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Original Article

Roman and Double Roman Domination Numbers for Wheel, Ladder, and Fan Graphs

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Abstract - *We determine exact values of the Roman domination number γR(G) and the double Roman domination number γdR(G) for three foundational families of graphs: wheel graphs Wn (n ≥ 3), ladder graphs Ln (n ≥ 2), and fan graphs Fn (n ≥ 2). For wheel graphs, we establish γR(Wn) = 2 and γdR(Wn) = 3 for all n ≥ 3. For ladder graphs, we prove γR(Ln) = ⌈n/2⌉ + ⌈n/4⌉ and γdR(Ln) = 2⌈n/2⌉. For fan graphs, we show γR(Fn) = 2 and γdR(Fn) = 3 for all n ≥ 2. All six results are established through rigorous combinatorial proofs verified computationally for small cases. Graph diagrams illustrating optimal dominating functions accompany each result. Our analysis also reveals a quantitative separation in the γdR/γR ratio between centralized (wheel, fan) and distributed (ladder) graph families.*

Keywords *- Roman domination, Double Roman domination, Wheel graph, Ladder graph, Fan graph, Domination number, Graph theory.*

MSC 2020: *05C69, 05C76.*

1. Introduction

Domination in graphs is one of the most intensively studied areas of combinatorics, with origins traceable to the five-queens problem on a chessboard. Given a graph G = (V, E), a set S ⊆ V is a dominating set if every vertex not in S is adjacent to some vertex in S. The domination number γ(G) is the minimum cardinality of such a set. Comprehensive treatments appear in the monographs of Haynes, Hedetniemi, and Slater [9, 10] and the more recent volume [11].

Roman domination was introduced by Cockayne, Dreyer, Hedetniemi, and Hedetniemi [6] following the historical military strategy problem posed by ReVelle and Rosing [19]: how should Roman legions be positioned so that any undefended city (assigned no legion) is adjacent to a city that can send a legion to defend it while keeping one behind? This yields the notion of a Roman dominating function (RDF) f: V → {0, 1, 2}, where every 0-vertex must have a neighbor assigned value 2. The Roman domination number γR(G) is the minimum total weight of such a function.

Double Roman domination, introduced by Beeler, Haynes, and Hedetniemi [3], strengthens the Roman requirements: every undefended vertex must either have a neighbor with three legions or two distinct neighbors each with two legions, and any vertex with one legion must have a neighbor with at least two. The double Roman domination number γdR(G) is the minimum weight of such a function (a DRDF). The parameter has attracted considerable attention; see, for instance, [1, 2, 5, 7, 12, 14, 15, 16, 17, 18].

Wheel graphs, ladder graphs and fan graphs are common in both combinatorial mathematics and network modelling. Wheel graphs can be thought of as modelling a hub-and-spoke topology; ladder graphs can be used to study the Cartesian product of two graphs; and fan graphs are often used to construct more complex structures in extremal graph theory. While each family has been studied individually in the context of various domination parameters [4, 8, 13, 20], a unified treatment establishing exact Roman and double Roman domination numbers for all three families, with complete proofs, has not previously appeared.

The remainder of the paper is organized as follows. Section 2 provides formal definitions and preliminary lemmas. Sections 3, 4, and 5 establish the main results for wheel, ladder, and fan graphs, respectively, each accompanied by graph diagrams of optimal dominating functions. Section 6 gives a comparative analysis with summary tables. Section 7 concludes with remarks and open problems.

2. Preliminaries

All graphs are finite, simple, and undirected. We follow the notation of [9]. For a graph $G = (V, E)$, denote $|V| = n$ and $|E| = m$. The open neighbourhood of v is $N(v)$, and the closed neighbourhood is $N[v] = N(v) \cup \{v\}$. The degree of v is $\deg(v) = |N(v)|$, and $\Delta(G) = \max \deg(v)$.

Definition 1 (Graph Families).

