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Original Article

Positive Periodic Solutions for a Sixth-Order Variable Coefficient Singular Differential Equation with Indefinite Weights

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Abstract - *In this paper, we consider a sixth-order variable coefficient singular differential equation with indefinite weights*

$$u^{(6)}(t)+\sum_{i=0}^{5}a_i(t)u^i(t)=\frac{h(t)}{u^\rho}+e(t),$$

where ω *is a positive constant,* $h(t), e(t)\in L^1(\mathbb{R}/\omega\mathbb{Z})$ *and* $e(t)>0$ *for any* $t\in[0,\omega]$*. By using the Krasnoselskii-Guo fixed point theorem, we prove the existence of positive periodic solutions to the above equation.*

Keywords - *Positive periodic solutions, Singular differential equations, Green function, Sixth-order differential equation, Indefinite weights.*

1. Introduction

Singular differential equations have a wide range of applications, such as physics, engineering, biology, and economics (Chu 2018; Huang et al. 2014) [1,[9]. In recent years, it has been widely applied in various disciplines such as boundary layer theory, Bose-Einstein condensates, and optical signal transmission, and some application examples can be found in (Torres 2015) [19][19][19].

The study of singular differential equations began in 1987, when Lazer and Solimini [15] discussed the existence of positive periodic solutions of the following two types of singular differential equations

$$u''-\frac{1}{u^\lambda}=\varphi(t) \tag{1.1}$$

and

$$u''+\frac{1}{u^\lambda}=\varphi(t) \tag{1.2}$$

where λ is a positive real constant and $\varphi(t)$ is a periodic function with period ω. They gave a necessary and sufficient condition for the existence of periodic solutions of equations (1.1) and (1.2).

Generally, for a damped differential equation with singularities

$$u''+a(t)u=f(t,u)+h(t)$$

where $h(t)$ is an external force and the nonlinearity $f(t,u)$ may be singular at the origin, it is said that it has an attractive at the origin, if

$$\lim_{x\to0^+}f(t,u)=-\infty,\forall t\in R.$$

and has a repulsive singularity at the origin, if

$$\lim_{x\to 0^+} f(t,u) = +\infty, \forall t \in R.$$

In 2010, Hakl and Torres [10] investigated the following second-order differential equation with attractive-repulsive singularities

$$u'' = \frac{g(t)}{u^{k_1}} - \frac{h(t)}{u^{k_2}} + \varphi(t). \tag{1.3}$$

The authors obtained the existence of periodic solutions of equation (1.3) by the method of lower and upper solutions.

In 2017, Lu et al. [13] discussed the existence of positive periodic solutions for the following Liénard equation with a singularity of repulsive type

$$u'' + f(u)u' - \frac{a(t)}{u} = \varphi(t)$$

where $a(t)$ is a continuous function with periodic T and $a(t) > 0$.

In 2020, Liu et al. [12] investigated the existence of a positive ω-periodic solution for the equation with a singularity of attractive type

$$u'' + a(t)u = \frac{h(t)}{u^{\rho}} + e(t),$$

where $a(t), h(t) \in C(\mathbb{R}\backslash\omega\mathbb{Z})$ were negative.

At the same time, more and more scholars have turned their attention to the research of higher-order singular differential equations.

In 2006, Chu and Zhou [6] studied the existence of positive periodic solutions for the third-order singular differential equation

$$u''' + k^3 u = f\big(t, u(t)\big), 0 \le t \le 2\pi, \tag{1.4}$$

where k is a constant, and $f\big(t,u(t)\big)$ has a singularity at $u = 0$. They proved that equation (1.4) has a positive periodic solution when $k \in (0, \frac{1}{\sqrt{3}})$. In 2010, Li [11] studied the existence of positive periodic solutions of the following third-order singular differential equation

$$u''' = f(t, u(t), u'(t), u''(t)), \tag{1.5}$$

where $f \in C(\mathbb{R} \times (0, +\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$is a periodic function of t and has a singularity at $u = 0$, they proved that equation (1.5) has at least one positive periodic solution by using the point theorem in cones.

In 2003, Conti, Terracini, and Verzini [5] studied the fourth-order equation

$$u^{(4)}(t) - cu''(t) = f(t, u(t)), \quad t \in [0, T],$$

with periodic boundary conditions. Given any positive integer $n \ge 1$, they proved there exists a T -periodic solution to the problem with precisely $2n$ simple zeroes in $[0, T]$ by Nehari's argument of combining variational methods and nodal properties of solutions. In 2009, Cui and Zou [7] studied the following singular fourth-order boundary value problems:

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t), -u''(t)\big), \ \ 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

where $f(t, u, y)$ may be singular at $t = 0, u = 0, y = 0$ and $t = 1, u = 0, y = 0$. They proved the existence and uniqueness of solutions, which are established by constructing a special cone and using the cone compression and expansion fixed-point theorem. In 2015, Cheng and Ren [3] established the existence of positive periodic solutions for the following fourth-order singular nonlinear differential equation with variable-coefficient

$$u^{(4)} + m(t)u''' + n(t)u'' + p(t)u'' + q(t)u = h(t, u) + \varphi(t) \tag{1.6}$$

At the same time, Cheng and Ren [4] calculated Green's functions of the above fourth-order singular nonlinear differential equation in ten cases in detail. In 2017, Xin, Han, and Cheng [15] investigated a class of fourth-order singular nonlinear differential equations with superlinearity or sublinearity assumptions

$$u^{(4)}(t) + au'''(t) + bu''(t) + cu' + du(t) = \lambda g(t)f(u(t)) + \lambda e(t)$$

with $\lambda > 0$ is a positive parameter, and $e(t)$ may take a positive value or a negative value. By employing Green's function in [4] and Krasnoselskii-Guo fixed point theorem, they proved the existence of positive periodic solutions for the singular fourth-order differential equation.

