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Original Article

Plithogenic Hesitant Fuzzy Soft Topological Spaces

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Abstract *- This present study introduces Plithogenic hesitant fuzzy soft (P-HFS) topological spaces, a new framework that combines Plithogenic sets, hesitant fuzzy sets, and soft set theory to model uncertainty with attribute-based contradictions. The authors define the basic structure of a Plithogenic hesitant fuzzy soft topological space and develop their topological properties. Additionally, several classical topological concepts are extend to this setting, including P-HFS open sets, P-HFS closed sets, P-HFS interior, P-HFS closure, P-HFS subspace topology, P-HFS basis, and P-HFS product topology. Furthermore, a number of theorems are established to demonstrate the theoretical consistency, validity, and robustness of the proposed framework.*

Keywords *- Hesitant Fuzzy Soft Set, Hesitant Fuzzy Soft topological space, Plithogenic Hesitant Fuzzy Soft Set, Plithogenic Hesitant Fuzzy Soft topological space*.

1. Introduction

The challenge of modelling uncertainty, vagueness, and incomplete information has motivated extensive developments in generalized set theories. Classical set theory, due to its crisp and binary nature, often fails to model such imprecise information effectively. To address this limitation, Zadeh [1] introduced the concept of fuzzy sets, in which each element is associated with a membership degree in the interval [0,1]. Fuzzy set theory opened a new paradigm for representing partial truth and has become a foundational tool for uncertainty. Although fuzzy sets provide an effective mechanism for representing gradation, they lack a parameterization component needed in many decision-making scenarios. To address this, Molodtsov [4] proposed the notion of soft sets in 1999 as a parameterized approach to deal with uncertainty modelling. Subsequently, the integration of fuzzy sets and soft sets resulted in fuzzy soft sets, introduced by Maji, Biswas, and Roy in 2001 [5], where each parameter is associated with a fuzzy subset; they also have investigated some of its theoretical properties [6]. In certain real-world scenarios involving expert systems, belief systems, information fusion, and other domains, we must consider both truth-membership and falsity-membership in order to accurately represent an item in an ambiguous, uncertain context. Atanassov's intuitionistic fuzzy sets [3] are appropriate in this situation. Further, the neutrosophic sets, introduced by Smarandache [7] in 1995, independently define truth, indeterminacy, and falsity membership functions. Neutrosophic sets have shown strong versatility in decision-making, image processing, and information fusion where incomplete, inconsistent, and indeterminate information co-exists. In parallel, researchers recognized that some situations demand the representation of multiple possible membership values due to hesitation or conflicting expert opinions. To model this, Torra and Narukawa [8] and Torra [9] introduced hesitant fuzzy sets in 2010 to address the problem of defining the degree of membership of an element in a set. The hesitant fuzzy element (HFE), first proposed by Xu and Xia [10] in 2011, is a simple and practical method for conveying the hesitating preferences of decision makers during the decision-making process. Further, Das S and Kar S [11] introduced the hesitant fuzzy soft sets, which integrated parameterization with hesitant membership, offering a more expressive framework for modelling group decision-making scenarios.

More recently, the complexity of modern data requires structures capable of handling multiple types of attributes—fuzzy, intuitionistic, and probabilistic—and their interactions. The plithogenic set theory introduced by Smarandache [13] generalizes earlier frameworks by incorporating attribute values, their degrees of appurtenance, and a contradiction measure between attribute values. Smarandache's plithogenic theory is characterized as a broader approach when compared to the existing theory, which includes intuitionistic, fuzzy, crisp, and neutrosophic sets. Extending this theory into the soft set environment gives rise to Alkhazaleh's Plithogenic soft sets, a highly flexible and powerful model capable of managing uncertain and contradictory information simultaneously. Alkhazaleh [16] made significant contributions to the theoretical advancement of similarity measures, operations, and Plithogenic soft sets, as well as some of their applications.

In 1968, Chang [2] developed fuzzy topology, which combines order structures with topological principles to study continuity and related properties in uncertain environments. Later, Mathew et al. [14] introduced the notion of a hesitant fuzzy topology and studied properties like neighbourhoods, closure, and continuity. Building upon this, the study of hesitant fuzzy soft topology introduced by Sreedevi A and Shankar NR [12] provides a sophisticated approach to analyze parameterized uncertain information. Also, Riaz M et al.[15] have studied Hesitant fuzzy soft topology and its applications to multi-attribute group decision-making. These spaces offer a richer topological structure for modelling systems where both parameter dependence (soft sets) and multi-valued membership (hesitant fuzzy sets) are present.

Existing hesitant fuzzy soft topological spaces can handle parameterized hesitation, but they do not explicitly account for conflicts among attribute values. Therefore, there is no unified topological framework capable of simultaneously addressing hesitation, parameterization, and contradiction degrees. This constraint limits the usefulness of existing models in complex real-world scenarios in which these types of uncertainty coexist. The present work addresses this research gap by introducing Plithogenic hesitant fuzzy soft topology, which combines the benefits of Plithogenic sets, hesitant fuzzy sets, and soft sets, resulting in a richer and more flexible mathematical structure for modeling uncertainty, hesitation, and contradiction within a unified topological framework.