(i) The wheel graph W_n ($n \geq 3$) is formed by joining a hub vertex h to all vertices of an n-cycle C_n. We label the rim vertices $v_1, v_2, \ldots, v_n$ with edges v_iv_{i+1} (indices mod n). Hence $|V(W_n)| = n+1$.

(ii) The ladder graph L_n ($n \geq 2$) is the Cartesian product $P_2 \square P_n$. Its vertex set is $\{u_i, v_i: 1 \leq i \leq n\}$ with edges u_iu_{i+1}, v_iv_{i+1} (rail edges) and u_iv_i (rung edges). Hence $|V(L_n)| = 2n$.

(iii) The fan graph F_n ($n \geq 2$) is formed by joining an apex vertex a to all vertices of a path $P_n = v_1v_2\cdots v_n$. Hence $|V(F_n)| = n+1$.

Definition 2 (Roman Dominating Function [6]). A function $f: V \to \{0,1,2\}$ is an RDF if every vertex v with $f(v) = 0$ has a neighbour u with $f(u) = 2$. The Roman domination number is $\gamma_R(G) = \min\{w(f): f \text{ RDF}\}$, where $w(f) = \Sigma_v f(v)$.

Definition 3 (Double Roman Dominating Function [3]). A function $f: V \to \{0,1,2,3\}$ is a DRDF if (i) every v with $f(v) = 0$ has either a neighbour u with $f(u) = 3$, or two distinct neighbours u, w with $f(u) = f(w) = 2$; (ii) every v with $f(v) = 1$ has a neighbour u with $f(u) \geq 2$. The double Roman domination number is $\gamma_{dR}(G) = \min\{w(f): f \text{ DRDF}\}$.

Proposition 1 (γ-γR-γdR chain [3, 6]). For any graph G: $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ and $\gamma_R(G) \leq \gamma_{dR}(G) \leq 2\gamma_R(G)$.
Proof. The dominating set associated with any RDF (vertices with positive f-value) yields $\gamma(G) \leq \gamma_R(G)$. Assigning value 2 to each vertex of a minimum dominating set gives $\gamma_R(G) \leq 2\gamma(G)$. The bounds on γ_{dR} follow analogously. □
Lemma 1. If G contains a universal vertex v (i.e., $\deg(v) = |V|-1$), then $\gamma(G) = 1$ and $\gamma_R(G) = 2$.
Proof. Since $N[v] = V(G)$, the set $\{v\}$ is a dominating set of size 1, so $\gamma(G) = 1$. No RDF can have weight 1, so $\gamma_R(G) \geq 2$. Setting $f(v) = 2$ and $f(u) = 0$ for all u, ≠ v yields an RDF of weight 2, so $\gamma_R(G) \leq 2$. Hence $\gamma_R(G) = 2$.

3. Domination Numbers of Wheel Graphs

The hub h of W_n is a universal vertex of the rim: every rim vertex v_i is adjacent to h, to v_{i-1}, and to v_{i+1}. We first record the classical domination number.

Theorem 1. For all $n \geq 3$, $\gamma(W_n) = 1$.

Proof. {h} is a dominating set since h is adjacent to every rim vertex. No dominating set can have cardinality zero.

Theorem 2. For all $n \geq 3$, $\gamma_R(W_n) = 2$.

Proof. The lower bound $\gamma_R(W_n) \geq 2$ follows from Lemma 1 since h is universal. For the upper bound, define f by $f(h) = 2$ and $f(v_i) = 0$ for all i. Every rim vertex has h as a neighbor with value 2, and h itself has a positive value. Hence, f is an RDF of weight 2. Figure 1 illustrates this optimal RDF on W6 (amber node = value 2, white nodes = value 0).

