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Original Article

Adomian Method for Solving Fractional Sturm-Liouville Problem with Eigenparameter-Dependent Boundary Conditions

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Abstract - *This research focused on computing eigenvalues and eigenfunctions of a class of fractional Sturm-Liouville boundary value problems with eigenparameter-dependent boundary conditions using the Adomian Decomposition Method (ADM). Fractional differential equations provide an effective mathematical tool for modeling and simulating non-classical dynamic phenomena in physics, engineering, and applied sciences. The main novelty of this work lies in the use of the Adomian Decomposition Method for computing both eigenvalues and eigenfunctions of fractional Sturm-Liouville boundary value problems, where the forcing term is an arbitrary function of x and y. The results indicate that the suggested method provides a simple and efficient alternative for the spectral analysis of fractional differential operators. At the end of the paper, the representative example is presented to demonstrate the applicability of the method.*

Keywords - *Adomian Decomposition Method, Eigenfunctions, Eigenvalues, Fractional differential equations, Sturm-Liouville problem.*

1. Introduction

Sturm-Liouville problem is one of the comprehensive fields of study in applied mathematics and mathematical physics. Determining the eigenvalues and eigenfunctions of boundary value problems is a matter of greatest importance in quantum mechanics, heat conduction, and vibration theory. Advances in the theory of fractional calculus have taken the Sturm-Liouville problems to a new type of problems, so-called fractional Sturm–Liouville problems. In general, the fractional Sturm-Liouville problems can be thought of as replacing the derivatives in Sturm-Liouville problems with fractional derivatives [1].

The Adomian Decomposition Method (ADM) is one of the effective and reliable approaches among the various techniques developed to solve linear and nonlinear differential equations. The most important feature of this method, first introduced by Adomian, is its simplicity of application. The method allows solutions to be expressed as rapidly convergent series without requiring linearization or discretization, making it particularly suitable for nonlinear and singular problems [2-4].

Numerous linear and nonlinear ordinary differential equations with singular initial and boundary value problems have been solved using ADM [5-7]. In particular, several studies have demonstrated its applicability to Sturm-Liouville-type problems, both in classical and fractional settings, yielding accurate eigenvalues and eigenfunctions with relatively low computational cost [8,10].

Motivated by these developments, the present study focuses on the application of the ADM to a class of fractional Sturm-Liouville problems with eigenparameter-dependent boundary conditions. The proposed approach provides the usage of the ADM to approximate the eigenvalues and the corresponding eigenfunctions of the problem. The results indicate that the method is not only simple to implement but also stable and convergent.

Before addressing the inverse of operator L, a brief recall of essential concepts from fractional calculus that are employed throughout this study will be given. In particular, first, the definitions of the Riemann–Liouville and Caputo fractional derivatives play a fundamental role in the subsequent analysis. are briefly reviewed.

2. Fractional Calculus

Definition 1.1. The fractional differential operator of order α is defined by

$$D_{0^+}^{\alpha} y(x) = \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-t)^{k-\alpha-1}\, y^{(k)}(t)dt \tag{2.1}$$

where $\alpha \in \mathbb{R}^+$, $k \in \mathbb{N}$ and satisfies the relation $k-1 < \alpha < k$.

Definition 1.2. The Riemann-Liouville fractional integral operator of order α is defined by

$$J_a^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, y(t)dt \tag{2.2}$$

where $y \in L_1[a,b]$ and $\alpha \in \mathbb{R}^+$.

Lemma 1. For $k \in \mathbb{N}, \alpha \in \mathbb{R}^+$, if $k-1 < \alpha < k$, and $y \in L_1[a,b]$ then

$$D_a^{\alpha} J_a^{\alpha} y(x) = y(x)$$

and

$$J_a^{\alpha} D_a^{\alpha} y(x) = y(x) - \sum_{m=0}^{k-1} y^{(m)}(c^0) \frac{x^m}{m!}$$

where $b > a \geq 0$ and $x > 0$.

