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Original Article

New Families of Prime Graphs via Vertex Expansions

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Abstract - *In graph labeling, 'prime labeling' is a process where the n vertices of a graph are assigned distinct integers from 1 to n such that the labels of any two adjacent vertices are relatively prime. The study of prime labeling for graphs obtained through structural operations has attracted considerable attention in recent years. In this paper, we investigate the existence of prime labeling for certain classes of graphs arising from vertex expansions. Specifically, we prove that the graphs obtained by expanding the intermediate vertices of a jewel graph without prime edge and the pendent vertices of a jellyfish graph without prime edge into paths admit prime labeling. Furthermore, we establish the existence of prime labeling for zigzag graphs. These results extend the family of known prime graphs and significantly contribute to the study of prime labeling under graph expansion operations.*

Keywords - *Prime Labeling, Graph Expansion, Jewel Graph, Jellyfish Graph, Zigzag Graph.*

1. Introduction

Graph theory is a fast-growing branch of discrete mathematics used in networks, communication, and cryptography. A key research area is graph labeling, which involves assigning integers to vertices or edges based on specific mathematical rules. Entringer initially proposed prime labeling, which Tout et al. [9] later formalized in their research paper. Over the years, the investigation into prime graphs has advanced substantially, with scholarly efforts primarily directed toward classifying which graph families accommodate such labeling. Among the diverse array of labeling methodologies, prime labeling stands out as a significant area of study, largely due to its rigorous number-theoretic requirements.

Definition 1.1.[9] A prime labeling of a graph G with n vertices is an injective function $f: V(G) \rightarrow \{1,2,\dots,n\}$ such that for every edge $uv \in E(G)$, $\gcd\big(f(u), f(v)\big) = 1$. A graph admitting such a labeling is called a prime graph.

This raises a logical question: do graph operations and structural changes preserve prime labeling?

Extensive research has been conducted to determine the primality of various fundamental graph families. However, understanding how to specific graph operations and structural expansions affect prime labeling remains a fertile ground for discovery. Motivated by this, the present study explores the existance of prime labeling in new graphs generated through vertex expansion operations.

For an extensive review of various graph labeling techniques, readers are directed to J. Gallian's dynamic survey [4]. Our work is primarily inspired by two key contributions: Kansagara et al. [6], who investigated prime labeling for various graphs resulting from the subdivision of edges, and Patel et al. [7], who established prime labeling for the union of several graphs, such as gear, crown, helm, and book graphs.

Unlike existing research, where a path is typically attached only at one end of a graph, our study introduces a completely different structural approach via vertex expansion. Specifically, we replace particular vertices entirely with a path. P_t. For instance, in the jewel graph, all intermediate vertices are replaced by a path. P_t. Similarly, in the Jellyfish graph, every pendant vertex is replaced by a path. P_t. This generalized vertex replacement approach sets our work apart from the existing literature.

In the domain of prime labeling, it is a well-known challenge that even a minor structural modification in a prime graph can either preserve or completely destroy its prime labeling property. The primary significance of our research lies in the fact that, despite introducing complex structural changes through vertex expansions, we successfully establish and preserve the prime labeling for these newly constructed graph families. This robust approach opens up new directions for future researchers to explore labeling properties in highly complex and modified network structures.

This section outlines the basic terminology and preliminary findings applied in our subsequent proofs.
Definition 1.2. A graph $H = (V', E')$ is defined as a subgraph of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Definition 1.3. A path graph P_n is defined by the vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{v_i v_{i+1} : 1 \le i < n\}$.

Definition 1.4. A cycle graph C_n $(n \ge 3)$ is a simple, closed loop of n vertices, with each vertex connected to exactly two others.

Definition 1.5. **[1]** The graph J_n, known as the jewel graph, originates from a 4-cycle with vertices u, x, v, y. It is characterized by a prime edge connecting x and y, along with a set of n intermediate vertices, each connected to both u and v.

Definition 1.6. **[1]** By excluding the prime edge from the jewel graph J_n, we obtain the jewel graph without a prime edge, denoted by J_n^*.

Definition 1.7. **[1]** The graph $J_{n,m}$, known as the jellyfish graph, originates from a 4-cycle with vertices u, x, v, y. It is characterized by a prime edge connecting x and y, along with a set of n pendent vertices attached to u and m pendent vertices attached to v.

Definition 1.8. **[1]** By excluding the prime edge from the jellyfish graph $J_{n,m}$, we obtain the jellyfish graph without a prime edge, denoted by $J_{n,m}^*$.

Definition 1.9. For $n \in \mathbb{N}$, the zigzag graph ZZ_n is defined as a graph with the vertex set $V(ZZ_n) = \{v_i, u_i : 1 \le i \le n\}$ and the edge set $E(ZZ_n) = \{v_i v_{i+1}, u_i u_{i+1} : 1 \le i < n\} \cup \{v_i u_{i+1} : i \equiv 1 (mod\ 2),\ 1 \le i < n\} \cup \{u_i v_{i+1} :\ i \equiv 0 (mod\ 2), 1 \le i < n\}$.
Fundamental lemmas derived from Euclid's division algorithm are established here for repeated use in the following sections.

Lemma 1.10. **[2]** For positive integers a, b, and k, $gcd(a, b) = 1$ under any of these conditions:
1. $a = 1$;
2. $a = 2^k$ and b is odd;
3. a and b are odd integers differing by 2^k;
4. a and b are consecutive.