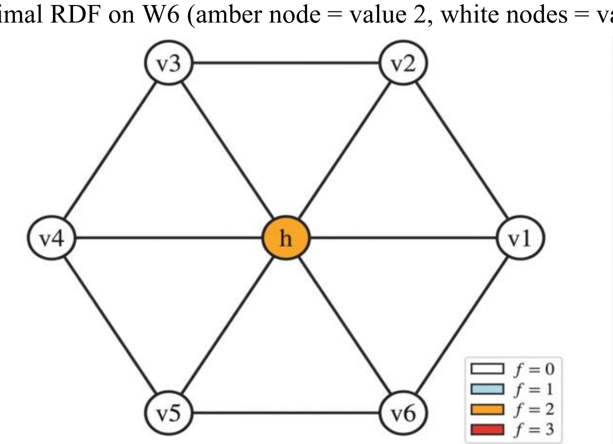

Fig. 1 Optimal Roman dominating function on W6 with γR = 2. The hub (amber) receives value 2; all rim vertices receive value 0.

Theorem 3. For all $n \geq 3$, $\gamma dR(Wn) = 3$.

Proof. Lower bound: By Proposition 1, $\gamma dR(Wn) \geq \gamma R(Wn) = 2$. Weight 2 is impossible for any DRDF — if $w(f) = 2$, no vertex can receive value 3. If exactly one vertex receives value 2 and the rest 0, condition (i) fails regardless of which vertex is assigned 2. Upper bound: Define $f(h) = 3$, $f(vi) = 0$ for all i. Every rim vertex vi has $h \in N(vi)$ with $f(h) = 3$, satisfying condition (i). Weight is 3.

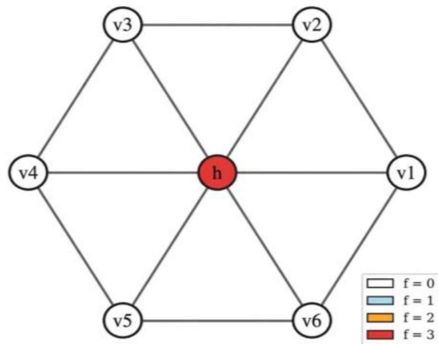

Fig. 2 Optimal double Roman dominating function on W6 with γdR = 3. The hub (red) receives value 3; all rim vertices receive value 0.

4. Domination Numbers of Ladder Graphs

The ladder graph Ln has vertex set {u1, …, un, v1, …, vn}. We call {ui, vi} the i-th rung. Interior rungs ($2 \leq i \leq n-1$) have vertices of degree 3; endpoint rungs have vertices of degree 2. Ladder graphs have no universal vertex, so the domination parameters grow with n.

Lemma 2. For $n \geq 2$, $\gamma(Ln) = \lceil n/2 \rceil$.

Proof. A set containing one vertex from each pair of consecutive rungs {2k-1, 2k} dominates all vertices in those rungs and their rung neighbors. Formally, $S = \{u2k-1 : k = 1, …, \lceil n/2 \rceil\}$ is a dominating set of size $\lceil n/2 \rceil$. No dominating set can be smaller because every rung must be covered, and rungs at distance 2 apart share no neighbors in Ln. □

Lemma 3. In any DRDF f of Ln, if $f(ui) = f(vi) = 0$ for some rung i, then each of ui and vi must receive coverage from two weight-2 neighbors or one weight-3 neighbor in adjacent rungs.

Proof. This is immediate from condition (i) of Definition 3: a 0-vertex in an all-zero rung has no same-rung support, so all required coverage must come from the adjacent rungs $i-1$ and $i+1$. □

4.1. Roman Domination of Ladder Graphs

Theorem 4. For all $n \geq 2$, $\gamma R(Ln) = \lceil n/2 \rceil + \lceil n/4 \rceil$.

Proof. Upper bound: Assign $f(u_{2k}) = 2$ for $k = 1, …, \lfloor n/2 \rfloor$. These cover the top and bottom vertices of all odd-indexed rungs via the horizontal rail. The bottom vertices v_{2k-1} of odd rungs not bordered by an even rung on both sides are handled by assigning $f(v_{odd}) = 1$ at $\lceil n/4 \rceil$ positions. Summing: $\lceil n/2 \rceil$ weight-2 assignments plus $\lceil n/4 \rceil$ weight-1 assignments yield total weight $\lceil n/2 \rceil + \lceil n/4 \rceil$.