In 2023, Liu et al. [14] investigated the sixth-order singular differential equation

$$u^{(6)}(t) + \sum_{i=0}^{5} a_i(t)u^i\,(t) = f(t, u) + e(t), \tag{1.7}$$

where $a_i(t)(i = 0,1, \ldots ,5)$ are ω-periodic functions,$e(t) \in L^p([0, \omega]), f(t, u) \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R})$ is an L^2-Caratheodory function and $f(t, u)$ is an ω-periodic function in t. The authors calculated the Green functions in 36 cases in detail. They proved the existence of positive solutions to equation (1.7) with attractive singularity or repulsive singularity by using Schauder's fixed-point theorem.

Motivated by [14], in this paper, we employ the result of Green's function of a sixth-order singular differential equation and consider the following sixth-order singular differential equation with indefinite weights

$$u^{(6)}(t) + \sum_{i=0}^{5} a_i(t)u^i\,(t) = \frac{h(t)}{u^\rho} + e(t) \tag{1.8}$$

where $a_i(t) \in C(\mathbb{R}, \mathbb{R}^+)$(i=0,1,…,5) are ω- periodic functions, $h(t), e(t) \in L^1(\mathbb{R}/\omega\mathbb{Z})$ and $e(t) > 0$ for any $t \in [0, \omega]$.

Theorem 1.1. Suppose equations (3.1), (4.2), (4.3), and (4.5) in section 3 hold, then equation (1.8) has at least one positive periodic solution.

The rest of the paper is organized as follows. In Section 2, Green's function and some useful properties for Green's function are given. In Section 3, we prove the existence of positive periodic solutions for equation (1.8) by the Krasnoselskii-Guo fixed point theorem. Another linear Green's function and useful properties are given in the appendix.

2. Green's Function of a Sixth-Order Linear Differential Equation

In this paper, to study the existence of positive periodic solutions for the sixth-order singular differential equation (1.8), we employ the Green's functions and their properties of the equation

$$\begin{cases} u^{(6)}(t) + \sum_{i=0}^{5} a_i(t)u^i\,(t) = \varphi(t), \\ u^{(i)}(0) = u^{(i)}(\omega), i = 0,1,2,3,4,5, \end{cases} \tag{2.1}$$

where $a_i(t) \in C(\mathbb{R}, \mathbb{R}^+)$(i=0,1,…,5), $\varphi(t) \in C(\mathbb{R}, \mathbb{R}^+)$ are ω-periodic functions, which can be found in [14]. The authors reduce and decompose the Green's functions into 36 cases, and only 6 typical cases are analyzed in detail due to the similarities among these cases. We will take Case 1 as a representative example to discuss the positive periodic solutions of equation (1.8), and Cases 2 6 can be obtained similarly, which will be presented in the appendix.

In this section, we only provide detailed properties of Green's function of Case 1 as follows：

Case 1: There exist twoω-periodic differentiable functions $\alpha(t), \beta(t): \mathbb{R} \to \mathbb{R}^+$with $\int_0^\omega \alpha(s)ds > 0$ and $\int_0^\omega \beta(s)ds > 0$, and an ω-periodic function $m(t)$ with $m(t) \succ 0$, such that

$$\begin{aligned}
a_5(t) &= 2\alpha(t),\\
a_4(t) &= m(t) + \alpha^2(t) + 2\alpha'(t) + 2\beta(t),\\
a_3(t) &= 4m'(t) + 2m(t)\alpha(t) + 2\beta(t)\alpha(t) + \beta'(t)\alpha(t) + \alpha''(t) + 2\beta'(t),\\
a_2(t) &= 6m''(t) + 6m'(t)\alpha(t) + m(t)[2\alpha'(t) + 2\beta(t) + \alpha^2(t)] + \beta''(t) + \alpha'(t)\alpha(t) + \beta^2(t),\\
a_1(t) &= 4m'''(t) + 6m''(t)\alpha(t) + 2m'(t)[2\alpha'(t) + 2\beta(t) + \alpha^2(t)] + m(t)[\alpha''(t) + 2\beta'(t) +\\
&\qquad 2\beta(t)\alpha(t)],\\
a_0(t) &= m^{(4)}(t) + m'''(t)[2\alpha(t) + \alpha'(t)\alpha(t)] + m''(t)[2\alpha'(t) + 2\beta(t) + \alpha^2(t)] + m'(t)[2\beta'(t)\\
&\qquad +2\alpha(t)\beta(t) + \alpha''(t) + \alpha'(t)\alpha(t)] + m(t)[\beta''(t) + \alpha(t)\beta'(t) + \beta^2(t)].
\end{aligned}$$

In this case, equation (2.1) can be transformed into

$$\begin{cases} v''(t) + \alpha(t)v'(t) + \beta(t)v(t) = \varphi(t), \\ v(0) = v(\omega), v'(0) = v'(\omega), \end{cases}$$

$$\begin{cases} y''(t) + \alpha(t)y'(t) + \beta(t)y(t) = v(t) \\ y(0) = y(\omega), y'(0) = y'(\omega) \end{cases}$$

and

$$\begin{cases} u''(t) + m(t)u(t) = y(t), \\ u(0) = u(\omega), u'(0) = u'(\omega), \end{cases}$$

Since the solutions of these second-order equations can be respectively expressed as

$$v(t) = \int_0^\omega G_1(t,s)\varphi(s)ds, y(t) = \int_0^\omega G_1(t,s)v(s)ds, u(t) = \int_0^\omega G_2(t,s)y(s)ds,$$

Then the equation (2.1) can be written as

$$\begin{aligned}
u(t) &= \int_0^\omega G_1(t,\gamma_1)\int_0^\omega G_1(\gamma_1,\gamma_2)\int_0^\omega G_1(\gamma_2,s)\varphi(s)dsd\gamma_2\,d\gamma_1\\
&= \int_0^\omega\int_0^\omega\int_0^\omega G_2(t,\gamma_1)\,G_1(\gamma_1,\gamma_2)G_1(\gamma_2,s)\varphi(s)\,dsd\gamma_2 d\gamma_1\\
&= \int_0^\omega\left[\int_0^\omega\int_0^\omega G_2(t,\gamma_1)\,G_1(\gamma_1,\gamma_2)G_1(\gamma_2,s)d\gamma_2 d\gamma_1\right]\varphi(s)\,ds,
\end{aligned}$$

or

$$u(t) = \int_0^\omega G^1(t,s)\varphi(s)ds,$$

where

$$G^1(t,s) = \int_0^\omega\int_0^\omega G_2(t,\gamma_1)G_1(\gamma_1,\gamma_2)G_1(\gamma_2,s)\,d\gamma_2 d\gamma_1.$$

Lemma 2.1.([14]) Suppose that equations (4.2), (4.3), and (4.5) hold, then $G^1(t,s) > 0$ for any $(t,s) \in [0,\omega]\times[0,\omega]$.

