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Original Article

On the Norm of Jordan Elementary Operators in Tensor Product of C^*-Algebras

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Abstract *- Elementary operators have been studied over years, with their norms being of significant interest in operator theory. The study includes the derivation of formulas that describe norms in terms of their coefficient operators, which is traced back to Stampfli`s Theorem that used the property of Numerical range as a foundation for the study of norms. Other properties of Elementary operators have been studied ever since, but little is known on Jordan Elementary operators in tensor products of C^*-algebras. This paper aims to extend the determination of norms of Jordan Elementary operators in the tensor product of C^*-algebra by determining the lower bound of the norm using the maximal numerical range by employing the technique of tensor product, finite rank operator and inner product. A lower bound of Jordan elementary operator in tensor product of C^*algebras is obtained.*

Keywords *- Maximal numerical range, Elementary operator, Rank-one-operator, Tensor Product, C^*- algebras.*

1. Introduction

Properties of elementary operators have been studied, including ranges, compactness, norm and ranks, and their expositions obtained. However, research in this area is still ongoing. Muiruri et al (2018) determined the norm of basic elementary operators in the tensor product using the properties of rank one operators, while Daniel (2018) used Stampfli`s Maximal numerical range to determine the lower bound of the norm of the basic elementary operators in tensor product of C^*-algebra. Muiruri (2024) further advanced this study by determining the norm of an elementary operator of length two in the tensor products of C^*-algebras.

This study extended the work of Muiruri (2024) and Daniel (2018) by determining the lower bound of Jordan elementary operator in tensor product of C^*-algebras. Properties of the tensor product, rank one operator and Maximal numerical range were applied in determining the lower bound of the norm of Jordan elementary operator in tensor product of C^*- algebras.

The following terminologies are relevant in this paper.

Definition 1.1 A C^*- *algebra* is a Banach algebra A with an involution satisfying the C^*- identity that is, $\forall\, a \in A$, $\|a^*a\| = \|a\|^2$.

Definition 1.2 let H be a Complex Hilbert space and $B(H)$ be the set of bounded linear operators on H. An elementary operator is an operator. $E_n\colon \mathrm{B}(H) \longrightarrow B(H)$ defined as $E_n(W) = \sum_{i=1}^{n} T_i W S_i$ for all $W \in B(H)$ where T_i, S_i are fixed operators in $B(H)$.

The following are examples of elementary operators:

i) Let H be a Complex Hilbert space and $B(H)$ be the set of bounded linear operators on H. A *basic elementary* operator is an operator. $E_1(W)\colon B(H) \to B(H)$, defined as $E(W) = TWS$ for all $W \in B(H)$, where S, T are fixed operators in $B(H)$. It is usually denoted as $M_{T,S}$.

ii) Let H be a Complex Hilbert space and $B(H)$ be the set of bounded linear operators on H. An *elementary* operator of length two is an operator. $E_2(W)\colon B(H) \to B(H)$, defined as $E_2(W) = T_1 W S_1 + T_2 W S_2$ for all $W \in B(H)$, where S, T are fixed operators in $B(H)$.

iii) A Jordan elementary operator $U_{T,S}\colon B(H) \longmapsto B(H), W \to TWS + SWT$ for all $W \in B(H)$ and S, T are fixed operators in $B(H)$.

Definition 1.3 Consider any vectors $y, z \in$ H. Then a *rank one operator,* $y \otimes {}'z \in$ B(H), is defined by $(y \otimes {}'z)x = \langle x, z\rangle y$ $\forall$ unit vectors $x \in H$.

Definition 1.4 A *finite rank one operator* $u \otimes {}'x: H \longrightarrow H$ is defined as $(u \otimes {}'x)y = u(y)x \; \forall \; y \in H$ where $u \in H^*$ and $x \in H$ is a unit vector with $\|u \otimes {}'x\| = |u(y)|$

Definition 1.5 The *Maximal numerical range,* $W_S(S^*T)$of (S^*T) relative to S is defined as, $W_S(S^*T) = \left\{\lambda \in \mathbb{C}: \exists\{x_n\} \in H, \|x_n\| = 1, \lim_{n\to\infty}\langle S^*Tx_n, x_n\rangle = \lambda, \lim_{n\to\infty} \|Tx_n\| = \|T\|\right\}$, where $S^*$is the Hilbert space adjoint of S.