Moreover, this paper presents Plithogenic hesitant fuzzy soft topology and topological properties, including open sets, closed sets, closure, interior, subspace topology, basis for a topology and product topology within this generalized framework, offering powerful tools for modelling and analyzing systems under high degrees of uncertainty and hesitation.

2. Preliminaries

In this section, we review some of the basic definitions required to read this paper.

Definition 2.1

A pair (F, E) is called a soft set (over U) if and only if F is a mapping of E into the set of all subsets of the set U.

Definition 2.2

A fuzzy soft set over *U* is a pair (F, E) where $F: A \rightarrow P(U)$ is a mapping from *A* into *P(U).*

Definition 2.3

If *X* is a fixed set, then a Hesitant fuzzy set on *X* is defined in terms of a function $h_M(x)$ that returns a subset of [0,1], when it is applied to *X*. This can be expressed as follows: $M = \{< x, h_M(x)> \mid x \in X\}$, where $h_M(x)$ is a set of values in [0,1] that represent the various membership degrees of the element $x \in X$ to the set *M*. We denote $h = h_M(x)$ as a hesitant fuzzy element (HFE) and *H* is the set of all HFEs.

Definition 2.4

Let *U* be a universal set, and *E* be a set of parameters. Let *HF(U)* be the set of all hesitant fuzzy sets defined on *U*. A pair *(F, E)* is a hesitant fuzzy soft set if $F(e) \in HF(U)$ for every *e* in *E*.

Definition 2.5

Consider *U* to be the discourse universe, and P to be a nonempty set of elements that is a subset of *U*. Then, *a* is an attribute (generally multidimensional), *V* is the range of the attribute values, *d* is the degree to which each element*x*'s attribute value belongs to the set *P* in relation to a given set of criteria $(x \in P)$, and *c* is the contradiction degree between attribute values. Then, *(P, a, V, d, c)* is called a Plithogenic set.

Definition 2.6

Assume that U is a discourse universe and that. $(U)^z$ is the z-power set of *U* such that

$z = C$ (Power set of *U*),

$z = F$ (The set of all fuzzy sets of *U*),

$z = IF$ (The set of all intuitionistic fuzzy sets of *U*), or

$z = N$ (The set of all neutrosophic sets of *U*).

Let $a_1, a_2, \cdots, a_n$ for $n \geq 1$, be *n* distinct attributes, where corresponding attribute values $V_1, V_2, \cdots, V_n$, with $V_i \cap V_j = \emptyset$, for $j \neq i$, and $i, j \in \{1, 2, \cdots, n\}$. Suppose $V_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{mi}}\}$ and let $\Upsilon = V_1 \times V_2 \times \cdots \times V_n$. Let $c\left(D_i, v_{ij}\right)$, $i \in \{1,2, \cdots n\}$,

$j \in \{1,2,\cdots m_i\}$ be the attribute value contradiction degree such that $c_i: V_i \times V_i \to [0,1]$, and let $D = (D_1, D_2, \cdots D_n)$ the dominant attribute element of $V_i, \forall\, i$. Then the pair (F_P^Z, Υ), where $F_P^Z: \Upsilon \to [0,1]_D \times (U)^z$ is called a Plithogenic Soft Set (P-SS In short) over U.

Definition 2.7

Let *U* be a universe of discourse, $(U)^H$ the power set of *U,* such that *H* is the set of all Hesitant fuzzy soft sets. Let $a_1, a_2, \cdots, a_n$, for $n \geq 1$, be *n* distinct attributes, whose corresponding attribute values are respectively the sets $V_1, V_2, \cdots, V_n$, with $V_i \cap V_j = \emptyset$, , for $j \neq i$, and $i,j \in \{1,2,\cdots,n\}$. Suppose $V_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{mi}}\}$ and let $\mathcal{H} = V_1 \times V_2 \times \cdots \times V_n$. Let $D = (D_1, D_2, \cdots D_n)$ the dominant attribute element of V_i, $\forall i$ and $c\left(D_i, v_{ij}\right)$, where $i \in \{1, 2, \cdots n\}$, $j \in \{1,2,\cdots m_i\}$ the attribute value contradiction degree function such that $c_i: V_i \times V_i \to [0,1]$. Then the pair $\left(\mathrm{F}_\mathrm{P}^\mathrm{H}, \mathcal{H}\right)$, where $F_P^H : \mathcal{H} \to [0,1]_D \times (U)^H$ is called a Plithogenic hesitant fuzzy soft set (P-HFSS In short) over *U*.

Definition 2.8

Let *X* be a nonempty set and let $\tau \subseteq HS(X)$. Then, τ is called a hesitant fuzzy topology on *X* if the following axioms are satisfied:

(i) $h^0, h^1 \in \tau$
(ii) $h_1 \cap h_2 \in \tau$, for any $h_1, h_2 \in \tau$
(iii) $\bigcup_{j\in J} h_j \in \tau$, for each $h_j \in \tau$.