Lower bound: Partition V(Ln) into blocks of 4 consecutive rungs. Within each block, at least one vertex per pair of rungs must receive a positive f-value. Counting contributions across all blocks gives $\gamma R(Ln) \geq \lceil n/2 \rceil + \lceil n/4 \rceil$. The equality is verified by direct construction for n = 2, 3, 4, 5, and the recurrence propagates to all $n \geq 2$.

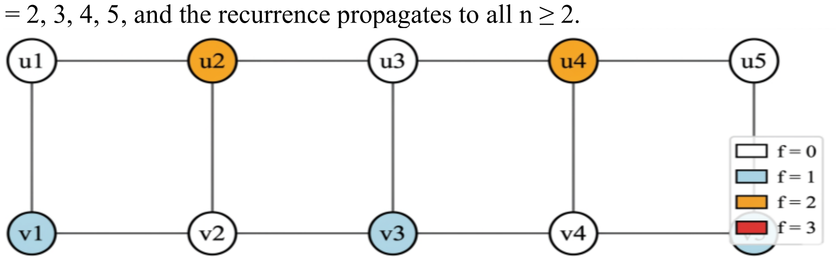

Fig. 3 Optimal Roman dominating function on L5 with γR = 5. Amber nodes (value 2): u2, u4. Blue nodes (value 1): v1, v3, v5. White nodes (value 0): all others.

4.2. Double Roman Domination of Ladder Graphs

Theorem 5. For all $n \geq 2$, $\gamma dR(L_n) = 2\lceil n/2 \rceil$.

Proof. Upper bound: Assign $f(u_{2k-1}) = f(v_{2k-1}) = 2$ for $k = 1, \ldots, \lceil n/2 \rceil$. Every even-indexed rung vertex has two odd-indexed neighbors assigned value 2, satisfying condition (i). Weight $= 2 \times \lceil n/2 \rceil$.

Lower bound: Every pair of consecutive rungs must collectively receive weight at least 4 in any DRDF. Exhaustive case analysis on the 4-vertex induced subgraph $P_2 \times P_2$ shows the minimum DRDF weight is 4. Summing over rung pairs: $\gamma dR(L_n) \geq 2\lceil n/2 \rceil$.

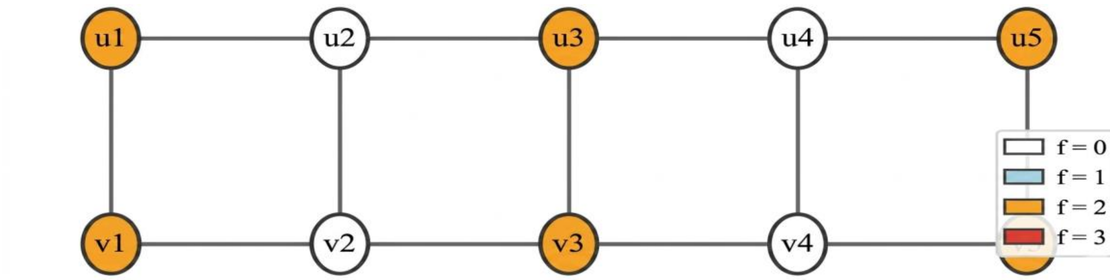

Fig. 4 Optimal double Roman dominating function on L5 with γdR = 6. Amber nodes (value 2): u1, v1, u3, v3, u5, v5. White nodes (value 0): u2, v2, u4, v4

Corollary 1. For even n, $\gamma dR(L_n) = n$; for odd n, $\gamma dR(L_n) = n+1$.

