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Original Article

Norm of the Sum of Two Basic Elementary Operators in the Tensor Product of C∗-Algebras

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Abstract *- In this paper, the Norm of the Sum of two basic elementary operators is determined in the tensor product of a C*-algebra. Using the Cauchy-Schwarz inequality and finite rank operators, the Norm of the Sum of two basic elementary operators is*

$$\left\|M_{A\otimes B,C\otimes D}+M_{E\otimes F,G\otimes H}|B(H\otimes K)\right\| = \|A\|\|B\|\|C\|\|D\| + \|E\|\|F\|\|G\|\|H\|$$

*in the tensor product of C*algebras.*

Keywords *- Basic elementary operator, finite rank operator, C*-algebra, and Tensor Product of C*-algebra.*

1. Introduction

Elementary operators are significant in operator theory, including the theory of bounded linear operators on Hilbert spaces and C*-algebras. The norm equalities and inequalities of basic elementary operators in Hilbert spaces, Banach algebras, and C*-algebras have been extensively studied. Norms of Elementary operators in tensor product spaces have been computed by finite rank operators, maximal numerical range, and the Cauchy-Schwarz inequality. In spite of these contributions, most of the existing results are either about a single basic elementary operator or about the sum of two basic elementary operators, either in the ordinary Hilbert space or Banach algebra context. Lack of attention is paid to the exact Norm of the sum of two basic elementary operators, especially in the case of the tensor product of C*-algebras. This creates a research gap because tensor product spaces have additional structural properties that may affect how elementary operators behave and how their norms are determined. The problem addressed in this study is therefore to determine the Norm of the sum of two basic elementary operators in the tensor product of C*-algebras. The study seeks to express this Norm in terms of the norms of the coefficient operators. To achieve this, the paper applies properties of tensor products, finite rank one operators, and the Cauchy-Schwarz inequality. This result contributes to the theory of elementary operators by extending known norm results to the tensor product setting of C*-algebras.

1.1. Tensor Products of C Algebras of Hilbert Spaces*

Definition 1.1.1. Tensor product. (Daniel et al., 2023)
Consider two complex Hilbert spaces, $Z=\{u_1,u_2\ldots\}$ and $L=\{v_1,v_1\ldots\}$ with inner products defined as $\langle u_1,u_2\rangle$ and $\langle v_1,v_2\rangle$ respectively. A tensor product of Z and L is a Hilbert space $Z\otimes L$ where $\otimes: Z\times L\rightarrow Z\otimes L, \otimes(u,v)\rightarrow u\otimes v$ is a bilinear mapping: The vectors $u\otimes v$ form a total subset of $Z\otimes L < u1\otimes v1, u2\otimes v2 > = < \mathrm{u}_1,\mathrm{v}_1 > < \mathrm{u}_2,\mathrm{v}_2 >, \forall\, \mathrm{u}_1,\mathrm{u}_2\in Z, v1, v2\in L$. This implies that
$\| u\otimes v\| = \| u\|\| v\|\ \forall\, u\in Z, v\in L$.If $E\in(Z)$, $F\in B(L)$, then $B(Z\otimes L)$ is a Hilbert space and for $E\otimes F\in B(Z\otimes L)$ we have $E\otimes F(u\otimes v)=Eu\otimes Fv\ \forall\, u\in Z, v\in L$.
Fundamental Properties of Operators in $(Z\otimes L)$ hold:
i).$(E\otimes F)(G\otimes H)=EG\otimes FH, \forall\, E\in B(Z), G\in B(Z)$ and $F\in B(L), H\in B(L)$.(Associativity and commutativity).
ii).$\| E\otimes F\| = \| E\|\| F\|\ \forall\, E\in(Z)$ and $F\in B(L)$ (Distributivity under tensor product).
iii). Linearity of the Tensor Product Map $\otimes(u,v)\rightarrow u\otimes v$ and its Properties:
$(u_1+u_2)\otimes v=(u_1\otimes v)+(u_2\otimes v)$
$(\psi u)\otimes v=\psi(u\otimes v)$.
$u\otimes(v_1+v_2)=u\otimes v_1+u\otimes v_2$
$u\otimes(\psi v)=\psi(u\otimes v)$.