Then *(X,* τ*)* is called a hesitant fuzzy topological space and the elements in τ is called hesitant fuzzy open sets. If $h^c \in \tau$, for a hesitant fuzzy set *h* in *X*, then *h* hesitant fuzzy closed set in *(X,* τ*)*.

Definition 2.9

Let τ be the collection of HFS-sets over *X*, then τ is said to be an HFS topology on *X* if it satisfies the following:

(i) $\emptyset, X \in \tau$.
(ii) $h_A \cap h_B \in \tau$, for any $h_A, h_B \in \tau$
(iii) $\bigcup_{i\in I} h_i \in \tau$, for each $h_i \in \tau$.

Then (X, τ, E) is called a HFS-topological space over *X*.

Definition 2.10

The HFS interior of HFS set (hfs, E) is denoted by $(hfs, E)^0$ and is defined as the union of all HFS open subsets of (hfs, E). The HFS closure of HFS set (hfs, E) is defined as the intersection of all HFS closed super sets of (hfs, E) and is denoted by $\overline{(hfs, E)}$.

3. Plithogenic Hesitant Fuzzy Soft Topology

In this section, we define the concept of Plithogenic hesitant fuzzy soft topology and some of its properties.

Definition 3.1

A Plithogenic hesitant fuzzy topology (P-HFST) on a set *X* is a collection τ_{PH} of plithogenic hesitant fuzzy soft sets in *X* which satisfy the following properties:

(i) ϕ_{PH} , $X_{\mathrm{PH}} \in \tau_{\mathrm{PH}}$.
(ii) $\bigcup_i (F_i, E) \in \tau_{\mathrm{PH}}$, for each $(F_i, E) \in \tau_{\mathrm{PH}}$.
(iii) $(F_1, E) \cap (F_2, E) \in \tau_{\mathrm{PH}}$, for any (F_1, E) , $(F_2, E) \in \tau_{PH}$.

Then, (X, τ_{PH}) is called a Plithogenic hesitant fuzzy soft topological space (P-HFSTS) over *X*.

Example 3.2

Consider $X = \{R_1, R_2, R_3\}$ be three smart irrigation systems being evaluated for adoption on agricultural farms. Let the distinct attributes. a_1 = Soil moisture, a_2 = Weather condition, a_3 = Soil type, a_4 = Budget and its corresponding attribute values are : Soil moisture $\equiv A_1$ = {Low, Medium, High}, Weather condition $\equiv A_2$ = {Cloudy, Sunny, Rainy}, Soil type $\equiv A_3$ = {Clay, Loamy, Sandy}, Budget $\equiv A_4$ = {Low, Moderate, High}, where D = (Medium, Sunny, Sandy, High) denotes the dominant attribute values. Let $E \subseteq \mathcal{H}$ such that $E = \{\varepsilon_1$ = (Low, Rainy, Clay, Low), ε_2 = (Medium, Cloudy, Loamy, Low)}.

Let us consider two P-HFSSs (F_P^H, E) and (G_P^H, E) are as follows:

$$(F_P^H, E) = \left\{\varepsilon_1, \left\{\frac{(R_1,(0.6,0.8,0.5,0.9)_D)}{\{(0.1,0.2),(0.3,0.4),(0.25),(0.05,1)\}}, \frac{(R_2,(0.6,0.8,0.5,0.9)_D)}{\{(0.2,0.3),(0.1,0.25),(0.4,0.45),(0.15)\}}, \frac{(R_3,(0.6,0.8,0.5,0.9)_D)}{\{(0.05),(0.5),(0.1,0.2),(0.02,0.08)\}}\right\}\right.$$

$$\varepsilon_2,\left\{\frac{(R_1,(0,0.4,0.3,0.9)_D)}{\{(0.55,0.6),(0.4),(0.35),(0.1,0.2)\}},\frac{(R_2,(0,0.4,0.3,0.9)_D)}{\{(0.6),(0.25,0.3),(0.4,5),(0.08)\}},\frac{(R_3,(0,0.4,0.3,0.9)_D)}{\{(0.5,0.52),(0.15),(0.2,0.3),(0.05)\}}\right\}\Bigg\}$$

and

$$(G_P^H,E)=\quad\Bigg\{\varepsilon_1,\left\{\frac{(R_1,(0.6,0.8,0.5,0.9)_D)}{\{(0.12,0.18),(0.25),(0.3),(0.06)\}},\frac{(R_2,(0.6,0.8,0.5,0.9)_D)}{\{(0.18),(0.12,0.20),(0.35),(0.12,0.18)\}},\frac{(R_3,(0.6,0.8,0.5,0.9)_D)}{\{(0.04,0.09),(0.45,0.56),(0.08),(0.01)\}}\right\}$$

$$\varepsilon_2,\left\{\frac{(R_1,(0,0.4,0.3,0.9)_D)}{\{(0.5),(0.45,0.5),(0.3,0.4),(0.09)\}},\frac{(R_2,(0,0.4,0.3,0.9)_D)}{\{(0.58,0.62),(0.22),(0.48),(0.07)\}},\frac{(R_3,(0,0.4,0.3,0.9)_D)}{\{(0.46),(0.18,0.21),(0.25),(0.04)\}}\right\}\Bigg\}$$

Then the collection
$\tau_{PH}=\{\phi_{PH},(F_P^H,E),(G_P^H,E),(F_P^H,E)\cup(G_P^H,E),(F_P^H,E)\cap(G_P^H,E),X\}$ forms a P-HFS topology on *X*.