Lemma 1.11. **[2]** For any integer $n > 1$ and arbitrary integers a, b, c, and d, the following properties hold:
1. If $a \equiv b\ (mod\ n)$ and $b \equiv c\ (mod\ n)$, then $a \equiv c\ (mod\ n)$.
2. If $a \equiv b\ (mod\ n)$ and $c \equiv d\ (mod\ n)$, then $a + c \equiv b + d\ (mod\ n)$ and $ac \equiv bd\ (mod\ n)$.
3. If $a \equiv b\ (mod\ n)$, then $a + c \equiv b + c\ (mod\ n)$ and $ac \equiv bc\ (mod\ n)$.

Theorem 1.12. **[8] (Bertrand's Postulate)** For every integer $n \ge 2$, there exists at least one prime p satisfying $n < p < 2n$.
While the prime labeling of standard jewel and jellyfish graphs was originally addressed by Chidambaram [3], that article was published in a non-mathematical venue. Therefore, for completeness and clarity, we provide independent and detailed proofs in this section. These demonstrations establish a rigorous theoretical basis for the present study and ensure that the paper remains mathematically self-contained.

Lemma 1.13. For any $n \in \mathbb{N}$, the jewel graph J_n admits a valid prime labeling.
Proof: Let J_n be the jewel graph with vertex set $V(J_n) = \{u, x, v, y, z_i : 1 \le i \le n\}$ and edge set $E(J_n) = \{ux, xv, vy, yu, xy, uz_i, vz_i :\ 1 \le i \le n\}$. It follows that $|V(J_n)| = p = n + 4$.

Since $p > 4$, we identify. p_1 as the largest prime number such that $p_1 \le p$. Define an ordered set $D = \{1,2, \dots, p\} \setminus \{1,2,3, p_1\}$ as $D = \{d_1, d_2, \dots, d_{p-4}\}$ with $d_1 < d_2 < \dots < d_{p-4}$.

We define a function $f: V(J_n) \to \{1,2, \dots, |V(J_n)|\}$ as follows:

$f(u) = 1, f(x) = 2, f(v) = p_1, f(y) = 3$, and $f(z_i) = d_i$ for each $1 \le i \le p - 4$.

Vertex u is trivially coprime to its neighbors since $f(u) = 1$. The core labels 2, 3, and $p_1 \ge 5$ are mutually coprime. Furthermore, Theorem 1.12 ensures $2p_1 > p$, guaranteeing no other multiples of p_1 exist in $\{1,2, \dots, p\}$. Consequently, $\gcd\big(f(v), f(z_i)\big) = \gcd(p_1, d_i) = 1$ for $1 \le i \le n$. Since the label set $\{1,2,3, p_1\} \cup D$ maps $\{1,2, \dots, p\}$ bijectively onto $V(J_n)$, f is a valid prime labeling.

Lemma 1.14. For any $n, m \in \mathbb{N}$, the jellyfish graph $J_{n,m}$ admits a valid prime labeling.
Proof: Let $J_{n,m}$ be the jellyfish graph with vertex set $V\big(J_{n,m}\big) = \{u, x, v, y, z_i : 1 \le i \le n + m\}$ and edge set $E\big(J_{n,m}\big) = \{ux, xv, vy, yu, xy\} \cup \{uz_i : 1 \le i \le n\} \cup \{vz_{n+j} : 1 \le j \le m\}$. It follows that $\big|V\big(J_{n,m}\big)\big| = p = n + m + 4$.

Since $p > 4$, we identify. p_1 as the largest prime number such that $p_1 \le p$. Define an ordered set $D = \{1,2, \dots, p\} \setminus \{1,2,3, p_1\}$ as $D = \{d_1, d_2, \dots, d_{p-4}\}$ with $d_1 < d_2 < \cdots < d_{p-4}$.

We define a function $f: V\big(J_{n,m}\big) \to \{1,2, \dots, |V(J_{n,m})|\}$ as follows:

$f(u) = 1, f(x) = 2, f(v) = p_1, f(y) = 3$, and $f(z_i) = d_i$ for each $1 \le i \le p - 4$.

Vertex u with $f(u) = 1$ is naturally coprime to all its neighbors. The core labels 2, 3, and $p_1 \ge 5$ are pairwise coprime. By Theorem 1.12, $2p_1 > p$, ensuring no other multiples of p_1 exist in $\{1,2, \dots, p\}$. Thus, $\gcd\Big(f(v), f\big(z_j\big)\Big) = \gcd\big(p_1, d_j\big) = 1$ for $n + 1 \le j \le n + m$. Since f maps $V(J_{n,m})$ bijectively onto $\{1,2, \dots, p\}$, it is a valid prime labeling.

This study establishes the prime labeling of the proposed graph structures by utilizing essential number-theoretic mechanisms, including divisibility, modular arithmetic, and coprimality. The constructive framework introduced here provides rigorous proofs that can be generalized to larger classes of graphs. Standard graph theory conventions are drawn from West [10] along with Gross and Yellen [5], and the necessary number-theoretic background is sourced from Burton [2].

2. Main Results

Lemma 2.1. Let $C_4 = (u, x, v, y)$ be a cycle of length 4. If n copies of the path P_t, where $n, t \in \mathbb{N}$ and $t > 1$, are joined to vertex u by a new edge at an end-vertex, the resulting graph admits a prime labeling.