3. Adomian Decomposition Approach to Fractional Sturm-Liouville Problem

This section contains the derivation of an algorithm for solving fractional Sturm–Liouville two-point boundary value problems. Consider the eigenvalue problem.

$$D^{\alpha}[p(x)y'(x)] + \lambda q(x)y(x) = F(x,y), \quad x \in (0,1) \tag{3.1}$$

$$(\alpha_1\lambda + \beta_1)y(0) + (\alpha_2\lambda + \beta_2)y'(0) = 0 \tag{3.2}$$

$$(\alpha_3\lambda + \beta_3)y(1) + (\alpha_4\lambda + \beta_4)y'(1) = 0 \tag{3.3}$$

where $0 < \alpha \leq 1$. In [8], Al-Mdallal et al. studied the homogeneous form of equation (3.1), and the special form of this equation is considered in [11]. In their paper, they used the regular boundary conditions. However, they did not consider the non-homogeneous equation with eigenparameter-dependent boundary conditions in their studies. By following the standard formulation of the Adomian decomposition method in operator form, equation (3.1) can be expressed in the general form.

$$Ly(x) = N \tag{3.4}$$

where $L = L(x, D^{\alpha}, D^1)$ is the highest order derivative term, and it is an invertible operator, and $N = N(\lambda, x, y, y')$ is a non-linear operator that contains all other terms. Here D^1 represents the ordinary (first-order) derivative. The operator L is chosen to be the simplest operator that is easily invertible; consequently,

$$L = L(x, D^{\alpha}, D^1) = D^{\alpha} p D^1 \tag{3.5}$$

As a consequence of Lemma 1 and the appropriate choice of the operator L, which is crucial for the effectiveness of the method, the inverse operator $L^{-1}$is obtained in the following form:

$$L^{-1} = \int_0^x \frac{1}{p(t)} J_0^{\alpha}(.)dt$$

When the inverse operator L^{-1} is applied to both sides of equation (3.4), and the boundary condition (3.2) is used, the left-hand side transforms into the following form

$$L^{-1}L\,y(x)=\int_0^x \frac{1}{p(t)}J_0^{\alpha}D^{\alpha}pD^1y(t)dt=\int_0^x \frac{1}{p(t)}\big(p(t)y'(t)-p(0)y'(0)\big)dt$$

$$=y(x)-y(0)+p(0)\left(\frac{\alpha_1\lambda+\beta_1}{\alpha_2\lambda+\beta_2}\right)y(0)\int_0^x \frac{1}{p(t)}dt$$

Hence, equation (3.4) takes the following form.

$$y(x)\ =y(0)-p(0)\left(\frac{\alpha_1\lambda+\beta_1}{\alpha_2\lambda+\beta_2}\right)y(0)\int_0^x \frac{1}{p(t)}dt+L^{-1}N(\lambda,x,y,y')$$

$$=y(0)-p(0)\left(\frac{\alpha_1\lambda+\beta_1}{\alpha_2\lambda+\beta_2}\right)y(0)\int_0^x \frac{1}{p(t)}dt+L^{-1}F(x,y)-L^{-1}\big(\lambda q(x)y(x)\big). \tag{3.6}$$

In the Adomian decomposition method, the solution of equation (3.1) is expressed as an infinite series of component functions.

$$y(x)=\sum_{n=0}^{\infty}y_n(x) \tag{3.7}$$

Moreover, the nonlinear operator N is decomposed into an infinite series of Adomian polynomials.

$$N=\sum_{n=0}^{\infty}A_n \tag{3.8}$$

where A_n denote the Adomian polynomials given by

$$A_n=\frac{1}{n!}\frac{d^n}{d\mu^n}\left[N\left(\sum_{i=0}^{\infty}\mu^i y_i\right)\right]_{\mu=0}. \tag{3.9}$$

Combining the equalities (3.6) and (3.8) and the linearity of the operator L^{-1} yield

$$\sum_{n=0}^{\infty}y_n(x)=\ y(0)-p(0)\left(\frac{\alpha_1\lambda+\beta_1}{\alpha_2\lambda+\beta_2}\right)y(0)\int_0^x \frac{1}{p(t)}dt+\sum_{n=0}^{\infty}L^{-1}A_n(x)-L^{-1}\left(\lambda q(x)\sum_{n=0}^{\infty}y_n(x)\right). \tag{3.10}$$

Therefore, the following recursive relation is obtained.

$$\begin{aligned} y_0(x)&=\ y(0)-p(0)\left(\tfrac{\alpha_1\lambda+\beta_1}{\alpha_2\lambda+\beta_2}\right)y(0)\int_0^x \tfrac{1}{p(t)}dt \\ y_{n+1}(x)&=\ \textstyle\sum_{n=0}^{\infty}L^{-1}A_n(x)-L^{-1}(\lambda q(x)\sum_{n=0}^{\infty}y_n(x)),\quad n\geq 0 \end{aligned} \tag{3.11}$$

where $A_n(x)=F\big(x,y_n(x)\big)$.