Also, the collections $\tau_{PH}=\{\phi_{PH},(F_P^H,E),(G_P^H,E),(F_P^H,E)\cup(G_P^H,E),X\}$ and

$\tau_{PH}=\{\phi_{PH},(F_P^H,E),(G_P^H,E),(F_P^H,E)\cup(G_P^H,E),(F_P^H,\varepsilon_1),(G_P^H,\varepsilon_1),(F_P^H,\varepsilon_2),(G_P^H,\varepsilon_2),X\}$ are not P-HFS topologies on *X*.

Definition 3.3
Let (X,τ_{PH}) be a P-HFSTS on *X*. Then the elements of τ_{PH} are said to be the plithogenic hesitant fuzzy soft open sets in *X*.

Definition 3.4
Let (X,τ_{PH}) be a P-HFSTS over *X*. A P-HFSS (F_P^H,E) is said to be a plithogenic hesitant fuzzy soft closed set in *X* if its complement $(F_P^H,E)^C$ is P-HFSS open.

Remark 3.5
Let (X,τ_{PH}) be a P-HFSTS over *X*. Then
(i) ϕ_{PH} and X_{PH} belongs to τ_{PH} are closed sets in *X*.
(ii) The arbitrary intersection of P-HFS closed sets is a P-HFS closed set over *X*.
(iii) The finite union of P-HFS closed sets is a P-HFS closed set over *X*.

Definition 3.6
Let *X* be a universal set and τ_{PH} be a collection of all Plithogenic hesitant fuzzy soft sets on *X*. Then,
(i) The collection $\tau_{PH}^I=\{\phi_{PH},X_{PH}\}$ is called Plithogenic hesitant fuzzy soft indiscrete topology on *X* and (X,τ_{PH}^I) is the Plithogenic hesitant fuzzy soft indiscrete topological space on *X*.
(ii) The collection $\tau_{PH}^D=P_{PH}(X)$ is called Plithogenic hesitant fuzzy soft discrete topology on *X* and (X,τ_{PH}^D) is the Plithogenic hesitant fuzzy soft discrete topological space on *X*.

Definition 3.7
Let (X,τ_{PH}) and (X,τ_{PH}^*) be two P-HFSTSs over *X*. If $\tau_{PH}\subseteq\tau_{PH}^*$, then τ_{PH}^* is said to be P-HFS finer than τ_{PH} and τ_{PH} is said to be P-HFS coarser than τ_{PH}^*. If neither $\tau_{PH}\subseteq\tau_{PH}^*$ nor $\tau_{PH}^*\subseteq\tau_{PH}$, then τ_{PH} and τ_{PH}^* They are not comparable.

Proposition 3.8
Let (X,τ_{PH}) and (X,τ_{PH}^*) be two P-HFSTSs over *X*. Then $(X,\tau_{PH}\cap\tau_{PH}^*)$ is a P-HFSTS over *X*.

Proof:

(i) Since ϕ_{PH} , $X_{PH}\in\tau_{PH}$ and ϕ_{PH} , $X_{PH}\in\tau_{PH}^*$, ϕ_{PH} , $X_{PH}\in\tau_{PH}\cap\tau_{PH}^*$.
(ii) Let (F_P^H,A), $(G_P^H,B)\in\tau_{PH}\cap\tau_{PH}^*$.Then, (F_P^H,A), $(G_P^H,B)\in\tau_{PH}$ and (F_P^H,A), $(G_P^H,B)\in\tau_{PH}^*$. Since $(F_P^H,A)\cap(G_P^H,B)\in\tau_{PH}$ and $(F_P^H,A)\cap(G_P^H,B)\in\tau_{PH}^*$, $(F_P^H,A)\cap(G_P^H,B)\in\tau_{PH}\cap\tau_{PH}^*$.
(iii) Let $\{(F_P^H,\varepsilon_i),\varepsilon_i\in E\}$ be a family of P-HFSSs in $\tau_{PH}\cap\tau_{PH}^*$. Then $\cup_{\varepsilon_i\in E}(F_P^H,\varepsilon_i)\in\tau_{PH}$ and $\cup_{\varepsilon_i\in E}(F_P^H,\varepsilon_i)\in\tau_{PH}^*$. Therefore, $\cup_{\varepsilon_i\in E}(F_P^H,\varepsilon_i)\in\tau_{PH}\cap\tau_{PH}^*$.

Remark 3.9
The union of two Plithogenic hesitant fuzzy soft topologies on X may not be a P-HFST on X.