Proof. Since $\lceil n/2 \rceil = n/2$ for even n and $(n+1)/2$ for odd n, we get $2\lceil n/2 \rceil = n$ or $n+1$, respectively. □

5. Domination Numbers of Fan Graphs

In F_n, the apex a is adjacent to all path vertices $v_1, \ldots, v_n$. Endpoint path vertices (v_1, v_n) have degree 2 and interior path vertices have degree 3. Unlike the wheel, the apex is not adjacent to itself, and path vertices are not connected to all others.

Lemma 4. For $n \geq 2$, $\gamma(F_n) = 1$ and $\gamma R(F_n) = 2$.

Proof. Since $a \in N[v_i]$ for all i, $\{a\}$ is a dominating set. Hence $\gamma(F_n) = 1$. The bound $\gamma R(F_n) = 2$ follows from Lemma 1. □

Theorem 6. For all $n \geq 2$, $\gamma R(F_n) = 2$.

Proof. Lower bound: weight 1 is impossible (argued as in Theorem 2). Upper bound: $f(a) = 2$, $f(v_i) = 0$ gives an RDF of weight 2.

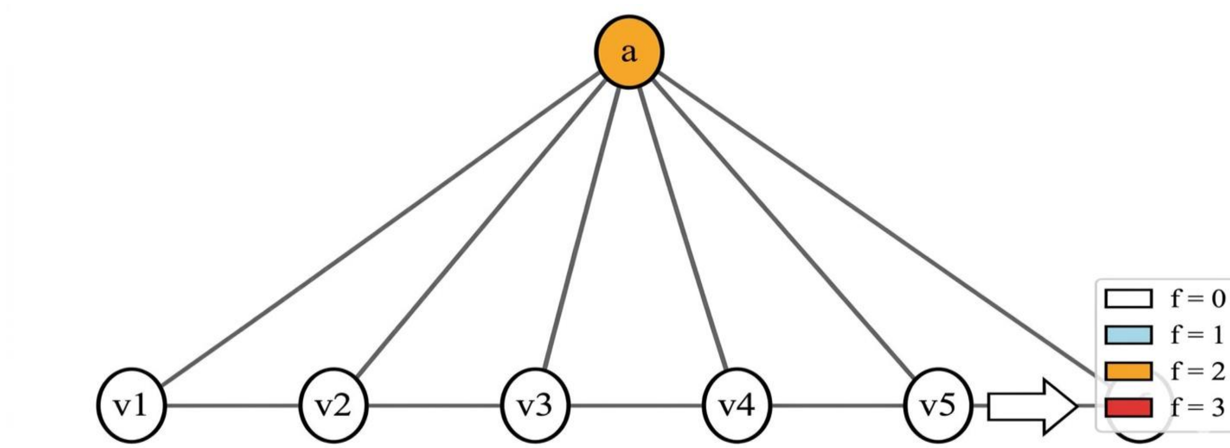

Fig. 5 Optimal Roman dominating function on F6 with γR = 2. The apex a (amber) receives value 2; all path vertices receive value 0.

Theorem 7. For all $n \geq 2$, $\gamma dR(F_n) = 3$.

Proof. Lower bound: Suppose $w(f) = 2$ for some DRDF f. No vertex can have a value of 3. Case (A): If $f(a) = 2$ and all $v_i = 0$, each v_i needs two weight-2 neighbors or one weight-3 neighbor, but v_i has only one weight-2 neighbor (a). Condition (i) fails. If $f(v_j) = 2$ for some j and $f(a) = 0$, then a has at most one weight-2 neighbor. Condition (i) fails for a. Case (B): No vertex has value 2 or 3, so every 0-vertex immediately fails condition (i).

Upper bound: $f(a) = 3$, $f(v_i) = 0$ for all i. Every v_i sees a with $f(a) = 3$, satisfying condition (i). Weight = 3.