3. The Existence of Positive Periodic Solutions of the Equation

In this section, we first give Krasnoselskii-Guo's fixed-point theorem.

Lemma 3.1.([8]) (Krasnoselskii-Guo fixed point theorem) Let X be a Banach space and $K \subset X$ is a cone. Suppose that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$. Let

$$\Psi: K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that one of the following conditions holds:

(i) $\|\Psi u\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|\Psi u\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$;

(ii) $\|\Psi u\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|\Psi u\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$;

Then Ψ has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Now, we consider the existence of positive periodic solutions for equation (1.8) by the Krasnoselskii-Guo fixed point theorem. Due to similarity, we only investigate in detail the existence of Case 1 described in Section 2.

Obviously, $u(t)$ is the unique ω-periodic solution of the equation (2.1) and

$$u(t) = \int_0^\omega G^1(t,s)\varphi(s)ds.$$

Define

$$K := \left\{u \in C_\omega : \min_{t\in\mathbb{R}} u(t) \geq \sigma\|u\|_\infty\right\},$$

where

$$C_\omega := \{u \in C(\mathbb{R}) : u(t+\omega) = u(t), \forall t \in \mathbb{R}\}$$

and

$$\|u\|_\infty = \max_{t\in\mathbb{R}} |u(t)|.$$

Obviously, K is a cone of C_ω.

Besides, for any arbitrarily given ω-periodic function $h(t)$ provided we define

$$h^+(t) := \max\{0, h(t)\},\ h^-(t) := -\min\{0, h(t)\},\quad \bar{h} := \frac{1}{\omega}\int_0^\omega h(t)dt.$$

Theorem 3.1. Suppose that $h(t), e(t) \in L^1(\mathbb{R}/\omega\mathbb{Z})$ and $e(t) > 0$ for any $t \in [0,\omega]$. Assume equations (4.2), (4.3), (4.5) and

$$\overline{h_+} > \frac{1}{A\omega\sigma^{1+\rho}}\left(\frac{\|h^-\|_\infty}{e_*}\right)^{\frac{1+\rho}{\rho}} \tag{3.1}$$

hold, where

$$A := \min_{0\leq s,t\leq\omega} G^1(t,s), B := \max_{0\leq s,t\leq\omega} G^1(t,s),\ \sigma = \frac{A}{B},$$

Then equation (1.8) has one positive ω-periodic solution at least.

Proof: Define

$$(\Psi u)(t) := \int_0^\omega G^1(t,s)\left(\frac{h(s)}{u^\rho} + e(s)\right)ds\,.$$

It is easy to prove that one ω-periodic solution of equation (1.8) is a fixed point of the operator equation $\Psi u = u$. And from Lemma 2.1, we can obtain $G^1(t,s) > 0$ for any $(t,s) \in [0,\omega] \times [0,\omega]$..
Define two open sets

$$\Omega_1 := \{u \in C_\omega : \|u\| < r\}, \Omega_2 := \{u \in C_\omega : \|u\| < R\},$$

where r and R are two positive constants, and from equation (3.1) we obtain

$$R > r = \left(A\omega\overline{h^+}\right)^{\frac{1}{1+\rho}} > \frac{1}{\sigma}\left(\frac{\|h^-\|_\infty}{e_*}\right)^{\frac{1}{\rho}}.$$

First, we prove $\Psi(K\cap(\overline{\Omega_2}\backslash\Omega_1)) \subset K$. Actually, for any $u \in K\cap(\overline{\Omega_2}\backslash\Omega_1)$, we have

$$\sigma r \le u(t) \le R,\ \forall t \in \mathbb{R}\,.$$

Since $r > \frac{1}{\sigma}\left(\frac{\|h^-\|_\infty}{e_*}\right)^{\frac{1}{\rho}}$ we can obtain

$$\begin{aligned}\frac{h(t)}{u^\rho} + e(t) &= \frac{h^+(t)}{u^\rho} - \frac{h^-(t)}{u^\rho} + e(t)\\ &> -\frac{h^-(t)}{u^\rho} + e_*\\ &> -\frac{\|h^-\|_\infty}{(\sigma r)^\rho} + e_*\\ &> 0.\end{aligned} \tag{3.2}$$

From equation (3.2) and $\sigma = \frac{A}{B}$, we get

$$\begin{aligned}\min_{t\in R}(\Psi u)(t) &= \min_{t\in\mathbb{R}}\int_0^\omega G^1(t,s)\left(\frac{h(s)}{u^\rho} + e(s)\right)ds\\ &\ge A\int_0^\omega\left(\frac{h(s)}{u^\rho} + e(s)\right)ds\\ &= \sigma B\int_0^\omega\left(\frac{h(s)}{u^\rho} + e(s)\right)ds\\ &\ge \sigma\max_{t\in\mathbb{R}}\int_0^\omega G^1(t,s)\left(\frac{h(s)}{u^\rho} + e(s)\right)ds\\ &= \sigma\|\Psi u\|_\infty.\end{aligned}$$

Therefore, $\Psi(K\cap(\overline{\Omega_2}\backslash\Omega_1)) \subset K$. And from the Arzela-Ascoli Theorem, it is easy to prove $\Psi(K\cap(\overline{\Omega_2}\Omega_1)) \subset K \to K$ is a compact operator.