The set of all vectors $\otimes(u,v)$, $u\epsilon Z$ and $v\epsilon L$, forms a total subset of $Z\otimes L$.

Definition 1.1.2. Elementary operator in a tensor product. (Muiruri et al, 2019)
Let Z and L be complex Hilbert spaces, and $B(Z \otimes L)$ denote the set of all bounded linear operators on the tensor product space $Z \otimes L$. For fixed elements $E \otimes F$ and $G \otimes H$ in $B(Z \otimes L)$, where $E, G \in B(Z)$ $F, H \in B(L)$,we define the elementary operator as follows:

$$E_n (Z \otimes L) = \sum_n (Ei \otimes Fi)(U \otimes V)(Gi \otimes Hi)$$

for every $U \otimes V \epsilon B(Z \otimes L)$, $Ei \otimes Fi$, $Gi \otimes Hi \in B(Z \otimes L)$.
Substituting $= 1$, , we obtain the basic elementary operator:

$$E(Z \otimes L) = (E \otimes F)(U \otimes V)(G \otimes H)$$

From equation (1), the basic elementary operator can be expressed as,
$E(Z \otimes L) = (E \otimes F)(U \otimes V)(G \otimes H) = (EUG) \otimes (FVH)$.

2. Norm of the Sum of Two Basic Elementary Operators

Different researchers have determined the Norm of basic elementary operators in different spaces using different approaches. For instance,(Boumazgour, 2008) used norm inequalities for 2×2 operator matrices to determine norm inequalities for the Sum of two basic elementary operators on a Hilbert space and obtained the norm inequality for the Sum of two basic elementary operators and proved the result as shown in Theorem 2.1;

Theorem 2.1: (Boumazgour, 2008)
If $A, B, C,$ and D are operators in $B(H)$, then,

$$\left\| M_{(A,B)} + M_{(C,D)} \right\| \leq \{(Max(\|B\|^2, \|D\|^2) + \|BD^*\|)(Max(\|A\|^2, \|C\|^2) + \|C^*A\|)\}^{\frac{1}{2}}$$

Okelo (2011) studied the structural properties of elementary operators. Okelo (2011) utilized Dvoretsky's theorem and its application in determining the Norm of a symmetrized two-sided multiplication operator on a C*-Algebras B(H), and the result in Lemma 2.2 and Theorem 2.3;

Lemma 2.2: (Okelo, 2011)
Let H be a Hilbert space, $B(H$be the algebra of bounded linear operators on H, and $M_{(A,B)}$ be a norm-attainable basic elementary operator. $M_{(A,B)} : B(H) \rightarrow B(H)$ defined by $M_{(A,B)}(X) = AXB : \forall X \in B(H)$ where A, B are norm-attainable operators fixed in B(H), then $||M_{(A,B)}|| = ||A||||B||$.

Theorem 2.3: Okelo (Okelo, 2011)
Let $x, y \in B(H)$ and $x \otimes y$ denote the tensor product of x and y, then

$$||x \otimes y + y \otimes x|| \leq \sqrt{2\|x\|^2\|y\|^2 + 2\|y * x\|^2}$$

Further, Okelo and Agure (2011) also used the finite rank operator to determine the Norm of the basic elementary operator and proved Lemma 2.4:

Lemma 2.4: (Okelo and Agure, 2011)
Let H be a Hilbert space, $B(H)$ and let B (H) be the algebra of bounded linear operators on H.
If $MA, B : B(H) \rightarrow B(H)$ is defined by $M_{A,B}B(X) = AXB$ where A, B are fixed elements are in $B(H)$ then

$$||M_{A,B}|B(X)| = ||A||||B||.$$

Bachir et al. (2012) determined the Norm of the Sum of two basic elementary operators in a Banach Algebra $B(H)$ by generalized maximal numerical range and spatial numerical range and proved Theorem 2.5:

Theorem 2.5: (Bachir et al., 2012)
Let $A_1A_2B_1B_2$ be operators in $B(E)$. If $||A_1||||A_2|| \in M_{(A_1)(A_2)} \cup M_{(A_2)A_1}$ and $||B_1||||B_2|| \in M_{(B_1)(B_2)} \cup M_{(B_2)B_1}$ then

$$\left\| M_{(A_1)(A_2)} + M_{(A_2)A_1} \right\| = ||A_1||||B_1|| + ||A_2||||B_2||$$

Boumazgour and Barraa (2015) determined the norm equality of a basic elementary operator for a special case when E is a Hilbert space, and S is a Schatten class
ideal of $L(E)$ and proved that

$$\left\| \mathrm{I} + M_{A,B} \right\| = 1\|A\|\|B\|$$

Boumazgour and Barraa (2015) further proved Lemma 2.6 and Theorem 2.7:

Lemma 2.6: (Boumazgour and Barraa, 2015)
Let $A, B \in L(E)$, and let S be a symmetric norm ideal of $L(E)$. Then
$$\|M_{S,A,B}\| = \|A\|\|B\|$$
Theorem 2.7: (Boumazgour and Barraa, 2015)
Let $A, B \in L(H)$, and suppose that $1 < p < \infty$. Then the following are equivalent:
$$\|I + M_{P,A,B}\| = 1 + \|A\|\|B\|$$
There exists $\lambda \in C$ with $|\lambda| = 1$ such that $\lambda\|A\| \in \sigma(A)$ and $\bar{\lambda}\|B\| \epsilon \delta(B_p)$
There exists $\lambda \in C$ with $|\lambda| = 1$ such that $\lambda\|A\| \in V(A)$ and $\bar{\lambda}\|B\| \epsilon V(B)$
Muiruri et al. (2019)determined the Norm of an elementary operator in a tensor product by employing the techniques of tensor products and finite rank one operators, and also expressed the Norm of an elementary operator in terms of its coefficient operators. Muiruri et al.,(2019) proved the Theorem 2.8:

Theorem 2.8:Muiruri et al. (2019)
Let H and K be complex Hilbert spaces, and $B(H \otimes K)$ be the set of all bounded linear operators on $H \otimes K$, then $\forall X \otimes Y \in B(H \otimes K)\ with\ ||X \otimes Y|| = 1$, we have:
$||M_{A\otimes B,C\otimes D}B(Z \otimes L)|| = ||A||||B||||C||||D||$ where A, B, and C, D are fixed elements in $B(H)$ and $B(K)$, respectively.

Corollary 2.9:(Muiruri et al.,2019)
Let H and K be complex Hilbert spaces, and $B(H \otimes K)$ be the set of bounded linear operators on $(H \otimes K)$. If for all $X \otimes Y \in B(H \otimes K)$ with $||X \otimes Y|| = 1$, then we have $||M_{A\otimes B,C\otimes D}|| = ||M_{A,C}||||M_{B,D}||$ where $M_{A,C}$ and $M_{B,D}$ are basic elementary operators on $B(H)$ and $B(K)$ respectively.
Daniel et al. (2022) determined the Norm of basic elementary operators in a tensor product by employing the techniques of maximal numerical range and proved Theorem 2.2.1.10:

Theorem 2.10:(Daniel et al., 2022)
Let Z and L be complex Hilbert spaces, and let $O_{E\otimes F,G\otimes H}$ be the basic elementary operator on $B(Z \otimes L)$, the set of bounded operators that are linear on a complex Hilbert space $Z \otimes L$. If $\forall U \otimes V \in B(Z \otimes L)$ with $||U \otimes V|| = 1, E, G \in B(Z), F, H \in B(L), \zeta \in W \circ (G), \xi \in W \circ (H)$ then;
$$\|O_{E\otimes F,G\otimes H} \backslash B(Z \otimes L)\| = \sup_{\zeta \in W_\circ(G)} \sup_{\xi \in W_\circ(H)} \{|\zeta||\xi|\|E\|\|F\|\}.$$

3. Methodology

To determine the Norm of the sum of two basic elementary operators, the study employed finite rank one operators and properties of tensor products of C^*-Algebras. The upper bound of the norms of both Sum of two basic elementary operators was determined using Cauchy–Bunyakovsky–Schwartz inequality(**Theorem 3.1**).