Definition 1.6 Let H and K be Hilbert spaces. The *tensor product* of H and K is a Hilbert Space $H \otimes K$ with a bilinear mapping; $\otimes: H \times K \longrightarrow H \otimes K$ with $(x, y) \mapsto x \otimes y$ such that;

i) The vectors $x \otimes y$ form a total subset $H \otimes K$.
ii) $\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle\langle y_1, y_2\rangle$ for all $x_1, x_2 \in H$ and $y_1, y_2 \in K$.

The linearity of the mapping $\otimes\,(x, y) = x \otimes y$ implies that $\otimes$ is linear with respect to the two coordinates. A tensor product of Hilbert spaces is itself a Hilbert space.

Definition 1.7 If $H \otimes K$ is the tensor product of the Hilbert spaces H and K and $B(H \otimes K)$ is the set of bounded linear operators on $H \otimes K$, then the *elementary operator.* T_n: $B(H \otimes K) \longrightarrow B(H \otimes K)$ is defined as;

$$T_n(X \otimes Y) = \sum_{i=1}^{n}(A_i \otimes B_i)(X \otimes Y)(C_i \otimes D_i) \text{ for all } (X \otimes Y) \in B(H \otimes K),$$

where $A \otimes B, C \otimes D$ are fixed elements of $B(H \otimes K)$ with $A, C \in B(K)$ and $B, D \in B(K)$.

Definition 1.8 The Jordan elementary operator $U_{A\otimes B, C\otimes D}$: $B(H \otimes K) \longrightarrow B(H \otimes K)$ is defined as;

$$U_{A\otimes B, C\otimes D}(X \otimes Y) = (A \otimes B)(X \otimes Y)(C \otimes D) + (C \otimes D)(X \otimes Y)(A \otimes B)$$

for $(X \otimes Y) \in B(H \otimes K)$, with $A \otimes B, C \otimes D$ being fixed elements of $B(H \otimes K)$ where $A, C \in B(H)$ and $B, D \in B(K)$.

Definition 1.9 The maximal numerical range, $W_{S_1\otimes S_2}\big((T_1 \otimes T_2)^*(S_1 \otimes S_2)\big)$, of $(T_1 \otimes T_2)^*(S_1 \otimes S_2)$ relative to $S_1 \otimes S_2$ is defined as;

$$W_{S_1\otimes S_2}\big((T_1 \otimes T_2)^*(S_1 \otimes S_2)\big) = \left\{\begin{matrix} \lambda \in \mathbb{C}: \exists\{x_n \otimes x_n\} \in H, \|x_n \otimes x_n\| = 1, \\ \lim_{n\to\infty}\langle (T_1 \otimes T_2)^*(S_1 \otimes S_2)(x_n \otimes x_n), (x_n \otimes x_n)\rangle = \lambda, \\ \lim_{n\to\infty} \|(S_1 \otimes S_2)(x_n \otimes x_n)\| = \|S_1\|\|S_2\| \end{matrix}\right\}$$

where $(T_1 \otimes T_2)^*$ is the Hilbert Adjoint of $(T_1 \otimes T_2)$.

Let $\{x_n \otimes x_n\}$ be a sequence of unit vectors on $H \otimes K$. Then $(x_n \otimes x_n) \otimes {}'(x_n \otimes x_n) \in B(H \otimes K)$ is a rank one operator on $H \otimes K$ for all unit vectors $x_n \otimes x_n \in H \otimes K$ defined by

$$\big((x_n \otimes x_n) \otimes {}'(x_n \otimes x_n)\big)(y_1 \otimes y_2) = \langle y_1 \otimes y_2, x_n \otimes x_n\rangle(x_n \otimes x_n)$$

for all $y_1 \otimes y_2 \in (H \otimes K)$.

2. Norms of Elementary Operators in C^*- Algebras

In this section, we recall some results that formed the foundation of the current work.

Theorem 2.1 (Okelo and Agure, 2014) If H is a Hilbert space and $B(H)$ is the set of all bounded linear operators in H. If $M_{A,B}$: B(H) $\to$ B(H) is a basic elementary operator with $M_{A,B}(X) = AXB$ for all $X \in B(H)$ where A, B are fixed in B(H), then we have,

$$\|M_{A,B}(X)\| = \|A\|\|B\| \text{ with } \|X\| = 1.$$

Theorem 2.2 (Boumazgour and Barra, 2001) If $U_{T,S}$ be the Jordan elementary operator with $T, S \in B(H)$ fixed with $S \neq 0$. Then we have

$$\|U_{T,S}\| = sup_{\lambda \in W_S(T^*S)} \left\{ \left\| \|S\|T + \frac{\bar{\lambda}}{\|S\|} + S \right\| \right\}$$

where $W_S(T^*S)$ is the Maximal Numerical range of T^*S relative to S.