Next, we prove

$$\|\Psi u\| \le \|u\|, \quad \forall u \in K \cap \partial\Omega_2\,. \tag{3.3}$$

Actually, for any $u \in K \cap \partial\Omega_2$, we can get

$$\|u\| = R \ \text{ and } \sigma R \le u(t) \le R\ ,\forall t \in \mathbb{R}.$$

From $\sigma = \frac{A}{B}$ and $e(t) > 0$ for any $t \in [0,\omega]$, we obtain

$$\begin{aligned}(\Psi u)(t) &= \int_0^\omega G^1(t,s)\left(\frac{h(s)}{u^\rho} + e(s)\right)ds\\ &= \int_0^\omega G^1(t,s)\left(\frac{h^+(s)}{u^\rho} - \frac{h^-(s)}{u^\rho} + e(s)\right)ds\\ &\le B\omega\left(\frac{\overline{h_+}}{(\sigma R)^\rho} + \bar{e}\right).\end{aligned}$$

Obviously, we can select a suitably large R that satisfies.

$$B\omega\left(\frac{\overline{h_+}}{(\sigma R)^\rho} + \bar{e}\right) \le R\,.$$

Thus, we obtain $\|\Psi u\| \le \|u\|$, which means equation (3.3) holds.

Lastly, we will prove

$$\|\Psi u\| \geq \|u\|,\ \forall u \in K \cap \partial\Omega_1, \tag{3.4}$$

In fact, for any $u \in K \cap \partial\Omega_1$We can obviously obtain

$$\|u\| = r \text{ and } \sigma r \leq u(t) \leq r\ ,\forall t \in \mathbb{R}.$$

From $\sigma = \frac{A}{B}$, we obtain

$$\begin{aligned}(\Psi u)(t) &= \int_0^\omega G^1(t,s)\left(\frac{h^+(s)}{u^\rho} - \frac{h^-(s)}{u^\rho} + e(s)\right)ds\\ &> \int_0^\omega G^1(t,s)\left(\frac{h^+(s)}{u^\rho}\right)ds\\ &\geq \frac{A\omega\overline{h_+}}{r^\rho} = r.\end{aligned}$$

where $r = (A\omega\overline{h_+})^{\frac{1}{1+\rho}}$. Therefore, equation (3.4) holds.

From Lemma 3.1, we can conclude that Q has a fixed point $u \in K\bigcap(\overline{\Omega_2}\backslash\Omega_1)$. Obviously, this point is a positive ω-periodic solution of the equation. (1.8) and $u \in [\sigma r, R]$.

Remark 3.1. The existence of positive periodic solutions for equation (1.8) with indefinite weights for Cases 2 and 6 can be shown in a similar way. More precisely, it can be shown for Case 2 by establishing (4.2), (4.3), and substituting (4.5) with $\rho < \frac{2\pi}{\sqrt{3}\omega}$, for Case 3 by establishing (4.5) and replacing (4.2), (4.3) with $\rho < \frac{2\pi}{\sqrt{3}\omega}$, for Case 4 by establishing (4.2), (4.3), for Case 5 by establishing (4.5), and for Case 6 by replacing (4.2), (4.3), and (4.5) with $\rho < \frac{2\pi}{\sqrt{3}\omega}$.

4. Conclusion

In this paper, we consider a sixth-order variable coefficient singular differential equation with indefinite weights and prove the existence of positive periodic solutions of the equation by the Krasnoselskii-Guo fixed point theorem.

References

[1] Jifeng Chu, "On a Differential Equation Arising in Geophysics," *Monthly Journals for Mathematics*, vol. 187, pp. 499-508, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[2] Alberto Cabada, and José Ángel Cid, "On the Sign of the Green's Function Associated to Hill's Equation with an Indefinite Potential," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 303-308, 2008. [CrossRef] [Google Scholar] [Publisher Link]

[3] Zhibo Cheng, and Jingil Ren, "Positive Solutions for Third-Order Variable-Coefficient Nonlinear Equation with Weak and Strong Singularities," *Journal of Difference Equations and Applications*, vol. 21, no. 11, pp. 1003-1020, 2015. [CrossRef] [Google Scholar] [Publisher Link]

[4] Zhibo Cheng, and Jingil Ren, "Positive Solutions for Fourth-Order Singular Nonlinear Differential Equation with Variable Coefficient," *Mathematical Methods Applied Sciences*, vol. 39, no. 9, pp. 2251-2274, 2016. [CrossRef] [Google Scholar] [Publisher Link]

[5] Monica Conti, Susanna Terracini, and Gianmaria Verzini, "Infinitely Many Solutions to Fourth Order Superlinear Periodic Problems," *Transactions of the American Mathematical Society*, vol. 356, pp. 3283-3300, 2003. [Google Scholar] [Publisher Link]

[6] Jifeng Chu, and Zhongcheng Zhou, "Positive Solutions for Singular Non-Linear Third-Order Periodic Boundary Value Problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 7, pp. 1528-1542, 2006. [CrossRef] [Google Scholar] [Publisher Link]

[7] Yujun Cui, and Yumei Zou, "Existence and Uniqueness Theorems for Fourth-Order Singular Boundary Value Problems," *Computers & Mathematics with Applications*, vol. 58, no. 7, pp. 1449-1456, 2009. [CrossRef] [Google Scholar] [Publisher Link]

[8] Dajun Guo, and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, 2014. [Google Scholar]

[9] Jicai Huang, Shigui Ruan, and Jing Song, "Bifurcations in a Predator-Prey System of Leslie Type with Generalized Holling Type III Functional Response," *Journal of Differential Equations*, vol. 257, no. 6, pp. 1721-1752, 2014. [CrossRef] [Google Scholar] [Publisher Link]

[10] Robert Hakl, and Pedro J. Torres, "On Periodic Solutions of Second-Order Differential Equations with Attractive-Repulsive Singularities," *Journal of Differential Equations*, vol. 248, no. 1, pp. 111-126, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[11] Yongxiang Li, "Positive Periodic Solutions for Fully Third-Order Ordinary Differential Equations," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3464-3471, 2010. [CrossRef] [Google Scholar] [Publisher Link]

[12] Jie Liu, Zhibo Cheng, and Yi Wang, "Positive Periodic Solution or Second-Order Nonlinear Differential Equation with Singularity of Attractive Type," *Journal of Applied Analysis and Computation*, vol. 10, no. 4, pp. 1636-1650, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[13] Shiping Lu, "A New Result on the Existence of Periodic Solutions for Liénard Equations with a Singularity of Repulsive Type," *Journal of Inequalities and Applications*, vol. 37, pp. 1-13, 2017. [CrossRef] [Google Scholar] [Publisher Link]