Theorem 3.1. Cauchy-Schwarz Inequality *(Ponnusamy and Silverman, 2006)*
If X is an inner product space.Then for all $x, y \in X$, then
$$|\langle x, y\rangle| \leq^2 \langle x, x\rangle\langle y, y\rangle.$$

Finite rank one operators (**Theorem 3.2**) was used to determine the lower bound of the Norm of the Sum of two basic elementary operators.

Theorem 3.2. (Bonyo and Agure, 2011)
For a complex Hilbert space H with dual H^* , then a finite rank one operator $u \otimes x$: $H \to H$ is defined as
$$(u \otimes x)y = u(y)x,$$
for all $y \in H$ where $u \in H^*$ and $x \in H$ is a unit vector with
$$\begin{aligned}||(u \otimes x)y|| &= \sup||(u \otimes x)y|| : y \in H, ||y|| \leq 1 \\ &= \sup||u(y)x|| : y \in H, ||y \leq 1 \\ &= \sup|u(y)|||x|| : y \in H, ||y|| \leq 1.\end{aligned}$$

4. Results and Discussion

4.1. Norm of the Sum of Two Basic Elementary Operators In Tensor Product Of C∗-Algebras

Theorem.

Let $H \otimes K$ be the tensor product of Hilbert spaces H and K, let $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. If $X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$ then,

$$\left\|M_{(A\otimes B,C\otimes D)} + M_{(E\otimes F,G\otimes H)}\right\| = \|A\|\|B\|\|C\|\|D\| + \|E\|\|F\|\|G\|\|H\|$$

where$M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}$ is the Sum of two basic elementary operators, such that $A, C, E, G \in B(H)$ and $B, D, F, H \in B(K)$.

Proof.

By definition,

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}|B(H \otimes K)\right\| = Sup\{\left\|(M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H})(X\otimes Y)\right\|$$

Such that $X\otimes Y \in B(H\otimes K), \|X\otimes Y\| = 1$

Hence,

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}|B(H \otimes K)\right\| \geq \left\|(M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H})(X\otimes Y)\right\|$$

for all $x \otimes y \in B(H \otimes K)$ with $\|X\otimes Y\| = 1$

Then for any $\varepsilon > 0$, there exists $X \otimes Y \in B(H \otimes K)$ with $\|X\otimes Y\| = 1$ such that.

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}|B(H \otimes K)\right\| - \varepsilon < \left\|(M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H})(X\otimes Y)\right\|$$

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}|B(H \otimes K)\right\| - \varepsilon < \|A\otimes B(X \otimes Y)C \otimes D + E \otimes F(X \otimes Y)G \otimes H\|$$

Using the tensor product of operators,

$$(A \otimes B)(X \otimes Y) = AX \otimes BY,$$

then

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}|B(H \otimes K)\right\| - \varepsilon < \|AXC\otimes BYD + EXG \otimes FYH\|$$

By the Cauchy-Schwarz inequality,

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}|B(H \otimes K)\right\| - \varepsilon \leq \|AXC\otimes BYD\| + \|EXG \otimes FYH\|$$

Clearly, using properties of the tensor product.

$\|AXC\otimes BYD\| = \|AXC\|\|BYD\|$

$\|EXG \otimes FYH\| = \|EXG\|\|FYH\|$

Since $\varepsilon \geq 0$ it was chosen arbitrarily, then

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}|B(H \otimes K)\right\| \leq \|AXC\|\|BYD\| + \|EXG\|\|FYH\|$$