4. Application of Method

Consider the fractional eigenvalue problem.

$$D_{0^+}^{\frac{2}{3}}[y'(x)]+\lambda xy(x)=xy(x),\qquad x\in(0,1) \tag{4.1}$$

with the boundary conditions

$$(\lambda+1)y(0)+\lambda y'(0)=0 \tag{4.2}$$

$$3y(1)-2y'(1)=0\,. \tag{4.3}$$

Equation (4.1) can be rewritten using operator notation as

$$Ly(x)=xy(x)-\lambda xy(x) \tag{4.4}$$

When the inverse transform L^{-1} is applied to (4.4), and the recursive relation is used

$$y(x) = y(0) - \left(\frac{\lambda+1}{\lambda}\right) y(0) \int_0^x dt + \int_0^x J_{0^+}^{\frac{2}{3}} \big(ty(t) - \lambda t y(t)\big) dt$$

$$= y(0)\left[1 - \left(\frac{\lambda+1}{\lambda}\right) x\right] + \frac{(1-\lambda)}{\Gamma\left(\frac{2}{3}\right)} \int_0^x \int_0^s (s-t)^{-\frac{1}{3}} t y(t) dt ds\,.$$

Using the equations (3.11)

$$y_0(x) = y(0)\left[1 - \left(\frac{\lambda+1}{\lambda}\right) x\right]$$

$$y_1(x) = \frac{(1-\lambda) y(0)}{\Gamma\left(\frac{2}{3}\right)} \int_0^x \int_0^s (s-t)^{-\frac{1}{3}} t dt ds - \left(\frac{\lambda+1}{\lambda}\right) \int_0^x \int_0^s (s-t)^{-\frac{1}{3}} t^2 dt ds.$$

The substitution $t = s - su$ gives

$$y_1(x) = \frac{(1-\lambda) y(0)}{\Gamma\left(\frac{2}{3}\right)} \left[\int_0^x s^{\frac{5}{3}} \beta\left(\frac{2}{3}, 2\right) ds - \left(\frac{\lambda+1}{\lambda}\right) \int_0^x s^{\frac{8}{3}} \beta\left(\frac{2}{3}, 3\right) ds\right]$$

and then

$$y_1(x) = (1-\lambda) y(0)\left[\frac{3}{8} \frac{\Gamma(2)}{\Gamma\left(\frac{8}{3}\right)} x^{\frac{8}{3}} - \left(\frac{\lambda+1}{\lambda}\right) \frac{3}{11} \frac{\Gamma(3)}{\Gamma\left(\frac{11}{3}\right)} x^{\frac{11}{3}}\right].$$

Here, the relation between Beta and Gamma functions is used. If the same operations are continued

$$y_2(x) = (1-\lambda)^2 y(0)\left[\frac{3^2\, \Gamma\left(\frac{6}{3}\right) \Gamma\left(\frac{14}{3}\right)}{8 \cdot 16 \cdot \Gamma\left(\frac{8}{3}\right) \Gamma\left(\frac{16}{3}\right)} x^{\frac{16}{3}} - \left(\frac{\lambda+1}{\lambda}\right) \frac{3^2\, \Gamma\left(\frac{9}{3}\right) \Gamma\left(\frac{17}{3}\right)}{11 \cdot 19 \cdot \Gamma\left(\frac{11}{3}\right) \Gamma\left(\frac{19}{3}\right)} x^{\frac{19}{3}}\right]$$

and

$$y_3(x) = (1-\lambda)^3 y(0)\left[\frac{3^3\, \Gamma\left(\frac{6}{3}\right) \Gamma\left(\frac{14}{3}\right) \Gamma\left(\frac{22}{3}\right)}{8 \cdot 16 \cdot 24\ \cdot \Gamma\left(\frac{8}{3}\right) \Gamma\left(\frac{16}{3}\right) \Gamma\left(\frac{24}{3}\right)} x^{\frac{24}{3}} - \left(\frac{\lambda+1}{\lambda}\right) \frac{3^3\, \Gamma\left(\frac{9}{3}\right) \Gamma\left(\frac{17}{3}\right) \Gamma\left(\frac{25}{3}\right)}{11 \cdot 19 \cdot 27 \cdot \Gamma\left(\frac{11}{3}\right) \Gamma\left(\frac{19}{3}\right) \Gamma\left(\frac{27}{3}\right)} x^{\frac{27}{3}}\right].$$

A simple induction argument shows that, in general, for $n = k+1$, it is natural to write.