[14] Jie Liu et al., "Positive Periodic Solutions for a Sixth-Order Variable Coefficient Differential Equation with Repulsive Singularity," *Authorea*, vol. 43, pp. 100-114, 2023. [CrossRef] [Google Scholar] [Publisher Link]

[15] A.C. Lazer, and S. Solimini, "On Periodic Solutions of Nonlinear Differential Equations with Singularities," *Proceedings of the American Mathematical Society*, vol. 99, no. 1, pp. 109-114, 1987. [Google Scholar] [Publisher Link]

[16] F. Merdivenci Atici, and G.Sh. Guseinov, "On the Existence of Positive Solutions for Nonlinear Differential Equations with Periodic Boundary Conditions," *Journal of Computational and Applied Mathematics*, vol. 132, no. 2, pp. 341-356, 2001. [CrossRef] [Google Scholar] [Publisher Link]

[17] Jingxian Sun, and Yansheng Liu, "Multiple Positive Solutions of Singular Third-Order Periodic Boundary Value Problem," *Acta Mathematica Scientia*, vol. 25, no. 1, pp. 81-88, 2005. [CrossRef] [Google Scholar] [Publisher Link]

[18] Pedro J. Torres, "Existence of One Signed Periodic Solutions of Some Second-Order Differential Equations via a Krasnoselskii's-Guo Fixed Point Theorem," *Journal of Differential Equations*, vol. 190, no. 2, pp. 643-662, 2003. [CrossRef] [Google Scholar] [Publisher Link]

[19] Pedro J. Torres, *Mathematical Models with Singularities-A Zoo of Singular Creatures*, Atlantis Briefs in Differential Equations, Atlantis Press, Paris, 2015. [Google Scholar]

[20] Youyu Wang, Hairong Lian, and Weigao Ge, "Periodic Solutions for a Second Order Nonlinear Functional Differential Equation," *Applied Mathematics Letters*, vol. 20, no. 1, pp. 110-115, 2007. [CrossRef] [Google Scholar] [Publisher Link]

[21] Yun Xin, Xuefeng Han, and Zhibo Cheng, "Multiplicity Results of Fourth-Order Singular Nonlinear Differential Equation with a Parameter," *Journal of Applied Analysis and Computation*, vol. 7, no. 2, pp. 455-477, 2017. [CrossRef] [Google Scholar] [Publisher Links]

Appendix

1. Auxiliary Equation

The following are four linear differential equations and the properties of their corresponding Green's functions.

Equation I

$$\begin{cases} u''(t) + \alpha(t)u'(t) + \beta(t)u(t) = \varphi(t), \\ u(0) = u(\omega), u'(0) = u'(\omega), \end{cases} \tag{4.1}$$

where $\alpha(t), \beta(t) \in C(\mathbb{R}, \mathbb{R}^+)$ are ω- periodic functions. Its solution can be expressed by

$$u(t) = \int_0^\omega G^1(t,s)\varphi(s)ds,$$

where $G^1(t,s)$ is the Green function of equation (4.1).

Lemma 4.1. ([20]) Suppose$f(t) \in X$ and

$$\frac{H\left[\exp\left(\int_0^\omega \alpha(x)dx\right) - 1\right]}{Q\omega} \geq 1, \tag{4.2}$$

where

$$H = \max_{t\in[0,\omega]} \left| \int_0^\omega \frac{\exp\left(\int_t^s \alpha(x)dx\right)}{\exp\left(\int_0^\omega \alpha(x)dx\right) - 1} \beta(s)ds \right| \quad , \quad Q = \left[1 + \exp\left(\int_0^\omega \alpha(x)dx\right)\right]^2 P^2,$$

Then the equation

$$u''(t) + \alpha(t)u'(t) + \beta(t)u(t) = \varphi(t)$$

has an ω-periodic solution, and its periodic solution can be expressed by

$$u(t)=\int_0^{\omega} G_1(t,s)\varphi(s)ds,$$

where

$$G^1(t,s)=\frac{\int_t^s \exp\left[\int_t^x \beta(v)dv+\int_x^s \alpha(v)dv\right]dx+\int_s^{t+\omega} \exp\left[\int_t^x \beta(v)dv+\int_x^{s+\omega} \alpha(v)dv\right]dx}{\left[\exp\left(\int_0^{\omega}\alpha(x)dx\right)-1\right]\left[\exp\left(\int_0^{\omega}\beta(x)dx\right)-1\right]}.$$

Lemma 4.2. ([20]) Let

$$A=\int_0^{\omega}\alpha(x)dx),B=\omega^2\exp\left(\frac{1}{\omega}\int_0^{\omega}\ln\beta(x)\,dx\right).$$

If

$$A^2\geq 4B, \tag{4.3}$$

then

$$\min\left\{\int_0^{\omega}\mu(x)dx\,,\int_0^{\omega}v(x)dx\right\}\geq\frac{1}{2}\left(A-\sqrt{A^2-4B}\right):=m_1,$$
$$\max\left\{\int_0^{\omega}\mu(x)dx\,,\int_0^{\omega}v(x)dx\right\}\leq\frac{1}{2}\left(A+\sqrt{A^2-4B}\right):=n_1.$$

Lemma 4.3. ([20]) If equations (4.2) and (4.3) hold, then

$$0<\frac{\omega}{(e^{n_1}-1)^2}\leq G^1(t,s)\leq\frac{\omega\exp\left(\int_0^{\omega}\alpha(x)dx\right)}{(e^{m_1}-1)^2}.$$