Proof. By joining one end-vertex of each of the n copies of the path P_t to the vertex u of the cycle C_4, we obtain the graph H. The vertex set and edge set of H are defined as follows: $V(H) = \{u, x, v, y\} \cup \{y_i : 1 \le i \le tn\}$ and $E(H) = \{ux, xv, vy, yu\} \cup \big\{uy_{(k-1)t+1} : 1 \le k \le n\big\} \cup \{y_i y_{i+1} : 1 \le i < tn, i \not\equiv 0 \ (mod\ t)\}$. It follows that $|V(H)| = p = nt + 4$ and $|E(H)| = nt + 4$.

Case 1: $\boldsymbol{t \equiv 0\ (mod\ 3)}$ and $\boldsymbol{n \in \mathbb{N}}$.
Define a function $f: V(H) \to \{1,2, \dots, |V(H)|\}$ as follows:
$f(u) = 3, f(x) = 2, f(v) = 1, f(y) = 4$, and $f(y_i) = i + 4$ for $1 \le i \le nt$.

For the edges $\{ux, xv, vy, yu\}$ in the 4-cycle, the vertex labels result in $\gcd(3,2) = \gcd(2,1) = \gcd(1,4) = \gcd(4,3) = 1$. Adjacent vertices y_i, y_{i+1} have labels $i + 4$ and $i + 5$, which are consecutive integers. By Lemma 1.10(4), it follows that $gcd(i + 4, i + 5) = 1$ for all $1 \le i < tn, i \not\equiv 0\ (mod\ t)$.

Observe that the vertex u is adjacent to $y_{(k-1)t+1}$ for $1 \le k \le n$. Let $f(u) = 3$ and $f\big(y_{(k-1)t+1}\big) = (k - 1)t + 5$. Given $t \equiv 0\ (mod\ 3)$, it follows that $(k - 1)t \equiv 0\ (mod\ 3)$. Thus, $f\big(y_{(k-1)t+1}\big) = (k - 1)t + 5 \equiv 5 \equiv 2\ (mod\ 3)$. Since $f\big(y_{(k-1)t+1}\big)$ is not a multiple of 3, $gcd\Big(f(u), f\big(y_{(k-1)t+1}\big)\Big) = 1$ for all $1 \le k \le n$. Hence, f defines a prime labeling of H for $t \equiv 0\ (mod\ 3)$ and all $n \in \mathbb{N}$.

***Case 2: $\boldsymbol{t \equiv 1\ (mod\ 3)}$* and $\boldsymbol{n \in \mathbb{N}}$.**
***Sub case 2.1: $\boldsymbol{t}$* is even.**
Define a function $f: V(H) \to \{1,2, \dots, |V(H)|\}$ as follows:

$f(u) = 3,\ f(x) = 2,\ f(y) = 4, f(v) = 1$, and $f(y_i) = i + 4$ for $1 \le i \le tn$.

Observe that vertex u is adjacent to y_{kt+1} for $0 \le k \le n-1$, where $f(y_{kt+1}) = kt + 1 + 4 = kt + 5$.

Given $t \equiv 1\ (mod\ 3)$, if $k \equiv 0\ or\ 2\ (mod\ 3)$ then $kt \equiv 0\ or\ 2\ (mod\ 3)$. Consequently, $f(y_{kt+1}) = kt + 5$ is congruent to either $0 + 5 \equiv 2\ (mod\ 3)$ or $2 + 5 \equiv 1\ (mod\ 3)$. Since $f(y_{kt+1})$ is not divisible by 3, it follows that $gcd\big(f(u), f(y_{kt+1})\big) = 1$ for all $k \equiv 0\ or\ 2\ (mod\ 3)$.

However, if $k \equiv 1\ (mod\ 3)$ and $t \equiv 1\ (mod\ 3)$, then $kt \equiv 1\ (mod\ 3)$. It follows that $f(y_{kt+1}) = kt + 5 \equiv 1 + 5 \equiv 0\ (mod\ 3)$. Since $f(y_{kt+1})$ is a multiple of 3 and $f(u) = 3$, we have $gcd\big(f(u), f(y_{kt+1})\big) = 3$. To satisfy the prime labeling condition, f must be modified here to ensure $gcd\big(f(u), f(y_{kt+1})\big) = 1$ for all $k \equiv 1\ (mod\ 3)$.

We define a modified function $g: V(H) \to \{1,2, \dots, |V(H)|\}$ to handle cases where $k \equiv 1\ (mod\ 3)$:

For each $k \in \{0,1, \dots, n-1\}$, if $k \equiv 1\ (mod\ 3)$, we set $g(y_{kt+1}) = f(y_{kt+2})$ and $g(y_{kt+2}) = f(y_{kt+1})$.

For all remaining vertices $c \in V(H)$, let $g(c) = f(c)$.

Observe that vertex u is adjacent to y_{kt+1} for $0 \le k \le n-1$.