Thus

$$\|AXC\| \leq \|A\|\|X\|\|C\| = \|A\|\|C\|: \|X\| = 1$$
$$\|BYD\| \leq \|B\|\|Y\|\|D\| = \|B\|\|D\|: \|Y\| = 1$$
$$\|EXG\| \leq \|E\|\|X\|\|G\| = \|E\|\|G\|: \|X\| = 1$$
$$\|FYH\| \leq \|F\|\|Y\|\|H\| = \|F\|\|H\|: \|Y\| = 1$$

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}|B(H \otimes K)\right\| = \|A\|\|B\|\|C\|\|D\| + \|E\|\|F\|\|G\|\|H\|$$

Conversely, let there exist unit vectors $e \otimes f \in H \otimes K$, where $e \in H$ and $f \in K$. Then,$\left\|(M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H})(X\otimes Y)(e\otimes f)\right\| \leq \left\|(M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H})\right\|\|(X\otimes Y)\|\|(e\otimes f)\|$

$$\leq \left\|(M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H})\right\|\|X\|\|y\|\|e\|\|f\|$$
$$\leq \left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}\right\|$$

Therefore,

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}\right\| \geq \left\|(M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H})(X\otimes Y)(e\otimes f)\right\|$$

That is,

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}\right\| \geq \|(A\otimes B)(X\otimes Y)(C\otimes D)(e\otimes f) + (E\otimes F)(X\otimes Y)(G\otimes H)(e\otimes f)\|$$
$$\geq \|AXCe\otimes BYDf + EXGe\otimes FYHf\|$$

Now squaring both sides,

$$\left\|M_{A\otimes B,C\otimes D} + M_{E\otimes F,G\otimes H}\right\|^2 \geq \|AXCe\otimes BYDf + EXGe\otimes FYHf\|^2$$
$$\geq \langle AXCe\otimes BYDf + EXGe\otimes FYHf, AXCe\otimes BYDf + EXGe\otimes FYHf\rangle$$

$$\geq \langle AXCe\otimes BYDf, AXCe\otimes BYDf\rangle + \langle AXCe\otimes BYDf, EXGe\otimes FYHf\rangle + \langle EXGe\otimes FYHf, AXCe\otimes BYDf\rangle + \langle EXGe\otimes FYHf, EXGe\otimes FYHf\rangle$$

Since

$$\langle x_1\otimes y_1, x_2\otimes y_2\rangle = \langle x_1, x_2\rangle\langle y_1, y_2\rangle$$

then

$$\langle AXCe\otimes BYDf + EXGe\otimes FYHf, AXCe\otimes BYDf + EXGe\otimes FYHf\rangle = \langle AXCe, AXCe\rangle\langle BYDf, BYDf\rangle + \langle AXCe, EXGe\rangle\langle BYDf, FYHf\rangle + \langle EXGe, AXCe\rangle\langle FYHf, BYDf\rangle + \langle EXGe, XGe\rangle\langle FYHf, FYHf\rangle$$
$$= \|AXCe\|^2\|BYDf\|^2 + 2\langle AXCe, EXGe\rangle\langle BYDf, FYHf\rangle + \|EXGe\|^2\|FYHf\|^2$$

Now let $\mu1, \mu2: H \to C$ and $\nu1, \nu2 : K \to C$ be bounded linear functionals. Choose vectors $y \in H$ and $z \in K$ define rank–one operators.

$$A = \mu1\otimes y, E = \mu1\otimes y, C = \nu1\otimes z, G = \nu1\otimes z$$

Then for all e ∈ H,

$Ae = (\mu1 \otimes y)e = \mu1(e)y, \|y\| = 1,$
$Ee = (\mu2 \otimes y)e = \mu2(e)y, \|y\| = 1$
$Ce = (\nu1 \otimes z)e = \nu1(e)z, \|z\| = 1$
$Ge = (\nu2 \otimes z)e = \nu2(e)z, \|z\| = 1$

Now,

$$\|A\| = Sup\{\|\mu1\otimes y\|: e\in H, \|e\| = 1\}$$
$$= Sup\{|\mu1(e)|\|y\|: e\in H, \|e\| = 1\}$$

Since $\|y\| = 1$ this simplifies
to $Sup\{|\mu1(e)|: e\in H, \|e\| = 1\}$=$|\mu1(e)|$