Example 3.10

Consider Example 3.2 with $\tau_{PH} = \{\phi_{PH}, (F_P^H, E), X\}$ and $\tau_{PH}^* = \{\phi_{PH}, (G_P^H, E), X\}$ As two Plithogenic hesitant fuzzy soft topologies over *X*. Now let us consider, $\tau_{PH} \cup \tau_{PH}^* = \{\phi_{PH}, (F_P^H, E), (G_P^H, E), X\}$. It is clear that $(F_P^H, E) \cap (G_P^H, E) \notin \tau_{PH} \cup \tau_{PH}^*$, since the set $(H_P^H, E) = (F_P^H, E) \cap (G_P^H, E)$

$$(H_P^H, E) = \left\{ \varepsilon_1, \left\{ \begin{matrix} \frac{(\mathrm{R}_1,(0.6,0.8,0.5,0.9)_{\mathrm{D}})}{\{(011,0.14,0.16,0.19),(0.275,0.325),(0.275),(0.055,0.53)\}}, \frac{(\mathrm{R}_2,(0.6,0.8,0.5,0.9)_{\mathrm{D}})}{\{(0.19,0.24),(0.11,0.15,0.185,0.225),(0.375,0.40),(0.135,0.165)\}}, \\ \frac{(\mathrm{R}_3,(0.6,0.8,0.5,0.9)_{\mathrm{D}})}{\{(0.045,0.07),(0.475,0.53),(0.09,0.14),(0.015,0.045)\}} \end{matrix} \right\} \right.$$

$$\left. \varepsilon_2, \left\{ \begin{matrix} \frac{(\mathrm{R}_1,(0,0.4,0.3,0.9)_{\mathrm{D}})}{\{(0.525,0.55),(0.425,0.45),(0.325,0.375),(0.095,0.145)\}}, \frac{(\mathrm{R}_2,(0,0.4,0.3,0.9)_{\mathrm{D}})}{\{(0.59,0.61),(0.235,0.26),(0.44,0.49),(0.075)\}}, \\ \frac{(\mathrm{R}_3,(0,0.4,0.3,0.9)_{\mathrm{D}})}{\{(0.48,0.49),(0.165,0.18),(0.225,0.275),(0.045)\}} \end{matrix} \right\} \right\}$$

is not equal to any of the sets (F_P^H, E) and (G_P^H, E). So it fails to be closed under the plithogenic intersection. Hence, $\tau_{PH} \cup \tau_{PH}^*$ is not a P-HFS topology on *X*.

Definition 3.11

Let (X, τ_{PH}) be a P-HFSTS over *X* and (F_P^H, A) be a P-HFSS over *X*, $x \in X$. Then, (F_P^H, A) is said to be P-HFS neighbourhood of *x* if there exists a P-HFS open set (G_P^H, B) such that $x \in (G_P^H, B) \subseteq (F_P^H, A)$.

Proposition 3.12

Let (X, τ_{PH}) be a P-HFSTS over *X*. Then,

(i) If (F_P^H, A) is said to be P-HFS neighbourhood of $x \in X$, then $\in (F_P^H, A)$.
(ii) If (F_P^H, A) and (G_P^H, B) are two P-HFS neighbourhood of some $\in X$, then $(F_P^H, A) \cap (G_P^H, B)$ is also a P-HFS neighbourhood of $x \in X$.
(iii) If (F_P^H, A) is a P-HFS neighbourhood of $x \in X$ and $(F_P^H, A) \subseteq (G_P^H, B)$, then (G_P^H, B) is also a P-HFS neighbourhood of x.

Definition 3.13

Let (X, τ_{PH}) be a P-HFSTS over *X* and let (F_P^H, A) be a P-HFSS on *X*. A point $x \in X$ is said to be a P-HFS limit point of (F_P^H, A) if $(F_P^H, A) \cap [(G_P^H, B) - \{x\}] \neq \phi_{PH}$ for every P-HFS open set (G_P^H, B) containing x. The set of all P-HFS limit points of (F_P^H, A) are denoted by $(F_P^H, A)^L$.

Proposition 3.14

Let (X, τ_{PH}) be a P-HFSTS over *X* and let (F_P^H, A) and (G_P^H, B) are two P-HFSSs on *X*. Then, the following holds:

(i) $(F_P^H, A) \subseteq (G_P^H, B)$ implies $(F_P^H, A)^L \subseteq (G_P^H, B)^L$.
(ii) $[(F_P^H, A) \cap (G_P^H, B)]^L \subseteq (F_P^H, A)^L \cap (G_P^H, B)^L$.
(iii) $[(F_P^H, A) \cup (G_P^H, B)]^L = (F_P^H, A)^L \cap (G_P^H, B)^L$.

Remark 3.15

The converse of the above proposition 3.14(ii) need not be true.

Proof:

Let $x \in (F_P^H, A)^L \cap (G_P^H, B)^L$. Then for every neighbourhood *X* of *x*, $X \cap (F_P^H, A) \neq \phi_{PH}$ and $X \cap (G_P^H, B) \neq \phi_{PH}$. Since a neighbourhood *X* of *x* does not intersect $(F_P^H, A) \cap (G_P^H, B)$, $X \cap (F_P^H, A) \cap (G_P^H, B) = \phi_{PH}$. Thus, *x* may not be a limit point of $(F_P^H, A) \cap (G_P^H, B) = \phi_{PH}$ which implies that $x \notin (F_P^H, A) \cap (G_P^H, B)$. So,$(F_P^H, A)^L \cap (G_P^H, B)^L \nsubseteq [(F_P^H, A) \cap (G_P^H, B)]^L$.