If $k \equiv 1\ (mod\ 3)$ and $t \equiv 1\ (mod\ 3)$, then $kt \equiv 1\ (mod\ 3)$. It follows that $g(y_{kt+1}) = f(y_{kt+2}) = kt + 6 \equiv 1 + 0 \equiv 1\ (mod\ 3)$. Since $g(y_{kt+1})$ is not divisible by 3 and $g(u) = 3$, we conclude that $gcd\big(g(u), g(y_{kt+1})\big) = 1$. Moreover, $gcd(g(y_{kt+2}), g(y_{kt+3})) = (kt + 5, kt + 7) = (kt + 5,2) = 1$, since t begin even implies $kt + 5$ is odd for all $k \in \mathbb{N}$.

In this case, the cycle labels remain coprime. Where y_i and y_{i+1} are adjacent and their labels are not modified, this pair consists of consecutive labels. Therefore, by Lemma 1.10, they are relatively prime. Hence, f defines a prime labeling of H for $t \equiv 1\ (mod\ 3)$, where t is even, and all $n \in \mathbb{N}$.

Sub case 2.2: t is odd and $t > 1$.
Let $t \in \{7,13,19,25,31, \dots\}$. We define an initial labeling function $f: V(H) \to \{1,2, \dots, |V(H)|\}$ as follows:

$$f(u) = 5,\ f(x) = 2,\ f(y) = 4, f(v) = 1,$$

- For the vertex set $\{y_j: 1 \le j \le t\}$: $f(y_j) = \begin{cases} j + 5, & for\ 1 \le j \le t-1 \\ 3 & for\ j = t \end{cases}$.
- For specific indices $k \in \{1,2, \dots, n\}$ determined by the residue of $t\ (mod\ 30)$, the labels for indices $(k-1)t + 1 \le j \le kt$ are defined as $f(y_j) = 2(k-1)t + t + 5 - j$, where the applicable values of k are as follows: If $t \equiv 7,13,19\ (mod\ 30)$, then $k \equiv 1\ (mod\ 5)$, with $k > 1$; and If $t \equiv 25\ (mod\ 30)$, then $1 < k \le n$.
- For all remaining indices j such that $t + 1 \le j \le nt$, let $f(y_j) = j + 4$.

To satisfy the prime labeling condition when $t \equiv 1\ (mod\ 30)$, we define the modified labeling function $g: V(H) \to \{1,2, \dots, |V(H)|\}$ based on f:

1. **Constant Vertex Labels:** $g(u) = 5,\ g(x) = 2,\ g(y) = 4, g(v) = 1$.
2. **Swapping Rules for k:** For $k \in \{2,3, \dots, n\}$, the labels of boundary vertices $y_{(k-1)t+1}$ are modified to prevent gcd conflicts:
   i) If $k \equiv 1\ (mod\ 10)$ and $k > 1$, we set $g\big(y_{(k-1)t+1}\big) = f(y_{(k-1)t+2})$ and $g\big(y_{(k-1)t+2}\big) = f(y_{(k-1)t+1})$.
   ii) If $k \equiv 6\ (mod\ 10)$, we set $g\big(y_{(k-1)t+1}\big) = f(y_{(k-1)t+3})$ and $g\big(y_{(k-1)t+3}\big) = f(y_{(k-1)t+1})$.
3. **Remaining Vertices:** For all other vertices $c \in V(H)$, let $g(c) = f(c)$.

In this case, the cycle labels remain coprime. Vertex u connects to the last internal vertex of each k-th path via edges. $y_{(k-1)t+1}u$ $(1 \le k \le n)$. We must show that $\gcd\Big(g\big(y_{(k-1)t+1}\big), g(u)\Big) = \gcd\big(g\big(y_{(k-1)t+1}\big), 5\big) = 1$.

Case A. For $k = 1$, the label is defined as $g(y_1) = f(y_1) = 6$. Clearly, $\gcd(6,5) = 1$. Furthermore, $gcd\big(f(y_{t-1}), f(y_t)\big) = \gcd(t - 1 + 5, 3) = \gcd(t + 4, 3) = 1$. This holds since $t \equiv 1 (mod\ 3)$, which implies that $t + 4 \equiv 5 \equiv 2\ (mod\ 3)$.

Case B. We consider when k is a "**reversing**" index. This applies when $t \equiv 7,13,19\ (mod\ 30)$ for $k \equiv 1\ (mod\ 5)$, and when $t \equiv 25\ (mod\ 30)$ for all $1 < k \le n$. For the specific indices where the sequence reverses, the function is defined as $f(y_j) = 2(k - 1)t + t + 5 - j$. We substitute the target index $j = (k - 1)t + 1$ into the formula to obtain $f\big(y_{(k-1)t+1}\big) = 2(k - 1)t + t + 5 + ((k - 1)t + 1) = kt + 4$.

Now, we must evaluate $gcd(kt + 4, 5)$. We can analyze this using modulo-5 arithmetic by breaking it down into two distinct cases for t:

***Condition B.*1.*:* When $t \equiv 7, 13, 19\ (mod\ 30)$ (Applies for $k \equiv 1\ (mod\ 5)$)**
Because $k \equiv 1\ (mod\ 5)$, it follows that $kt \equiv t\ (mod\ 5)$. Therefore, $kt + 4 \equiv t + 4\ (mod\ 5)$.