Thus, the Norm of A is

$$\|A\| = Sup\{\|\mu1 \otimes \text{Y e}\|: e\in H, \|e\| = 1\}$$
$$=|\mu1(e)|$$

Similarly, the Norm of E, G, and C is

$\|E\| = |\mu1(e)|, \|G\| = |\nu2(e)|, \|C\| = |\nu1(e)|.$

From equation (2),

$\|AXCe\|^2 = \|(C\mu1 \otimes y)\, X\, (\nu1 \otimes z)e\|^2$
$=\|(C\mu1 \otimes y)\, X\, \nu1(e)z\|^2$
$=\|\nu1(e)\, (C\mu1 \otimes y)\, Xz\|^2$
$= |\nu1(e)|^2\|\mu1(X(z))\, y\|^2$
$= |\nu1(e)|^2|\mu1(X(z))|^2\|y\|^2$
$=\|A\|^2\|C\|^2$

Thus, using the same concept,

$\|BYDf\|^2 = \|B\|^2\|D\|^2$
$\|EXGe\|^2 = \|E\|^2\|G\|^2$
$\|FYHf\|^2 = \|F\|^2\|H\|^2$

Also,

$\langle AxCe, ExGe\rangle = \langle(\mu1 \otimes y)\, x\, (\nu1 \otimes z)e, (\mu2 \otimes y)\, x\, (\nu2 \otimes z)e\rangle$
$= \langle(\mu1 \otimes y)\, x\, (\nu1(e)z), (\mu2 \otimes y)\, x\, (\nu2(e)z)\rangle$
$= \langle\nu1(e)(\mu1 \otimes y)\, xz, \nu2(e)(\mu2 \otimes y)\, xz\rangle$
$= \langle\nu1(e)\mu1(x(z))\, y, \nu2(e)\mu2(x(z))\, y\rangle$
$= \nu1(e)\, \mu1(x(z))\, \nu2(e)\, \mu2(x(z))\, \langle y, y\rangle$
$= \nu1(e)\, \mu1(x(z))\, \nu2(e)\, \mu2(x(z)).$

But $\nu1(e), \mu1(x(z)), \nu2(e)$, and $\mu2(x(z))$ are positive real numbers. Hence, $\nu_1(e) = |\nu_1(e)| = \|C\|$,

$\mu 1(x(z)) = |\mu 1(x(z))| = \|A\|,$
$\nu 2(e) = |\nu 2(e)| = \|G\|, \mu 2(x(z)) = |\mu 2(x(z))| = \|E\|.$
Thus,

$$\langle AXCe, EXGe\rangle = \|A\|\|C\|\|E\|\|G\|$$

But the norms of $A, C, E$and G are scalars, hence

$$\langle AXCe, EXGe\rangle = \|A\|\|E\|\|C\|\|G\| \quad (3)$$

Similarly,

$$\langle ByDf, FyHf\rangle = \|B\|\|D\|\|F\|\|H\| \quad (4)$$

Now, substituting equations (3) and (4) into equation (2), then

$$\|M_{A\otimes B, C\otimes D} + M_{E\otimes F, G\otimes H}|B(H \otimes K)\|^2 \geq \|A\|^2\|B\|^2\|C\|^2\|D\|^2 + 2\|A\|\|B\|\|C\|\|D\|\|E\|\|F\|\|G\|\|H\| + \|E\|^2\|F\|^2\|G\|^2\|H\|^2$$

Factorising,

$$\|M_{A\otimes B, C\otimes D} + M_{E\otimes F, G\otimes H}|B(H \otimes K)\|^2 \geq \{\|A\|\|B\|\|C\|\|D\| + \|E\|\|F\|\|G\|\|H\|\}^2.$$

Thus, obtaining the square root of both sides

is $\left\|M_{A\otimes B, C\otimes D} + M_{E\otimes F, G\otimes H}|B(H \otimes K)\right\| \geq \|A\|\|B\|\|C\|\|D\| + \|E\|\|F\|\|G\|\|H\|.$ (5)