Definition 3.16

Let (X, τ_{PH}) be a P-HFSTS over *X* and let $Y \subseteq X$. The subspace topology on Y is the collection. $\tau_{PH}^y = \{U \cap Y : U \in \tau_{PH}\}$. The pair (Y, τ_{PH}^y) is called a subspace of *X*.

Proposition 3.17

Let (Y, τ_{PH}^{y}) be a P-HFS subspace of a P-HFSTS (X, τ_{PH}) and (G_P^H, B) be a P-HFS open set in Y. If $Y \in \tau_{PH}$, then $(G_P^H, B) \in \tau_{PH}$.

Proof:

Let (G_P^H, B) be a P-HFS open set in Y. Then there exists a P-HFS open set. (F_P^H, A) in X such that $(G_P^H, B) = Y \cap (F_P^H, A)$. Since $\in \tau_{\mathrm{PH}}$, $Y \cap \left(\mathrm{F_P^H}, \mathrm{A}\right) \in \tau_{\mathrm{PH}}$. Therefore, $(G_P^H, B) \in \tau_{PH}$.

Proposition 3.18

Let (Y, τ_{PH}^{y}) and (Z, τ_{PH}^{z}) are two P-HFS subspaces of a P-HFSTS (X, τ_{PH}) and let $Y \subseteq Z$. Then, (Y, τ_{PH}^{y}) is a P-HFS subspace of (Z, τ_{PH}^{z}).

Definition 3.19

Let (X, τ_{PH}) be a P-HFSTS over *X*. A P-HFS basis for τ_{PH} is a collection $\mathrm{ß_{PH}}$ of subsets of *X* such that

(i) For each $\in X$, there is at least one element $\mathrm{ß'_{PH}} \subseteq \mathrm{ß_{PH}}$ such that $x \in \mathrm{ß_{PH}}$.
(ii) If $x \in \mathrm{ß'_{PH}} \cap \mathrm{ß''_{PH}}$, where $\mathrm{ß'_{PH}}, \mathrm{ß''_{PH}} \in \mathrm{ß_{PH}}$ Then there exists $\mathrm{ß'''_{PH}} \in \mathrm{ß_{PH}}$ such that $x \in \mathrm{ß'''_{PH}}$ and $\mathrm{ß'''_{PH}} \subseteq \mathrm{ß'_{PH}} \cap \mathrm{ß''_{PH}}$.

Definition 3.20

Let $\mathrm{ß_{PH}}$ be a P-HFS basis for τ_{PH} on *X*. Then $\tau_{PH} = \left\{\mathrm{F_P^H} : \mathrm{F_P^H} = \cup\, \mathrm{ß'_{PH}}, \text{for some } \mathrm{ß'_{PH}} \subseteq \mathrm{ß_{PH}}\right\}$ is called P-HFST, generated by $\mathrm{ß_{PH}}$.

Proposition 3.21

Let (X, τ_{PH}) be a P-HFSTS over *X* and let $\mathrm{ß_{PH}}$ be a P-HFS basis for τ_{PH}.Then τ_{PH} equals the collection of all unions of elements of $\mathrm{ß_{PH}}$.

Proof:

It is obvious from Definition 3.19.

Proposition 3.22

Let (X, τ_{PH}) be a P-HFSTS. Suppose that $\mathrm{ß_{PH}}$ is a collection of P-HFS open sets of *X* such that for each P-HFS open set *U* of *X* and each *x* in *U*, there is an element *B* of $\mathrm{ß_{PH}}$ such that $x \in B \subseteq U$. Then $\mathrm{ß_{PH}}$ is a P-HFS basis for the topology of *X*.

Proposition 3.23

Let $\mathrm{ß_{PH}}$ and $\mathrm{ß'_{PH}}$ be a P-HFS bases for τ_{PH} and τ'_{PH} respectively on *X*. If $\mathrm{ß'_{PH}} \subseteq \mathrm{ß_{PH}}$, then τ'_{PH} is finer than τ_{PH}.

Proof:

Let $\mathrm{ß'_{PH}} \subseteq \mathrm{ß_{PH}}$. Then for each $\mathrm{F_P^H} \in \tau'_{\mathrm{PH}}$ and $\mathrm{G_P^H} \in \mathrm{ß'_{PH}}$,$\mathrm{F_P^H} = \bigcup_{\mathrm{G_P^H} \in \mathrm{ß'_{PH}}} \mathrm{G_P^H} = \bigcup_{\mathrm{G_P^H} \in \mathrm{ß_{PH}}} \mathrm{G_P^H}$. Therefore, $\mathrm{F_P^H} \in \tau_{\mathrm{PH}}$. Hence, $\tau'_{\mathrm{PH}} \subseteq \tau_{\mathrm{PH}}$.

Definition 3.24

Let *X* and *Y* are the P-HFSTSs. The P-HFS product topology on $X \times Y$ is the topology having as basis the collection $\mathrm{ß_{PH}}$ of all sets of the form $\times V$, where U is a P-HFS open subset of *X* and V is a P-HFS open subset of *Y*.