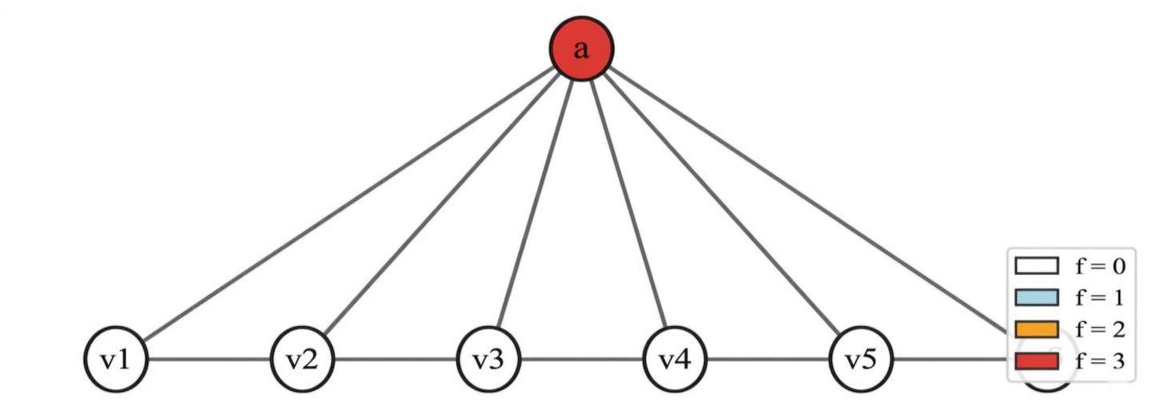

Fig. 6 Optimal double Roman dominating function on F6 with γdR = 3. The apex a (red) receives value 3; all path vertices receive value 0

Remark. The structural symmetry between Wn and Fn is notable: both have a single high-degree vertex whose neighbourhood covers all remaining vertices. The key difference is that in Wn the hub is also part of a cycle (the rim), whereas in Fn the apex is connected only to a path. Nevertheless, both achieve the same optimal values γR = 2 and γdR = 3.

6. Comparative Analysis

Table 1 consolidates the main results. Table 2 gives the asymptotic ratios among the three domination parameters.

Table 1. Summary of domination parameters

Graph	Order	γ(G)	γR(G)	γdR(G)
Wn (n≥3)	n+1	1	2	3
Ln (n≥2)	2n	$\lceil n/2\rceil$	$\lceil n/2\rceil+\lceil n/4\rceil$	$2\lceil n/2\rceil$
Fn (n≥2)	n+1	1	2	3

Table 2. Ratios of domination parameters (n → ∞ for Ln).