Evaluating $t\ (mod\ 5)$ for each specific valid case:
- − If $t \equiv 7\ (mod\ 30)$, then $t \equiv 2\ (mod\ 5) \Rightarrow t + 4 \equiv 6 \equiv 1\ (mod\ 5)$.
- − If $t \equiv 13\ (mod\ 30)$, then $t \equiv 3\ (mod\ 5) \Rightarrow t + 4 \equiv 7 \equiv 2\ (mod\ 5)$.
- − If $t \equiv 19\ (mod\ 30)$, then $t \equiv 4\ (mod\ 5) \Rightarrow t + 4 \equiv 8 \equiv 3\ (mod\ 5)$.

***Condition B.*2.*:* When $t \equiv 25\ (mod\ 30)$ (Applies for all $1 < k \le n$)**
If $t \equiv 25\ (mod\ 30)$, then t is exactly divisible by 5 ($t \equiv 0\ (mod\ 5)$). For any k, kt will also be a multiple of 5 ($kt \equiv 0\ (mod\ 5)$). Therefore, $kt + 4 \equiv 4\ (mod\ 5)$.

In all valid scenarios evaluated across both conditions, $kt + 4 \not\equiv 0\ (mod\ 5)$. Because $kt + 4$ is not a multiple of the prime 5, it follows that $\gcd(kt + 4, 5) = 1$.

Case C. We analyze the "remaining" indices where $k \not\equiv 1\ (mod\ 5)$ and $k > 1$. For these indices, we are given $f(y_i) = i + 4$. Substituting $i = (k - 1)t + 1$, we obtain $f\big(y_{(k-1)t+1}\big) = (k - 1)t + 5$. Observe that $\gcd\Big(f\big(y_{(k-1)t+1}\big), f(u)\Big) = \gcd\big((k - 1)t + 5, 5\big) = \gcd\ ((k - 1)t, 5)$. For the gcd to be greater than 1, the product $(k - 1)t$ must be divisible by 5. However, consider its factors:

- t is not divisible by 5, as established in Case B ($t\ (mod\ 5) \in \{2,3,4\}$).
- Since $k \not\equiv 1\ (mod\ 5)$, it follows that $(k - 1) \not\equiv 0\ (mod\ 5)$.

Since 5 is a prime number and neither factor is a multiple of 5, their product $(k - 1)t$ is not divisible by 5. This implies that: $\gcd\Big(f\big(y_{(k-1)t+1}\big), f(u)\Big) = \gcd\big((k - 1)t, 5\big) = 1$.

For $\boldsymbol{t \equiv 1\ (mod\ 30)}$**:** The default formula $f\big(y_{(k-1)t+1}\big) = (k - 1)t + 5$ yields a multiple of 5 precisely when $k \equiv 1\ (mod\ 5)$ (which includes $k \equiv 1\ (mod\ 10)$ and $k \equiv 6\ (mod\ 10)$). The modified function g averts this conflict via vertex swapping:

- If $k \equiv 1\ (mod\ 10)$ (where $k > 1$), $y_{(k-1)t+1}$ is swapped with $y_{(k-1)t+2}$. Thus, $g\big(y_{(k-1)t+1}\big) = (k - 1)t + 6$. Since $k \equiv 1\ (mod\ 5)$ and $t \equiv 1\ (mod\ 5)$, $(k - 1)t + 6 \equiv (0)(1) + 6 \equiv 1\ (mod\ 5)$.
- If $k \equiv 6\ (mod\ 10)$, $y_{(k-1)t+1}$ is swapped with $y_{(k-1)t+3}$. Thus, $g\big(y_{(k-1)t+1}\big) = (k - 1)t + 7$. Since $k \equiv 1\ (mod\ 5)$ and $t \equiv 1\ (mod\ 5)$, $(k - 1)t + 7 \equiv (0)(1) + 7 \equiv 2\ (mod\ 5)$.

In all subcases, $g(y_{(k-1)t+1})$ is not divisible by 5. Hence, $\gcd\big(g\big(y_{(k-1)t+1}\big), f(u)\big) = 1$.

The path edges are of the form. $y_i y_{i+1}$ for $i \not\equiv 0\ (mod\ t)$.

- **Standard Paths:** Where the function $f(y_i) = i + 4$ or the reversed sequence $f(y_j)$ applies without modification, adjacent vertices receive consecutive integer labels (either ascending or descending) since any two consecutive integers are relatively prime, $gcd(a, a + 1) = 1$**.**
- **Swapped Paths (**when $t \equiv 1\ (mod\ 30)$**):** The function g introduces swaps that disrupt the consecutive sequence. We must

adjacencies at the swap boundaries:

i) When $k \equiv 1\ (mod\ 10)\ (k > 1)$: The swap occurs between $y_{(k-1)t+1}$ and $y_{(k-1)t+2}$. The only new critical adjacency is $y_{(k-1)t+2}$ and $y_{(k-1)t+3}$. Their labels are $g\left(y_{(k-1)t+2}\right) = (k-1)t + 5$ and $g\left(y_{(k-1)t+3}\right) = (k-1)t + 7$. Hence, $gcd\ ((k-1)t + 5, (k-1)t + 7) = gcd\ ((k-1)t + 5,2)$. Since t is odd and $k \equiv 1\ (mod\ 10)$, k is odd, so $(k-1)t$ is even. Therefore, $(k-1)t + 5$ is odd, implying $gcd((k-1)t + 5,2) = 1$.