From equations (1) and (5), then

$$\left\|M_{A\otimes B, C\otimes D} + M_{E\otimes F, G\otimes H}|B(H \otimes K)\right\| = \|A\|\|B\|\|C\|\|D\| + \|E\|\|F\|\|G\|\|H\|.$$

4.2. Discussion

The findings of this research show a great step forward in the knowledge of the Norm of the sum of two fundamental elementary operators of the C*-algebras' tensor product. Several factors are responsible for the better results as compared to previous work:

4.2.1. Closed-Form Norm Expression

The operator norms have been investigated in previous works; they were mostly given in inequalities or in partial characterizations (Boumazgour, 2008; Okelo, 2011). This research, on the other hand, is able to obtain a closed-form expression for the Norm of the sum, explicitly connect it with the Norm of the coefficient operators, and also determine the Norm of the sum. This will give not only a better theoretical understanding but will also be useful for performing calculations.

4.2.2. Use of Finite Rank Operators

The finite rank one operators are used to obtain upper and lower bounds of the Norm. It is different from the generalized numerical range approach (Bachir et al., 2012) or the maximal numerical range approach (Daniel et al., 2022), where the derived Norm is attainable and verifiable, which might reduce the approximation needed.

4.2.3. Extension to Tensor Product Spaces

Muiruri et al. (2019) discussed the norms in a tensor product space, but in their analysis, they considered only a single elementary operator, not multiple operators. This research gap is filled by our approach, which yields a formula that describes the interaction of two operators in the tensor product scenario, broadening its scope of applicability to a larger family of operator structures.

4.2.4. Analytical Simplicity and Generality

The methodology can be applied to any bounded linear operators defined on Hilbert spaces that are a tensor product. Unlike previous results (Boumazgour & Barraa, 2015), it is not restricted to special classes of operators like Schatten ideals or symmetric norm ideals.

4.2.5. Improved Theoretical Insight

In addition to giving numerical accuracy, this study provides insight into the behavior of operators when composing under tensor products. It gives a clear characterization of the relation between the properties of individual operators and the corresponding properties of the combined Norm, which was not discussed much in the previous literature.

In conclusion, this discussion highlights that the combination of finite rank operator techniques, Cauchy-Schwarz inequality, and tensor product properties not only improves norm determination for sums of basic elementary operators but also fills a critical gap in the current literature on operator theory in C*-algebra tensor products.

5. Conclusion

From the study, it can be concluded that using a finite rank operator, the Norm of the Sum of two basic elementary operators in the tensor product of C*Algebras is $\|M_{\mathrm{A\otimes B,C\otimes D}} + M_{\mathrm{E\otimes F,G\otimes H}}|B(H \otimes \mathrm{K})\| = \|A\|\|B\|\|C\|\|D\| + \|E\|\|F\|\|G\|\|H\|$.

Conflicts of Interest

The authors declare no conflicts of interest.

Notations

- H, K, Z, L: Complex Hilbert spaces where operators act.
- $B(H), B(K), B(Z), B(L)$: Algebra of bounded linear operators on H, K, Z, L respectively.
- $\otimes$: Tensor product operation for both spaces and operators.
- $E, F, G, H, U, V, Ei, Fi, Gi, Hi$: Bounded linear operators in $B(H), B(K)$, or their tensor product space.
- S: Schatten class ideal, a class of compact operators whose singular values are p-summable.
- Basic elementary operator: Operator of the form $E(X) = (E \otimes F)(U \otimes V)(G \otimes H)$ with fixed operators E, F, G, H, and variable X in $B(H \otimes K)$.
- Cauchy-Schwarz inequality: Inequality $|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle\langle y, y\rangle}$ in inner product spaces.
- $u \otimes x$: Rank-one operator mapping $y \in H$ to $u(y)x$, where $u \in H*$ and $x \in H$.
- $||\cdot||$: Norm of an operator, typically the operator norm induced by the Hilbert space norm.
- $\langle\cdot,\cdot\rangle$: Inner product in a Hilbert space.

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