The work of Okelo and Agure, and that of Boumazgour and Barra, formed the foundation for the results by King'ang'i (2017) as shown below:
Theorem 2.3 (Kingangi, 2017) If E_2 be an elementary operator of length two on $B(H)$ then,

$$\|E_2\| \geq sup_{\lambda \in W_{B_1}(B_2{}^*B_1)} \left\{ \left\| \|B_1\|A_1 + \frac{\bar{\lambda}}{\|B_1\|} A_2 \right\| \right\}$$

, where $A_i, B_i$are fixed elements of B(H) for $i = 1,2$.
The results in this section were important in the derivation of the results in this paper.

3. Norms of Elementary Operators in the Tensor Product of C^*- Algebras

Muiruri (2018) determined the norm of the basic elementary operator in a tensor product of C^* -algebra using rank one operators as shown below:

Theorem 3.1 (Muiruri, 2018)Let H and K are complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $(H \otimes K)$. Then, for all $X \otimes Y \in$ B($H \otimes K$) with $\|X \otimes Y\| = 1$ we have,

$$\|M_{A\otimes B, C\otimes D}\| = \|A\|\|B\|\|C\|\|D\|$$

Where A, C and B, D are fixed elements in $B(H)$ and $B(K)$ respectively.

Muiruri further came up with the corollary below as a consequence of the result above.

Corollary 3.2 (Muiruri, 2018) Let H and K be complex Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. The for all $X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$ we have,

$$\|M_{A\otimes B, C\otimes D}\| = \|M_{A,C}\|\|M_{C,D}\|$$

where $M_{A,C}$ and $M_{C,D}$ are basic elementary operators in $B(H)$ and $B(K)$ respectively.

In 2024, Muiruri determined the norm of an elementary operator of length two in a tensor product of C^*-algebras using the concept of rank one operators shown as below:

Theorem 3.3 (Muiruri, 2024) Let $H \otimes K$ be a tensor product of Hilbert spaces and $B(H \otimes K)$ be the set of bounded linear operators on $H \otimes K$. Then, for all $X \otimes Y \in B(H \otimes K)$ with $\|X \otimes Y\| = 1$ we have,

$$\|U_{\mathrm{A\otimes B, C\otimes D}}\| = 2\|A\|\|B\|\|C\|\|D\|$$

Where $U_{\mathrm{A\otimes B, C\otimes D}}$ is the Jordan Elementary operator and $A, C \in \mathrm{B}(H)$ and $B, D \in \mathrm{B}(K)$.

Now, as our main result, we investigate the bounds of the norm of an elementary operator of length two in a tensor product of C∗-algebras using the concept of finite rank operator and properties of tensor product of C*-algebras. The methods used in the above results were necessary in the computation of our results.

4. Results

This section presents the main result in which the lower bound of the norm of Jordan elementary operator in tensor product of C^*-Algebra is determined.

Theorem 4.1: Let $U_{T_1\otimes T_2,S_1\otimes S_2}$ be the Jordan elementary operator with $T_1\otimes T_2, S_1\otimes S_2 \in B(H\otimes K)$ fixed with $S_1\otimes S_2 \neq 0$. Then

$$\left\|U_{T_1\otimes T_2,S_1\otimes S_2}\right\| \geq sup_{\lambda\in W_{S_1\otimes S_2}(T_1\otimes T_2)^*(S_1\otimes S_2)}\left\{\left\|\|S_1\|\|S_2\|T_1\otimes T_2+\frac{\bar{\lambda}}{\|S_1\|\|S_2\|}S_1\otimes S_2\right\|\right\}$$

where $W_{S_1\otimes S_2}(T_1\otimes T_2)^*(S_1\otimes S_2)$ is the maximal numerical range of $(T_1\otimes T_2)^*(S_1\otimes S_2)$ relative to $(S_1\otimes S_2)$ and $(T_1\otimes T_2)^*$ is the Hilbert adjoint of $(T_1\otimes T_2)$.

Proof.