ii) When $k \equiv 6\ (mod\ 10)$: The swap occurs between $y_{(k-1)t+1}$ and $y_{(k-1)t+3}$, creating a critical adjacency between $y_{(k-1)t+3}$ and $y_{(k-1)t+4}$ with labels $g\left(y_{(k-1)t+3}\right) = (k-1)t + 5$ and $g\left(y_{(k-1)t+4}\right) = (k-1)t + 8$. Thus, $gcd((k-1)t + 5, (k-1)t + 8) = \gcd\ ((k-1)t + 5,3)$. Given $t \equiv 1\ (mod\ 30)$, we have $t \equiv 1\ (mod\ 3)$. Since $k \equiv 6\ (mod\ 10)$ implies k is an even multiple of 3 (i.e., $k \equiv 0\ (mod\ 3)$). It follows that $(k-1)t \equiv (2)(1) \equiv 2\ (mod\ 3)$. Consequently, $(k-1)t + 5 \equiv 2 + 5 \equiv 1\ (mod\ 3)$. Because $(k-1)t + 5$ is not divisible by 3, we concluded that $\gcd\ ((k-1)t + 5,3) = 1$.

Where y_i and y_{i+1} are adjacent and their labels are not modified, this pair consists of consecutive labels. Therefore, by Lemma 1.10, they are relatively prime. Hence, for all $n \in \mathbb{N}$ and odd t, f and g define prime labelings of H when $t \equiv 7,13,19,25\ (\ mod\ 30)$ and $t \equiv 1\ (\ mod\ 30)$ (for $t > 1$), respectively.

Hence, H is prime for all $n \in \mathbb{N}$ and $m \equiv 1\ (mod\ 3)(m > 1)$.

***Case 3:* $\boldsymbol{t \equiv 2\ (\ mod\ 3)}$ and $\boldsymbol{n \in \mathbb{N}}$.**
Define a function $f: V(H) \to \{1,2, \dots, |V(H)|\}$ as follows:

$$f(u) = 3, f(x) = 2, f(v) = 1, f(y) = 4,$$

and for the subscripted vertices y_i, where $(k-1)t + 1 \le i \le kt$ for $1 \le k \le n$:

$$f(y_i) = \begin{cases} 2(k-1)t + t + 5 - i, & k \equiv 0\ (mod\ 3); \\ i + 4, & otherwise. \end{cases}.$$

In this case, the cycle labels remain coprime. For $k \equiv 0\ (mod\ 3)$, the k-th copy of the path P_t is labeled in reverse order so that adjacent vertices y_i, y_{i+1} remain consecutive and relatively prime. If $k \not\equiv 0\ (mod\ 3)$, the labeling follows Case 1, ensuring the vertices remain relatively prime as previously proved.

Observe that the vertex u is adjacent to $y_{(k-1)t+1}$ for $1 \le k \le n$. Let $f(u) = 3$.

If $k \equiv 0\ (mod\ 3)$, then $f\left(y_{(k-1)t+1}\right) = 2(k-1)t + t + 5 - (k-1)t - 1 = (k-1)t + t + 4 = kt + 4$. Since $k \equiv 0\ (mod\ 3)$, it follows that $kt \equiv 0\ (mod\ 3)$ for any given t (including $t \equiv 2\ (\ mod\ 3)$). Thus, $kt + 4 \equiv 0 + 4 \equiv 1\ (\ mod\ 3)$. Since $f\left(y_{(k-1)t+1}\right) \equiv 1\ (mod\ 3)$, the label $f\left(y_{(k-1)t+1}\right)$ is not a multiple of 3. Because $f(u) = 3$ is prime, we conclude that $gcd\left(f(u), f\left(y_{(k-1)t+1}\right)\right) = 1$ for all $k \equiv 0\ (mod\ 3)$.

If $k \not\equiv 0\ (mod\ 3)$, then $k \equiv 1\ or\ 2\ (mod\ 3)$. In this case, $f\left(y_{(k-1)t+1}\right) = (k-1)t + 5$.

Since $t \equiv 2\ (\ mod\ 3)$ and $k - 1 \equiv 0\ or\ 1\ (mod\ 3)$, we have two subcases:

- If $k - 1 \equiv 0\ (mod\ 3)$, then $(k-1)t \equiv 0\ (mod\ 3)$.
- if $k - 1 \equiv 1\ (mod\ 3)$, then $(k-1)t \equiv (1)(2) \equiv 2\ (mod\ 3)$.

Consequently, $(k-1)t + 5$ is congruent to either $0 + 5 \equiv 2\ (mod\ 3)$ or $2 + 5 \equiv 1\ (mod\ 3)$. Since $f\left(y_{(k-1)t+1}\right) \equiv 2\ or\ 1\ (mod\ 3)$, the label is not a multiple of 3. Given that $f(u) = 3$ is prime, it follows that $gcd\left(f(u), f\left(y_{(k-1)t+1}\right)\right) = 1$ for all $k \not\equiv 0\ (mod\ 3)$. Hence, f defines a prime labeling of H for $t \equiv 2\ (\ mod\ 3)$ and all $n \in \mathbb{N}$.

From the results of Cases 1, 2 and 3, it is concluded that the graph H admits a prime labeling for all $n, m \in \mathbb{N}$. Hence, H is a prime graph.

Illustration 2.1. Figure 1 illustrates a prime labeling for the graph obtained by joining 4 copies of a path P_5 to a vertex of the cycle C_4.