$$y_{k+1}(x) = (1-\lambda)^{k+1} y(0)\left[\frac{3^{k+1}\, \Gamma\left(\frac{6}{3}\right) \Gamma\left(\frac{14}{3}\right) \Gamma\left(\frac{22}{3}\right) \cdots \Gamma\left(\frac{6+8k}{3}\right)}{8 \cdot 16 \cdot 24\ \cdots (8+8k) \cdot \Gamma\left(\frac{8}{3}\right) \Gamma\left(\frac{16}{3}\right) \Gamma\left(\frac{24}{3}\right) \cdots \Gamma\left(\frac{8+8k}{3}\right)} x^{\left(\frac{8+8k}{3}\right)}\right.$$
$$\left. - \left(\frac{\lambda+1}{\lambda}\right) \frac{3^{k+1}\, \Gamma\left(\frac{9}{3}\right) \Gamma\left(\frac{17}{3}\right) \Gamma\left(\frac{25}{3}\right) \cdots \Gamma\left(\frac{9+8k}{3}\right)}{11 \cdot 19 \cdot 27 \cdots (11+8k) \cdot \Gamma\left(\frac{11}{3}\right) \Gamma\left(\frac{19}{3}\right) \Gamma\left(\frac{27}{3}\right) \cdots \Gamma\left(\frac{11+8k}{3}\right)} x^{\left(\frac{11+8k}{3}\right)}\right].$$

Substituting the functions $y_0(x)$, $y_1(x)$, $y_2(x)$, $y_3(x)$ into the assumed solution, the approximation of $y(x)$ is obtained and expressed in the following series form:

$$y(x) \cong \sum_{n=0}^{3} y_n(x)\,.$$

By inserting $y(x)$ into the second boundary condition (4.3), the eigenvalues are found by solving the following characteristic equation:

$$\Delta(\lambda) = \frac{0.0046581055\lambda^4 - 0.08158919\lambda^3 + 0.2110422899\lambda^2 + 1.20738574 14\lambda - 0.3414969469}{\lambda}.$$

Therefore, the first three eigenvalues can be computed as

$$\lambda_1 = -2.7957550516201377, \lambda_2 = 0.2713028347346597, \lambda_3 = 8.084716554791001.$$

The first three eigenvalues λ_1, λ_2 and λ_3 are presented in Table 1.

Table 1. The approximations to the first three eigenvalues

n	$\lambda_{1,n}$	$\lambda_{2,n}$	$\lambda_{3,n}$
3	−2.7957550516201377	0.2713028347346597	8.084716554791001
5	−2.7508610663049553	0.27129698800468705	7.221507847315261
10	−2.7507214210880258	0.27129698782232553	7.215993603097323
15	−2.7507214210872135	0.2712969878223252	7.21599360354564
20	−2.7507214210872135	0.27129698782232525	7.215993603545639

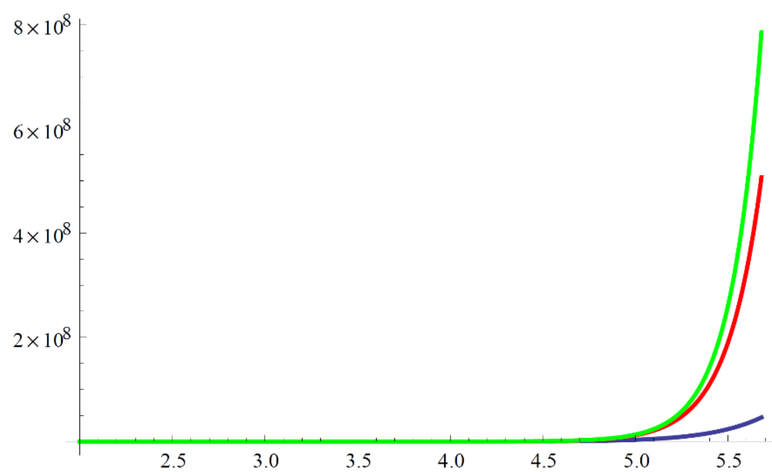
The results in Table 1, computed at `WorkingPrecision`$= 16$ in MATHEMATICA, demonstrate that the proposed method achieves rapid convergence with a minimal number of terms included in the series. To further evaluate the numerical stability and sensitivity of the proposed method, an error analysis was performed by comparing the results with higher precision. Table 2 shows the absolute errors calculated for different values of n using `WorkingPrecision`$= 32$ and `WorkingPrecision`$= 64$.