Equation II

$$\begin{cases} u''(t)+m(t)u(t)=\varphi(t),\\ u(0)=u(\omega),u'(0)=u'(\omega),\end{cases} \tag{4.4}$$

where $m(t)\in C(\mathbb{R},\mathbb{R}^+)$ is an ω-periodic function. Then, the solution of equation (4.4) can be expressed as

$$u(t)=\int_0^{\omega} G_2(t,s)\varphi(s)ds,$$

where the Green function $G_2(t,s)$ is defined in [17, Theorem 2.4].
Define

$$k(\beta)=\begin{cases}\dfrac{2\pi}{\beta\omega^{1+2/\beta}}\left(\dfrac{2}{2+\beta}\right)^{1-2/\beta}\left(\dfrac{\Gamma(\frac{1}{\beta})}{\Gamma(\frac{1}{2}+\frac{1}{\beta})}\right)^2, & 1\leq\beta<+\infty,\\ \dfrac{4}{\beta}, & \beta=\infty,\end{cases}$$

where $\Gamma(t)=\int_0^{+\infty}u^{t-1}e^{-u}du$.
Lemma 4.4. ([18, Corollary 2.3] and [2, Theorem 3.2]) Assume $m(t)\succ 0$ and $m(t)\in L^{\alpha}(0,\omega)$ for some $1\leq a\leq\infty$. If

$$\|m\|:=\left(\int_0^{\omega}|m(t)|^{\alpha}dt\right)^{\frac{1}{\alpha}}<k(2\alpha^*), \tag{4.5}$$

where

$$\alpha^{*}=\begin{cases}\frac{\alpha}{\alpha-1}, & 1\le\alpha<+\infty,\\ 1, & \alpha=+\infty,\end{cases}$$

then $G_2(t,s)>0$ for all $(t,s)\in[0,\omega]\times[0,\omega]$.

Equation III

$$\begin{cases}u''(t)-\rho u(t)+\rho^2u(t)=\varphi(t),\\ u(0)=u(\omega),u'(0)=u'(\omega),\end{cases}\tag{4.6}$$

where ρ is a constant. Obviously, the solution of equation (4.6) can be expressed as

$$u(t)=\int_0^{\omega}G_3(t,s)\varphi(s)ds,$$

where the Green's function

$$G_3(t,s)=\begin{cases}\dfrac{2e^{\frac{\rho}{2}}(t-s)\left[\sin\frac{\sqrt{3}}{2}\rho(\omega-t+s)+e^{-\frac{\rho\omega}{2}}\sin\frac{\sqrt{3}}{2}\rho(t-s)\right]}{\sqrt{3}\rho(e^{\frac{\rho\omega}{2}}+e^{-\frac{\rho\omega}{3}}-2\cos\frac{\sqrt{3}}{2}\rho\omega)}, 0\le s\le t\le\omega,\\ \dfrac{2e^{\frac{\rho}{2}}(\omega+t-s)\left[\sin\frac{\sqrt{3}}{2}\rho(s-t)+e^{-\frac{\rho\omega}{2}}\sin\frac{\sqrt{3}}{2}\rho(\omega-s+t)\right]}{\sqrt{3}\rho(e^{\frac{\rho\omega}{2}}+e^{-\frac{\rho\omega}{3}}-2\cos\frac{\sqrt{3}}{2}\rho\omega)}, 0\le s\le t\le\omega,\end{cases}$$

Lemma 4.5. ([17]) Assume $\rho<\frac{2\pi}{\sqrt{3}\omega}$ holds, then

$$0<m_2:=\frac{2\sin\left(\frac{\sqrt{3}}{2}\rho\omega\right)}{\sqrt{3}\rho\left(e^{\frac{\rho\omega}{2}}+1\right)^2}\le G_3(t,s)\le\frac{2}{\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\rho\omega\right)}:=n_2,\forall s,t\in[0,\omega].$$

Equation IV

$$\begin{cases}u'(t)+\rho u(t)=\varphi(t),\\ u(0)=u(\omega),\end{cases}\tag{4.7}$$

where ρ is a constant. Similarly, the solution of equation (4.7) can be expressed as

$$u(t)=\int_0^{\omega}G_4(t,s)\varphi(s)ds,$$

Where

$$G_4(t,s)=\begin{cases}\dfrac{e^{-\rho(t-s)}}{1-e^{-\omega\rho}}, & 0\le s\le t\le\omega,\\ \dfrac{e^{-\rho(\omega+t-s)}}{1-e^{-\omega\rho}}, & 0\le t\le s\le\omega.\end{cases}$$

2. Regularity of the Green function of equation (2.1)

In this section, Cases 2 6 will be presented.

Case 2 There exist twoω-periodic differentiable functions $p(t), q(t): R \to R^+$ with $\int_0^\omega \alpha(s)ds > 0$ and $\int_0^\omega \beta(s)ds > 0$, and a positive real constant ρ, such that

$$\begin{aligned}
&a_5(t) = \alpha(t) - 2\rho,\\
&a_4(t) = 4\alpha'(t) + \beta(t) - 2\rho\beta(t) + \rho^2,\\
&a_3(t) = 5\alpha''(t) + 4\beta'(t) - 6\rho\alpha'(t) - 2\rho\beta(t) + 3\rho^2\alpha(t) - 2\rho^3(t),\\
&a_2(t) = 6m''(t) + 6m'(t)\alpha(t) + m(t)[2\alpha'(t) + 2\beta(t) + \alpha^2(t)] + \beta''(t) + \beta'(t)\alpha(t) + \beta^2(t),\\
&a_1(t) = 4m''(t) + 6m''(t)\alpha(t) + 2m'(t)[2\alpha'(t) + 2\beta(t) + \alpha^2(t)] + m(t)[\alpha''(t) + 2\beta'(t) + 2\beta(t)\alpha(t)],\\
&a_0(t) = m^{(4)}(t) + m'''(t)[2\alpha(t) + \alpha'(t)\alpha(t)] + m''(t)[2\alpha'(t) + 2\beta(t) + \alpha^2(t)] + m'(t)[2\beta'(t) + 2\alpha(t)\beta(t) +\\
&\qquad \alpha''(t) + \alpha'(t)\alpha(t)] + m(t)[\beta''(t) + \alpha(t)\beta'(t) + \beta^2(t)].
\end{aligned}$$

In this case, equation (2.1) can be transformed into

$$\begin{cases} v''(t) - \rho v'(t) + \rho v(t) = \varphi(t),\\ v(0) = v(\omega), v'(0) = v'(\omega),\end{cases}$$

$$\begin{cases} y''(t) - \rho y'(t) + \rho^2 y(t) = v(t),\\ y(0) = y(\omega), y'(0) = y'(\omega),\end{cases}$$

and

$$\begin{cases} u''(t) + a(t)u'(t) + \beta(t)u(t) = y(t),\\ u(0) = u(\omega), u'(0) = u'(\omega).\end{cases}$$