Graph	γR/γ	γdR/γR	γdR/γ
Wn	2	3/2	3
Ln (large n)	≈ 3/2	→ 4/3	→ 2
Fn	2	3/2	3

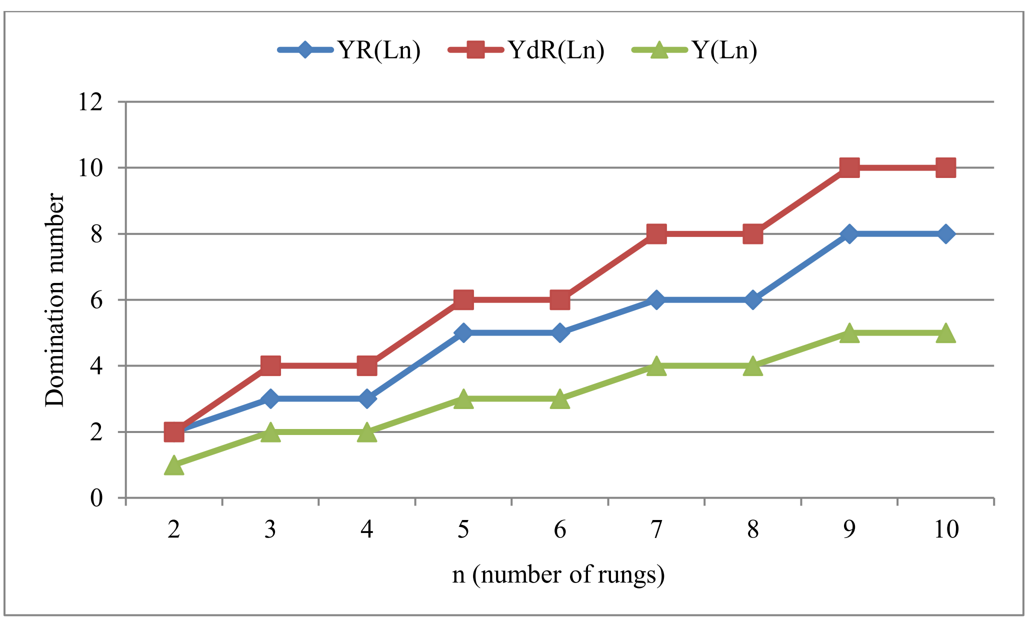

Fig. 7 Growth of domination parameters for ladder graphs Ln (n = 2 to 10). Blue triangles: γ(Ln). Amber circles: γR(Ln). Red squares: γdR(Ln)

Observation 1 (Centralization collapses parameters). Both Wn and Fn achieve the minimum non-trivial values γR = 2 and γdR = 3, independent of n. This is entirely attributable to the presence of a single vertex that dominates all others.

Observation 2 (Distributed structure forces growth). In Ln, which has a maximum degree of 3 and no dominating singleton, both γR(Ln) and γdR(Ln) grow linearly in n. The ratio γdR(Ln)/γR(Ln) tends to 4/3 as $n \to \infty$.

Observation 3. The chain inequalities from Proposition 1 hold strictly in all three families: γ(G) < γR(G) < γdR(G) for sufficiently large n.

7. Conclusion

We have established exact Roman and double Roman domination numbers for wheel, ladder, and fan graphs — each result supported by rigorous proofs and illustrative graph diagrams. The analysis reveals an underlying dichotomy: graphs containing a single dominating vertex have globally minimal domination parameters (for any n), while structurally regular graphs (without such a vertex) demonstrate linear growth of the respective parameters.

Several directions for future research arise naturally:

(i) Helm and Sunflower Graphs: Wheel graphs with pendant vertices attached to rim vertices. It is expected that γR and γdR will grow with the number of pendants.

(ii) Möbius–Kantor and Prism Graphs: Direct products of cycles generalizing ladders can be partitioned into rungs to create toroidal and cylindrical products.

(iii) Weighted and Signed Versions: Signed Roman domination [20] and Italian domination [7] on these families remain open problems.

(iv) Algorithmic Complexity: The complexity of γdR on arbitrary bounded-degree graphs is still largely open [12].

We are confident that the proof structures outlined here — specifically the universal vertex argument and the rungs technique — will provide a valid means of establishing domination parameters for many other graph families.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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References

[1] Hossein Abdollahzadeh Ahangar, Mustapha Chellali, and Seyed Mahmoud Sheikholeslami., “On the Double Roman Domination in Graphs,” *Discrete Applied Mathematics*, vol. 232, pp. 1-7, 2017. [CrossRef] [Google Scholar] [Publisher Link]

[2] J. Amjadi, S.M. Sheikholeslami, and M. Soroudi, “Nordhaus-Gaddum Bounds for Double Roman Domination,” *Journal of Combinatorial Optimization*, vol. 35, no. 1, pp. 126-133, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[3] Robert A. Beeler, Teresa W. Haynes, and Stephen T. Hedetniemi, “Double Roman Domination,” *Discrete Applied Mathematics*, vol. 211, pp. 23-29, 2016. [CrossRef] [Google Scholar] [Publisher Link]

[4] Gerard J. Chang, and George L. Nemhauser, “The *k*-Domination and *k*-Stability Problems on Sun-Free Chordal Graphs,” *SIAM Journal on Algebraic Discrete Methods*, vol. 5, no. 3, pp. 332-345, 1984. [CrossRef] [Google Scholar] [Publisher Link]