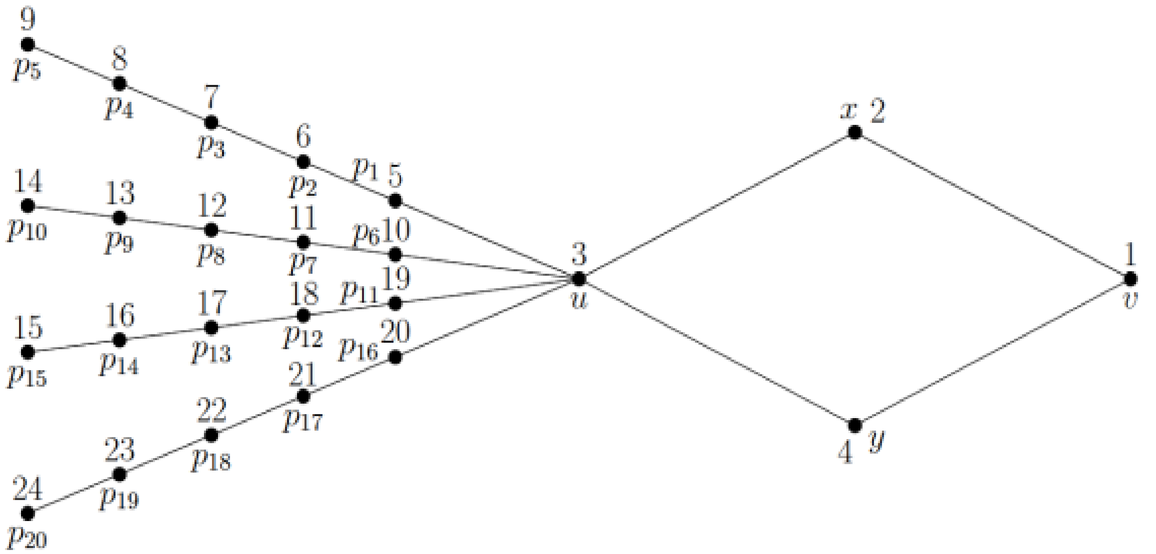

Fig. 1 Prime labeling for the graph obtained by joining 4 copies of the path P_5 to a vertex of the cycle C_4

Theorem 2.1. The graph $\boldsymbol{G}$, constructed by expanding every intermediate vertex of the jewel graph without a prime edge $\boldsymbol{J_n^*}$ into a path $\boldsymbol{P_t}$, admits a prime labeling for all $\boldsymbol{n}, \boldsymbol{t} \in \mathbb{N}$.

Proof. Let G be defined as above with vertex set $V(G) = \{u, x, v, y\} \cup \{y_i : 1 \leq i \leq tn\}$ and edge set $E(G) = \{ux, xv, vy, yu\} \cup \{uy_{(k-1)t+1} : 1 \leq k \leq n\} \cup \{y_{kt}v : 1 \leq k \leq n\} \cup \{y_i y_{i+1} : 1 \leq i < tn, i \ (mod\ t) \neq 0\}$. It follows that $|V(G)| = p = nt + 4$ and $|E(G)| = n(t+1) + 4$.

For $t = 1$, the graph G is isomorphic to the standard jewel graph with the prime edge xy removed. By Lemma 1.13, the standard jewel graph admits a prime labeling. Since deleting an edge from a prime graph inherently preserves the prime labeling condition, the resulting graph, which is exactly the jewel graph without prime edge xy (i.e., G for $t = 1$), is a prime graph.

For $t > 1$, consider the subgraph H of G, constructed by attaching n copies of the path. P_t to the vertex u of the cycle (u, x, v, y), with vertex set $V(H) = \{u, x, v, y\} \cup \{y_i : 1 \leq i \leq tn\}$ and edge set $E(H) = \{ux, xv, vy, yu\} \cup \{uy_{(k-1)t+1} : 1 \leq k \leq n\} \cup \{y_i y_{i+1} : 1 \leq i < tn, i \ (mod\ t) \neq 0\}$. Unlike G, where paths bridge both u and v to form a closed structure, H features paths incident only to u, thereby excluding all edges connected to v found in the original graph G.

We observe that, $V(G) = V(H)$. This implies that the order of both graphs is identical, $|V(G)| = V(H) = p = nt + 4$. Since the vertex sets are the same, we can adopt the labeling function $f: V(H) \rightarrow \{1, 2, \ldots, |V(H)|\}$ defined in Lemma 2.1 for the graph G. From the construction of H, for every edge $e = x_1 x_2 \in E(H)$, we have $\gcd\big(f(x_1), f(x_2)\big) = 1$.

The graph G differs from H only in its edge set, specifically: $E(G) = E(H) \cup \{y_{kt}v : 1 \leq k \leq n\}$. The additional edges in G are those connecting the end-vertices of the n copies of the path. P_t(denoted by y_{kt}) to the vertex v. To prove that f is a prime labeling for G, we only need to verify the adjacency condition for these new edges.

In the labeling defined in Lemma 2.1, the vertex v is assigned the label $f(v) = 1$. Consequently, for all new edges $y_{kt}v$ in G, $\gcd\big(f(y_{kt}), f(v)\big) = \gcd(f(y_{kt}), 1) = 1$. Thus, all adjacent vertices in G have labels that are relatively prime. This confirms that the function f is a valid prime labeling for G. Hence, G is a prime graph.