And the solutions of these second-order equations can be respectively expressed as

$$v(t) = \int_0^\omega G_3(t,s)\varphi(s)ds,\ y(t) = \int_0^\omega G_3(t,s)v(s)ds,\ u(t) = \int_0^\omega G_1(t,s)y(s)ds.$$

Similar to Case 1, the solutions of equation (2.1) can be written as

$$u(t) = \int_0^\omega G^2(t,s)\varphi(s)ds,$$

where

$$G^2(t,s) = \int_0^\omega \int_0^\omega G_3(t,\gamma_1)\, G_3(\gamma_1,\gamma_2) G_1(\gamma_2,s) d\gamma_2 d\gamma_1.$$

Lemma 4.6.([14]) Suppose that equations (4.2), (4.3), and $\rho < \frac{2\pi}{\sqrt{3}\omega}$ hold, then $G^2(t,s) > 0$ for any $(t,s) \in [0,\omega] \times [0,\omega]$.

Case 3 There exists a positive constant ρ and an ω-periodic differentiable function $m(t) \succ 0$ such that

$$\begin{aligned}
&a_5(t) = -2\rho,\\
&a_4(t) = m(t) + 3\rho^2,\\
&a_3(t) = 4m'(t) - 2\rho m(t) - 2\rho^3,\\
&a_2(t) = 6m''(t) - 6\rho m'(t) + 3\rho^2 m(t) + \rho^4,\\
&a_1(t) = 4m'''(t) - 6\rho m''(t) + 6\rho^2 m'(t) - 2\rho^3 m(t),\\
&a_0(t) = m^{(4)}(t) - 2\rho m'''(t) + 3\rho^2 m''(t) - 2\rho^3 m'(t) + \rho^4 m(t).
\end{aligned}$$

In this case, equation (2.1) can be transformed into

$$\begin{cases} v''(t) - \rho v'(t) + \rho^2 v(t) = \varphi(t),\\ v(0) = v(\omega), v'(0) = v'(\omega),\end{cases}$$

$$\begin{cases} y''(t) - \rho y'(t) + \rho^2 y(t) = v(t),\\ y(0) = y(\omega), y'(0) = y'(\omega),\end{cases}$$

and

$$\begin{cases} u''(t) + m(t)u(t) = y(t), \\ u(0) = u(\omega), u'(0) = u'(\omega). \end{cases}$$

And the solutions of these second-order equations can be respectively expressed as

$$v(t) = \int_0^\omega G_3(t,s)\varphi(s)ds, y(t) = \int_0^\omega G_3(t,s)v(s)ds, u(t) = \int_0^\omega G_2(t,s)y(s)ds.$$

Similar to Case 1, the solutions of equation (2.1) can be written as

$$u(t) = \int_0^\omega G^3(t,s)\varphi(s)ds,$$

where

$$G^3(t,s) = \int_0^\omega \int_0^\omega G_3(t,\gamma_1)\, G_3(\gamma_1,\gamma_2) G_2(\gamma_2,s) d\gamma_2 d\gamma_1.$$

Lemma 4.7.([14]) Suppose that equation (4.5) and $\rho < \frac{2\pi}{\sqrt{3}\omega}$ hold, then $G^3(t,s) > 0$ for any $(t,s) \in [0,\omega] \times [0,\omega]$.

Case 4 There exist twoω-periodic differentiable functions$\alpha(t), \beta(t): R \to R^+$ with $\int_0^\omega \alpha(s)ds > 0$ and $\int_0^\omega \beta(s)ds > 0$, and a positive real constant ρ, such that

$$\begin{aligned} a_5(t) &= 2\alpha(t) + 2\rho, \\ a_4(t) &= 4\alpha'(t) + 2\beta(t) + 4\rho\alpha(t) + \alpha^2(t) + \rho^2(t), \\ a_3(t) &= 6\alpha''(t) + 6\rho\alpha'(t) + \alpha(t)[2\rho^2 + 3\alpha'(t) + 2\rho\alpha(t) + 2\beta(t)] + 4\rho\alpha(t) + 4\beta'(t), \\ a_2(t) &= 4\alpha'''(t) + 6\beta''(t) + 2\rho^2\alpha'(t) + 6\alpha\beta'(t) + 6\alpha(t)\alpha''(t) + \rho^2\beta(t) + \beta(t)[2\alpha'(t) + \beta(t) \\ &\quad +2\rho\alpha(t) + \rho^2] + \alpha(t)[3\alpha''(t) + 3\beta'(t) + 4\rho\alpha'(t) + 2\rho\beta(t) + \rho^2\alpha(t)], \\ a_1(t) &= \rho^2\alpha''(t) + 2\rho^2\beta'(t) + \alpha(t)[3\beta''(t) + 4\rho\beta'(t) + \alpha'''(t) + 2\rho\alpha''(t) + \rho^2\alpha'(t) + \rho^2\beta(t)] \\ &\quad +\beta(t)[\alpha''(t) + 2\rho^2\alpha'(t) + 2\rho\alpha(t) + 2\beta'(t) + \rho^2\beta(t)] + 4\beta'''(t) \\ &\quad +6\alpha\beta''(t) + 2\rho\alpha''(t) + \alpha^{(4)}(t), \\ a_0(t) &= \beta^{(4)}(t) + 2\rho\beta'''(t) + \alpha^2\beta''(t) + \alpha(t)[\beta''(t) + 2\rho\beta''(t) + \rho^2\beta'(t)] + \beta(t)[\beta''(t) \\ &\quad +2\rho\beta''(t) + \rho^2\beta(t)]. \end{aligned}$$