Table 2. Error analysis of the first three eigenvalues for $n = 5, 10$, and 20 using different working precisions

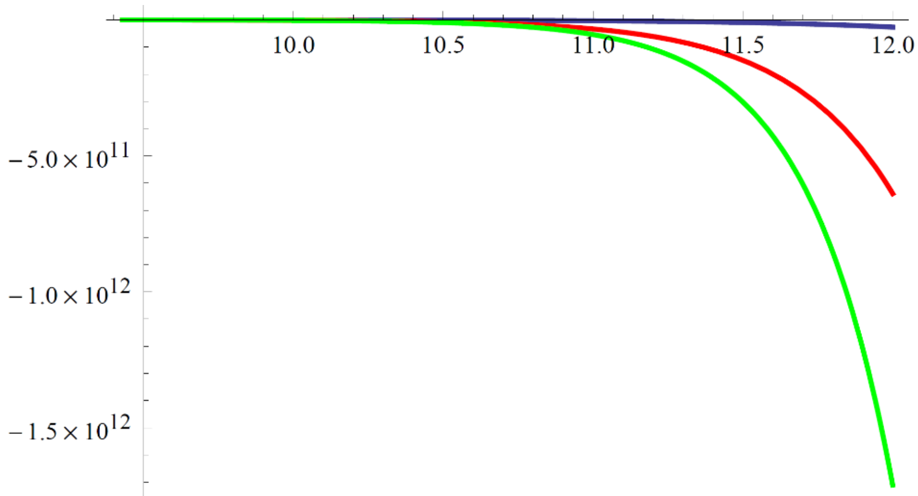
Eigenvalues	n	Error32	Error64
$\lambda_{1,n}$	5	$8.881784197001252 \times 10^{-16}$	2.83876×10^{-27}
	10	$4.884981308350689 \times 10^{-15}$	2.1656×10^{-28}
	20	$1.77635683940025 \times 10^{-15}$	4.9536×10^{-28}
$\lambda_{2,n}$	5	$5.551115123125783 \times 10^{-16}$	9.4786×10^{-28}
	10	$1.221245327087672 \times 10^{-15}$	4.7536×10^{-28}
	20	$1.110223024625156 \times 10^{-15}$	7.8237×10^{-28}
$\lambda_{3,n}$	5	$3.552713678800501 \times 10^{-15}$	9.89101×10^{-27}
	10	$9.769962616701378 \times 10^{-15}$	2.83866×10^{-27}
	20	$1.865174681370263 \times 10^{-14}$	7.5521×10^{-28}

This comparison shows how the accuracy of the eigenvalues improves as the computational precision is increased. This further demonstrates the numerical stability of the method and its convergence behavior under varying precision constraints.

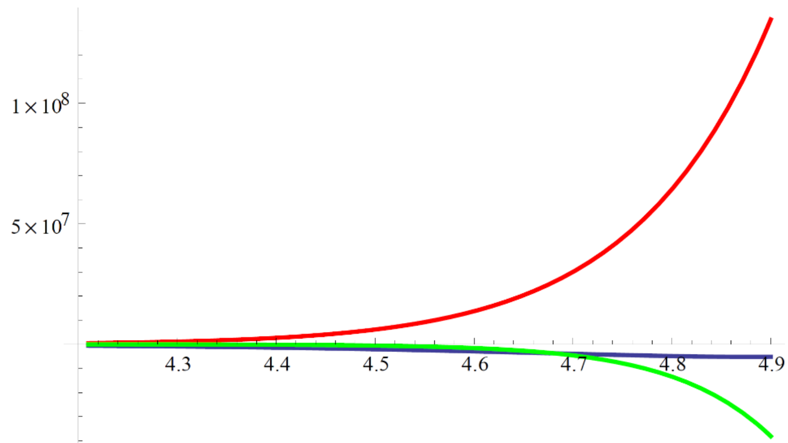
To further validate the method, the associated eigenfunctions are now examined through graphical approximations presented in the next figures.



(a)



(b)



(c)

Fig. 1 The approximations to the eigenfunctions, taking $n = 10, 15, 20$ associated with the first eigenvalue (a), the second eigenvalue (b), the third eigenvalue (c). Blue ($n = 10$), red ($n = 15$), green ($n = 20$)

5. Conclusion

In this study, the eigenvalues and eigenfunctions of a class of fractional Sturm-Liouville problems with eigenparameter-dependent boundary conditions were computed using the Adomian Decomposition Method (ADM). The most significant contribution of this work lies in the consideration of the non-homogeneous equation with a general forcing term $F(x, y)$ and the linear appearance of eigenvalues within the boundary conditions. This study provides a significant extension to the spectral analysis of such equations, which are limitedly studied in the literature.

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