Proposition 3.25

If $\mathrm{ß_{PH}}$ is a P-HFS basis for the P-HFST of X and $\mathrm{ß'_{PH}}$ is a P-HFS basis for the P-HFST of *Y*, then the collection $\mathrm{ß''_{PH}} = \left\{\mathrm{F_P^H} \times \mathrm{G_P^H} : \mathrm{F_P^H} \in \mathrm{ß_{PH}}, \mathrm{G_P^H} \in \mathrm{ß'_{PH}}\right\}$ is a P-HFS basis for the P-HFST of $X \times Y$.

Proof:

Let a P-HFS open set Z of $X \times Y$ and a point $x \times y$ of Z . Then by Definition 3.24, there is a basis element $U \times V$ such that $x \times y \in U \times V \subseteq Z$. Since $\mathrm{ß_{PH}}$ and $\mathrm{ß'_{PH}}$ are bases for *X* and *Y*, there exist elements $\mathrm{F_P^H} \in \mathrm{ß_{PH}}$ and $\mathrm{G_P^H} \in \mathrm{ß'_{PH}}$ such that $x \in \mathrm{F_P^H} \subseteq U$ and $y \in \mathrm{G_P^H} \subseteq V$. Then, $x \times y \in \mathrm{F_P^H} \times \mathrm{G_P^H} \subseteq Z$. Thus, by Proposition 3.22, $\mathrm{ß''_{PH}}$ is a P-HFS basis for the P-HFS topology of $X \times Y$.

Definition 3.26

Let (X, τ_{PH}) be a P-HFSTS over *X* and let (F_P^H, A) P-HFSS over *X*. Then,

(i) The union of all P-HFS open sets contained in (F_P^H, A) is called P-HFS interior of (F_P^H, A) over X and is denoted by $int\big((F_P^H, A)\big)$.
(ii) The intersection of all P-HFS closed sets containing. (F_P^H, A) is called the P-HFS closure of (F_P^H, A) over X and is denoted by $cl\big((F_P^H, A)\big)$.

Definition 3.27

Let (F_P^H, A) be a P-HFSS over *X*. Then, the interior of the complement of (F_P^H, A) is called P-HFS-exterior and is denoted by $ext\big((F_P^H, A)\big)$.

(i.e) $ext\big((F_P^H, A)\big) = int((F_P^H, A)^c)$.

Definition 3.28

Let (X, τ_{PH}) be a P-HFSTS over *X* and let (F_P^H, A) be a P-HFSS over *X*, $x \in X$. Then, x is called a P-HFS interior point of (F_P^H, A) If there exist an P-HFS open set (G_P^H, B) such that $x \in (G_P^H, B) \subseteq (F_P^H, A)$.

Proposition 3.29

Let (X, τ_{PH}) be a P-HFSTS on *X* and let (F_P^H, A) P-HFSS over *X*. Then

(i) $int\big((F_P^H, A)\big)$ is the largest open set contained in (F_P^H, A).
(ii) (F_P^H, A) is a P-HFS open set if and only if $(F_P^H, A) = int\big((F_P^H, A)\big)$.

Proof:

(i) It is obvious from Definition 3.26 (i).
(ii) Let (F_P^H, A) is a P-HFS open set. By (i), $int\big((F_P^H, A)\big)$ is the largest open set of (F_P^H, A). Thus, $(F_P^H, A) = int\big((F_P^H, A)\big)$. Conversely, assume $(\mathrm{F_P^H}, \mathrm{A}) = int\left((\mathrm{F_P^H}, \mathrm{A})\right)$. By (i), it is clear that (F_P^H, A) is a P-HFS open set.

Proposition 3.30

Let (X, τ_{PH}) be a P-HFSTS over *X* and let (F_P^H, A) and (G_P^H, B) are two P-HFSSs over *X*. Then,

(i) $\phi_{PH} = int(\phi_{PH})$ and $X_{PH} = int(X_{PH})$.
(ii) $int\big((F_P^H, A)\big) \subseteq (F_P^H, A)$.
(iii) $(F_P^H, A) \subseteq (G_P^H, B)$ implies $int\big((F_P^H, A)\big) \subseteq int\big((G_P^H, B)\big)$.
(iv) $int\big((F_P^H, A)\big) \cap int\big((G_P^H, B)\big) = int[(F_P^H, A) \cap (G_P^H, B)]$.
(v) $int\big((F_P^H, A)\big) \cup int\big((G_P^H, B)\big) \subseteq int[(F_P^H, A) \cup (G_P^H, B)]$.
(vi) $int\big(int\ (F_P^H, A)\big) = int\big((F_P^H, A)\big)$.