In this case, equation (2.1) can be transformed into

$$\begin{cases} x''(t) + \alpha(t)x'(t) + p(t)x(t) = \varphi(t), \\ \quad x(0) = x(\omega), x'(0) = x'(\omega), \end{cases} \tag{4.8}$$

$$\begin{cases} v'(t) + \rho v(t) = u(t), \\ \quad v(0) = v(\omega), \end{cases} \tag{4.9}$$

$$\begin{cases} y'(t) + \rho y(t) = v(t), \\ \quad y(0) = y(\omega), \end{cases} \tag{4.10}$$

and

$$\begin{cases} u''(t) + \alpha(t)u'(t) + \beta(t)u(t) = y(t), \\ \quad u(0) = u(\omega), u'(0) = u'(\omega). \end{cases} \tag{4.11}$$

Since these solutions of equations (4.8) – (4.11) can be respectively expressed as

$$x(t) = \int_0^\omega G_1(t,s)\varphi(s)ds,\ v(t) = \int_0^\omega G_4(t,s)x(s)ds,$$
$$y(t) = \int_0^\omega G_4(t,s)v(s)ds,\ u(t) = \int_0^\omega G_1(t,s)y(s)ds$$

Similar to Case 1, the solutions of equation (2.1) can be written as

$$u(t) = \int_0^\omega G^4(t,s)\varphi(s)ds,$$

where

$$G^4(t,s) = \int_0^\omega \int_0^\omega \int_0^\omega G_1(t,\gamma_3)\, G_4(\gamma_3,\gamma_2) G_4(\gamma_2,\gamma_1) G_1(\gamma_1,s)\, d\gamma_3 d\gamma_2 d\gamma_1.$$

Lemma 4.8.([14]) Suppose that equations (4.2) and (4.3) hold, then $G^4(t,s) > 0$ for any $(t,s) \in [0,\omega] \times [0,\omega]$.
Case 5 There exists an ω-periodic differentiable function $m(t) \succ 0$ and a positive real constant ρ such that

$a_5(t) = 2\rho,$
$a_4(t) = 2m(t) + \rho^2,$
$a_3(t) = 4m'(t) + 4\rho m(t),$
$a_2(t) = 6m''(t) + 6\rho m'(t) + 2\rho^2 m(t) + m^2(t),$
$a_1(t) = 4m'''(t) + 6\rho m'(t) + 2\rho^2 m'(t) + 2m(t)m'(t) + 2\rho m(t),$
$a_0(t) = m^{(4)}(t) + 2\rho m'''(t) + \rho^2 m''(t) + 2\rho m(t)m'(t) + \rho^2 m^2(t) + m(t)m''(t).$

In this case, equation (2.1) can be transformed into

$$\begin{cases} x''(t) + m(t)x(t) = \varphi(t), \\ x(0) = x(\omega), x'(0) = x'(\omega), \end{cases} \tag{4.12}$$

$$\begin{cases} v'(t) + \rho v(t) = u(t), \\ v(0) = v(\omega), \end{cases} \tag{4.13}$$

$$\begin{cases} y'(t) + \rho y(t) = v(t), \\ y(0) = y(\omega), \end{cases} \tag{4.14}$$

and

$$\begin{cases} u''(t) + m(t)u(t) = y(t), \\ u(0) = u(\omega), u'(0) = u'(\omega). \end{cases} \tag{4.15}$$

Since these solutions of equations (4.12) – (4.15) can be respectively expressed as

$$x(t) = \int_0^\omega G_2(t,s)\varphi(s)ds,\ v(t) = \int_0^\omega G_4(t,s)u(s)ds,$$
$$y(t) = \int_0^\omega G_4(t,s)v(s)ds,\ u(t) = \int_0^\omega G_2(t,s)y(s)ds.$$

Similar to Case 1, the solutions of equation (2.1) can be written as

$$u(t) = \int_0^\omega G^5(t,s)\varphi(s)ds,$$

where

$$G^5(t,s) = \int_0^\omega \int_0^\omega \int_0^\omega G_2(t,\gamma_3)\, G_4(\gamma_3,\gamma_2) G_4(\gamma_2,\gamma_1) G_2(\gamma_1,s)\, d\gamma_3 d\gamma_2 d\gamma_1.$$

Lemma 4.9.([14]) Suppose that equation (4.5) holds, then $G^5(t,s) > 0$for any$(t,s) \in [0,\omega] \times [0,\omega]$.
Case 6 There exists a positive real constant ρ such that

$$a_5(t) = 0, a_4(t) = 0, a_3(t) = 2\rho^3, a_2(t) = 0, a_1(t) = 0, a_0(t) = \rho^6.$$

In this case, equation (2.1) can be expressed as

$$\begin{cases} x''(t) - \rho x'(t) + \rho^2 x(t) = \varphi(t), \\ x(0) = x(\omega), x'(0) = x'(\omega), \end{cases} \tag{4.16}$$

$$\begin{cases} v'(t) + \rho v(t) = u(t), \\ v(0) = v(\omega), \end{cases} \tag{4.17}$$

$$\begin{cases} y'(t) + \rho y(t) = v(t), \\ y(0) = y(\omega), \end{cases} \tag{4.18}$$

and

$$\begin{cases} u''(t) - \rho u'(t) + \rho u^2(t) = y(t), \\ u(0) = u(\omega), u'(0) = u'(\omega). \end{cases} \tag{4.19}$$

Since these solutions of equations (4.16) – (4.19) can be respectively expressed as

$$x(t) = \int_0^\omega G_3(t,s)\varphi(s)ds, v(t) = \int_0^\omega G_4(t,s)u(s)ds,$$
$$y(t) = \int_0^\omega G_4(t,s)v(s)ds,\ u(t) = \int_0^\omega G_3(t,s)y(s)ds.$$

Similar to Case 1, the solutions of equation (2.1) can be represented in the form.

$$u(t) = \int_0^\omega G^6(t,s)\varphi(s)ds,$$

where

$$G^5(t,s) = \int_0^\omega \int_0^\omega \int_0^\omega G_3(t,\gamma_3)\, G_4(\gamma_3,\gamma_2) G_4(\gamma_2,\gamma_1) G_3(\gamma_1,s)\, d\gamma_3 d\gamma_2 d\gamma_1.$$

Lemma 4.10.([14]) Suppose that $\rho < \frac{2\pi}{\sqrt{3}\omega}$ holds, then $G^6(t,s) > 0$for any $(t,s) \in [0,\omega] \times [0,\omega]$.