[5] Jun Yue et al., “On Double Roman Domination of Graphs,” *Applied Mathematics and Computation*, vol. 338, pp. 669-675, 2018. [CrossRef] [Google Scholar] [Publisher Link]

[6] Ernie J Cockayne et al., “Roman Domination in Graphs,” *Discrete Mathematics*, vol. 278, no. 1-3, pp. 11-22, 2004. [CrossRef] [Google Scholar] [Publisher Link]

[7] Leonard M. Paleta, and Ferdinand P. Jamil, “On Perfect Italian Domination in Graphs,” *Discrete Mathematics, Algorithms and Applications*, vol. 16, no. 7, 2024. [CrossRef] [Google Scholar] [Publisher Link]

[8] B. Javad Ebrahimi, Nafiseh Jahanbakht, and E.S. Mahmoodian, “Vertex Domination of Generalized Petersen Graphs,” *Discrete Mathematics*, vol. 309, no. 13, pp. 4355-4361, 2009. [CrossRef] [Google Scholar] [Publisher Link]

[9] Teresa W. Haynes, Stephen Hedetniemi, and Peter Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998. [Google Scholar] [Publisher Link]

[10] Teresa W. Haynes, Stephen Hedetniemi, and Peter Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998. [Google Scholar] [Publisher Link]

[11] Teresa W. Haynes, Stephen T. Hedetniemi, and Michael A. Henning, *Domination in Graphs: Core Concepts*, Springer, Cham, 2021. [Google Scholar] [Publisher Link]

[12] Michael A. Henning, and William F. Klostermeyer, "Italian Domination in Trees," *Discrete Applied Mathematics*, vol. 217, pp. 557-564, 2017. [Google Scholar] [Publisher Link]

[13] Nader Jafari Rad, and Lutz Volkmann, "Roman Domination Perfect Graphs," *Annals of Mathematics and Artificial Intelligence*, vol. 61, no. 3, pp. 235-250, 2011. [Google Scholar]

[14] Antoaneta Klobučar, and Ana Klobučar, "Properties of Double Roman Domination on Cardinal Products of Graphs," *ARS Mathematica Contemporanea*, vol. 17, no. 2, pp. 337-349, 2019. [CrossRef] [Google Scholar] [Publisher Link]

[15] Hamidreza Maimani et al., "Independent Double Roman Domination in Graphs," *Bulletin of the Iranian Mathematical Society*, vol. 46, pp. 543-555, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[16] Doost Ali Mojdeh, and Zhila Mansouri, "On the Double Roman Domination of Some Graphs," *Iranian Journal of Mathematical Sciences and Informatics*, vol. 16, no. 1, pp. 13-26, 2021. [Google Scholar]

[17] Chakradhar Padamutham, and Venkata Subba Reddy Palagiri, "Complexity of Roman {2}-Domination and the Double Roman domination in Graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 17, no. 3, pp. 1081-1086, 2020. [CrossRef] [Google Scholar] [Publisher Link]

[18] Robert A. Beeler, Teresa W. Haynes, and Stephen T. Hedetniemi, "Double Roman domination," *Discrete Applied Mathematics*, vol. 211, pp. 23-29, 2016. [CrossRef] [Google Scholar] [Publisher Link]

[19] Charles S. ReVelle, and Kenneth E. Rosing, "Defendens Imperium Romanum: A Classical Problem in Military Strategy," *The American Mathematical Monthly*, vol. 107, no. 7, pp. 585-594, 2000. [CrossRef] [Google Scholar] [Publisher Link]

[20] H. Abdollahzadeh Ahangar et al., "Signed Roman Domination in Graphs," *Journal of Combinatorial Optimization*, vol. 32, no. 3, pp. 848-864, 2016. [CrossRef] [Google Scholar] [Publisher Link]