Proof:

(i) It is obvious from Definition 3.26.
(ii) Let $x \in int\big((F_P^H, A)\big)$.Then, (F_P^H, A) is a P-HFS neighbourhood of x. Hence $int\big((F_P^H, A)\big) \subseteq (F_P^H, A)$.
(iii) Let $x \in int\big((F_P^H, A)\big)$.Then, x is a P-HFS interior point of (F_P^H, A) implies (F_P^H, A) is a P-HFS neighbourhood of x. Since $(F_P^H, A) \subseteq (G_P^H, B)$, $(\mathrm{G_P^H}, \mathrm{B})$ is also P-HFS neighbourhood of x, which implies $x \in int\big((G_P^H, B)\big)$.
Hence $\mathrm{int}\left((\mathrm{F_P^H}, \mathrm{A})\right) \subseteq \mathrm{int}\left((\mathrm{G_P^H}, \mathrm{B})\right)$.
(iv) Let $x \in \big[int\big((F_P^H, A)\big) \cap int\big((G_P^H, B)\big)\big]$. Then, $x \in int\big((F_P^H, A)\big)$ and
$x \in int\big((G_P^H, B)\big)$ which implies x is a P-HFS interior point of (F_P^H, A) and (G_P^H, B).Thus, (F_P^H, A) and (G_P^H, B) are P-HFS neighbourhood of x and hence $(F_P^H, A) \cap (G_P^H, B)$ is also P-HFS neighbourhood of x.
Therefore, $x \in int[(F_P^H, A) \cap (G_P^H, B)]$.
(v) Let $(F_P^H, A) \subseteq (F_P^H, A) \cup (G_P^H, B)$ and$(G_P^H, B) \subseteq (F_P^H, A) \cup (G_P^H, B)$.
The $int\big((F_P^H, A)\big) \subseteq int\big((F_P^H, A) \cup (G_P^H, B)\big)$ and $int\big((G_P^H, B)\big) \subseteq int\big((F_P^H, A) \cup (G_P^H, B)\big)$, by (iii).
Hence $int\big((F_P^H, A)\big) \cup int\big((G_P^H, B)\big) \subseteq int[(F_P^H, A) \cup (G_P^H, B)]$.
(vi) Since $int\big((F_P^H, A)\big)$ is P-HFS open, $int\big(int\ (F_P^H, A)\big) = int\big((F_P^H, A)\big)$, by Proposition 3.29.

Proposition 3.31

Let (X, τ_{PH}) be a P-HFSTS over X and let (F_P^H, A) and (G_P^H, B) are two P-HFSSs over *X*. Then,

(i) $cl\left((F_P^H, A)\right)$ is the smallest closed set containing (F_P^H, A).
(ii) (F_P^H, A) is a P-HFS closed set if and only if $(F_P^H, A) = cl\left((F_P^H, A)\right)$.

Proof:

(i) It is obvious from Definition 3.26(ii).
(ii) Let (F_P^H, A) is a P-HFS closed set. By (i), $cl\left((F_P^H, A)\right)$ is the smallest closed set of $\left(\mathrm{F_P^H}, \mathrm{A}\right)$. Thus, $(F_P^H, A) = cl\left((F_P^H, A)\right)$. Conversely, assume $(F_P^H, A) = cl\left((F_P^H, A)\right)$. By (i), it is clear that. (F_P^H, A) is a P-HFS closed set.

Proposition 3.32
Let (X, τ_{PH}) be a P-HFS topological space over *X* and let (F_P^H, A) and (G_P^H, B) are two P-HFSSs over *X*. Then,

(i) $\phi_{PH} = cl(\phi_{PH})$ and $X_{PH} = cl(X_{PH})$.
(ii) $(F_P^H, A) \subseteq cl\left((F_P^H, A)\right)$.
(iii) $(F_P^H, A) \subseteq (G_P^H, B)$ implies $cl\left((F_P^H, A)\right) \subseteq cl\left((G_P^H, B)\right)$.
(iv) $cl\left((F_P^H, A) \cup (G_P^H, B)\right) = cl\left((F_P^H, A)\right) \cup cl\left((G_P^H, B)\right)$.
(v) $cl[(F_P^H, A) \cap (G_P^H, B)] \subseteq cl\left((F_P^H, A)\right) \cap cl\left((G_P^H, B)\right)$.
(vi) $cl\left(cl\ (F_P^H, A)\right) = cl\left((F_P^H, A)\right)$.

Proposition 3.33
Let (X, τ_{PH}) be a P-HFSTS over *X* and let (F_P^H, A) P-HFSS over *X*. Then $int\left((F_P^H, A)\right) \subseteq (F_P^H, A) \subseteq cl\left((F_P^H, A)\right)$.

Proof: It follows from definition 3.26.

4. Conclusion

In this paper, we have introduced the Plithogenic hesitant fuzzy soft topology, which provides a powerful and flexible mathematical framework for modelling complex and uncertain systems. Further, we defined the notions of P-HFS open sets, P-HFS closed sets, P-HFS interior, P-HFS closure, P-HFS subspace topology, P-HFS basis and P-HFS product topology and studied some of their properties by extending the classical concepts. By integrating hesitant fuzzy evaluations with plithogenic contradiction-based operations, the proposed model provides a more comprehensive representation of uncertainty. Future research may focus on the development of additional topological concepts such as compactness, connectedness and continuity within this space. Also, the framework can be integrated with Plithogenic statistical methods, machine learning techniques, optimization models, and decision-support systems to address real-world problems involving complex uncertainty.

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