Maximal numerical range of $(T_1\otimes T_2)^*(S_1\otimes S_2)$ relative to $(S_1\otimes S_2)$

is defined as

$$W_{S_1\otimes S_2}\big((T_1\otimes T_2)^*(S_1\otimes S_2)\big)=\left\{\begin{array}{c}\lambda\in\mathbb{C}:\exists\{x_n\otimes x_n\}\in H,\|x_n\otimes x_n\|=1,\\ \lim\limits_{n\to\infty}\langle (T_1\otimes T_2)^*(S_1\otimes S_2)(x_n\otimes x_n),(x_n\otimes x_n)\rangle=\lambda,\\ \lim\limits_{n\to\infty}\|(S_1\otimes S_2)(x_n\otimes x_n)\|=\|S_1\|\|S_2\|\end{array}\right\}$$

By definition

$$\left\|U_{T_1\otimes T_2,S_1\otimes S_2}\big((y_1\otimes y_2)\otimes' (S_1\otimes S_2)(x_n\otimes x_n)\big)(x_n\otimes x_n)\right\| \leq \left\|U_{T_1\otimes T_2,S_1\otimes S_2}\right\|\|S_1\|\|S_2\|.$$

$$\begin{aligned}\left\|U_{T_1\otimes T_2,S_1\otimes S_2}\right\|\|S_1\|\|S_2\| &\\ \geq \|((T_1\otimes T_2)&((y_1\otimes y_2)\otimes'(S_1\otimes S_2)(x_n\otimes x_n))(S_1\otimes S_2))(x_n\otimes x_n)\\ &+((S_1\otimes S_2)((y_1\otimes y_2)\otimes'(S_1\otimes S_2)(x_n\otimes x_n))(T_1\otimes T_2))(x_n\otimes x_n)\|\end{aligned}$$

$$=\|\langle (S_1\otimes S_2)(x_n\otimes x_n),(S_1\otimes S_2)(x_n\otimes x_n)\rangle (T_1\otimes T_2)(y_1\otimes y_2)+\langle (T_1\otimes T_2)(x_n\otimes x_n),(S_1\otimes S_2)(x_n\otimes x_n)\rangle (S_1\otimes S_2)(y_1\otimes y_2)\|$$

$$=\|\|(S_1\otimes S_2)(x_n\otimes x_n)\|^2(T_1\otimes T_2)(y_1\otimes y_2)+\langle (x_n\otimes x_n),(T_1\otimes T_2)^*(S_1\otimes S_2)(x_n\otimes x_n)\rangle (S_1\otimes S_2)(y_1\otimes y_2)\|$$

Thus,

$$\begin{aligned}\left\|U_{T_1\otimes T_2,S_1\otimes S_2}\right\| \geq \frac{1}{\|S_1\|\|S_2\|}&\|\|(S_1\otimes S_2)(x_n\otimes x_n)\|^2(T_1\otimes T_2)(y_1\otimes y_2)\\ &+\langle (x_n\otimes x_n),(T_1\otimes T_2)^*(S_1\otimes S_2)(x_n\otimes x_n)\rangle (S_1\otimes S_2)(y_1\otimes y_2)\|\end{aligned}$$

Taking limits as $n\to\infty$, we obtain,

$$\left\|U_{T_1\otimes T_2,S_1\otimes S_2}\right\| \geq \left\|\|S_1\|\|S_2\|(T_1\otimes T_2)(y_1\otimes y_2)+\frac{\bar{\lambda}}{\|S_1\|\|S_2\|}(S_1\otimes S_2)(y_1\otimes y_2)\right\|$$

Consider the set $\left\{\left\|\|S_1\|\|S_2\|(T_1\otimes T_2)+\frac{\bar{\lambda}}{\|S_1\|\|S_2\|}(S_1\otimes S_2)\right\|:\lambda\in W_{S_1\otimes S_2}(T_1\otimes T_2)^*(S_1\otimes S_2),\|y_1\otimes y_2\|=1\right\}$

Then we have,

$$\left\|U_{T_1\otimes T_2,S_1\otimes S_2}\right\| \geq \sup\left\{\left\|\|S_1\|\|S_2\|(T_1\otimes T_2)+\frac{\bar{\lambda}}{\|S_1\|\|S_2\|}(S_1\otimes S_2)\right\|:\lambda\in W_{S_1\otimes S_2}(T_1\otimes T_2)^*(S_1\otimes S_2),\|y_1\otimes y_2\|=1\right\}$$

5. Conclusion

This paper employed the techniques of maximal numerical range to investigate the norm of Jordan elementary operators in tensor product of C^* -algebras. A lower bound of this operator was obtained.

An attempt can be made to extend the research to Jordan Triple systems and to Jordan Elementary Operators of arbitrary finite length in tensor products.

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