq a, -\alpha \leq \theta \leq \alpha$.

In this case, we need the powers of $\frac{a}{r}$ to be positive or powers of $\frac{r}{a}$ to be negative so that the corresponding series will converge. To obtain negative powers of $\left(\frac{r}{a}\right)^{-s}$ (4.3) and (4.4) will be used to produce (4.11) and (4.12). When the asymptotic nature of (4.12) is considered, the dominant terms for $n = 1$ will produce (4.13). Then, using (3.1) and (4.13) gives the stresses. $\sigma_{\theta z}(r, \theta)$ (4.14) and (4.15) $\sigma_{rz}(r, \theta)$. The dominant term in (4.16) and (4.17) occurs when $n = 1$, producing (4.18) $W(r, \theta)$, then the corresponding stresses are obtained from (3.1) and (4.18) to produce (4.19) and (4.20) $\sigma_{\theta z}(r, \theta)$ and $\sigma_{rz}(r, \theta)$ respectively.

4.3. Fields at the Crack Tip

To investigate the fields at the crack tip, we introduced a local polar coordinate system. (ρ, φ) With origin at the tip, with the help of the cosine rule $r^2 = a^2 + \rho^2 - 2a\rho \cos(\pi - \varphi)$. Taken advantage of the integration properties of integrals of continuous functions on a symmetric interval in the two symmetric integrals associated with $(-i\infty, i\infty)$. We were able to come up with equation (4.46), which is the field equation at the crack tip or K_{III} (SIF).

5. Conclusion

The infinite Mellin transform was used to derive two half-known functions that were connected by a Wiener-Hopf equation. The Wiener-Hopf technique was employed to get the transformed displacement. Different fracture parameter responses at different regions of the wedge were obtained due to the presence of the crack.

- a) This research has shown that the stress intensity factor is independent of material property but linearly depends on the concentrated load. Therefore, tradespeople should understand that, irrespective of the structural material, there is a certain fracture response dependent on the application of load.
- b) This research shows that along the crack surface of the material, there is no continuity of displacement. This can lead to structural failure and waste of resources when applied to work without remedy.
- c) The result shows that the failure of such wedges described here does not depend on the choice of material but on loading.

- d) Finally, we have provided a closed-form solution for the determination of stress and displacement for such wedges as described in this work.

Recommendation

This method can be extended to wedges with other types of loading, such as distributed load. Further investigation can be carried out to determine the maximum value of the length of the crack 'a' for which the material can sustain reliable form in service.

Contribution to knowledge

- a) This research has shown that the stress intensity factor is independent of material property but depends on the concentrated load. Therefore, tradespeople should understand that, irrespective of the structural material, there is a certain fracture response to the application of load.
- b) This research shows that along the crack, the displacement is not continuous, indicating that failure and waste of resources can set in unless remedial measures are taken before the material is applied in work.
- c) The result shows that the failure of such wedges described here does not depend on the choice of material but on loading.
- d) We have provided a closed-form solution for the displacement everywhere in the wedge, which will enhance the determination of stresses at the appropriate parts of the material

Declarations

Consent for Publication

The authors have approved the manuscript for submission, and it has not been published or submitted for publication elsewhere.

Availability of Data and Materials

All data generated or analysed during this study are included in this published article.

Competing Interests

There is no competing interest among the authors.

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Authors' Contributions

NFA, NCA designed and conducted research, analyzed data; NFA wrote the paper; AAFO, AAC had primary responsibility for the final content. All authors read and approved the final manuscript.

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