Illustration 2.2. Figure 2 illustrates a prime labeling for the graph constructed by expanding every intermediate vertex of the jewel graph without a prime edge J_5^* into a path P_3.

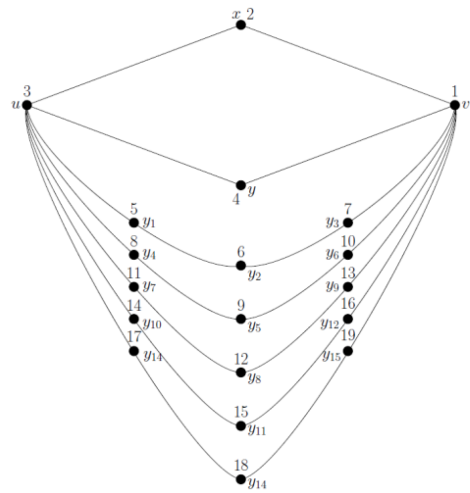

Fig. 2 A prime labeling for the graph constructed by expanding every intermediate vertex of J_5^* into a path P_3

Theorem 2.2. The graph $\boldsymbol{G}$, constructed by expanding every pendant vertex of the jellyfish graph without a prime edge $\boldsymbol{J_{n,m}^*}$ into a path $\boldsymbol{P_t}$, admits a prime labeling for all $\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{t} \in \mathbb{N}$.

Proof. Let G be defined as above with vertex set $V(G) = \{u, x, v, y\} \cup \{y_i : 1 \leq i \leq t(n+m)\}$ and edge set $E(G) = \{xy, ux, xv, vy\} \cup \{uy_{ti+1} : 0 \leq i \leq n-1\} \cup \{vy_{ti+1} : n \leq i \leq n+m-1\} \cup \{y_i y_{i+1} : 1 \leq i < t(n+m), i \ (mod\ t) \neq 0\}$. It follows that $|V(G)| = p = t(n+m) + 4$ and $|E(G)| = t(n+m) + 4$.

For $t = 1$, the graph G is isomorphic to the standard jellyfish graph with the prime edge xy removed. By Lemma 1.14, the standard jellyfish graph admits a prime labeling. Since deleting an edge from a prime graph inherently preserves the prime labeling condition, the resulting graph, which is exactly the jellyfish graph without prime edge xy (i.e., G for $t = 1$)-is a prime graph.

For $t > 1$, consider the subgraph H of G, constructed by attaching n copies of the path. P_t to the vertex u of the cycle (u, x, v, y), with vertex set $V(H) = \{u, x, v, y\} \cup \{y_i : 1 \leq i \leq tn\}$ and edge set $E(H) = \{ux, xv, vy, yu\} \cup \left\{uy_{(k-1)t+1} : 1 \leq k \leq n\right\} \cup \{y_i y_{i+1} : 1 \leq i < tn, i \ (mod\ t) \neq 0\}$. In graph G, n and m copies of the path P_t are attached to vertices u and v of the base cycle (u, x, v, y), respectively. In contrast, H is a subgraph of G that retains only the n paths attached to vertex u, omitting the paths attached to vertex v.

By Lemma 2.1, the subgraph H admits a prime labeling with the bijective function $f: V(H) \to \{1, 2, \dots, tn + 4\}$. To establish that the entire graph G is a prime graph, we extend the labeling f to a new bijection function $h: V(G) \to \{1, 2, \dots, t(n+m) + 4\}$. The set of newly introduced vertices in G is given by the difference set: $V(G) \backslash V(H) = \{y_i : tn + 1 \leq i \leq t(n+m)\}$.

We define the extended function h as follows: $h(w) = f(w)$ for all $w \in V(H)$ and $h(y_i) = i + 4$ for all $y_i \in V(G) \backslash V(H)$.

Clearly, h is a bijection since it maps the newly added tm vertices to the remaining sequence of integers $\{tn + 5, \dots, t(n+m) + 4\}$.

To prove that h is a prime labeling for G, we must show that $\gcd(h(a), h(b)) = 1$ for all $ab \in E(G)$. Since H is a subgraph of G and h preserves the labeling of H, the coprimality condition is inherently satisfied for all edges in $E(H)$ by Lemma 2.1.

Therefore, it is sufficient to verify the condition strictly for newly added edges, $E(G) \backslash E(H) = E_1 \cup E_2$, where $E_1 = \{vy_{ti+1} : n \leq i \leq n+m-1\}$ and $E_2 = \{y_i y_{i+1} : tn + 1 \leq i < t(n+m), i \ (mod\ t) \neq 0\}$.

For any edge $vy_{ti+1} \in E_1$, since $h(v) = f(v) = 1$, we trivially have $gcd(h(v), h(y_{ti+1})) = 1$. For any internal edge $y_i y_{i+1} \in E_2$, the endpoints receive consecutive integer labels, yielding $gcd(h(y_i), h(y_{i+1})) = gcd(i + 4, i + 5) = 1$. Because $gcd(h(a), h(b)) = 1$ holds for all newly introduced edges in $E(G) \backslash E(H)$, h is a prime labeling. Hence, G is a prime graph.

Illustration 2.3. Figure 3 illustrates a prime labeling for the graph constructed by expanding every pendant vertex of the jellyfish graph without a prime edge $J^*_{4,3}$ into a path P_4.

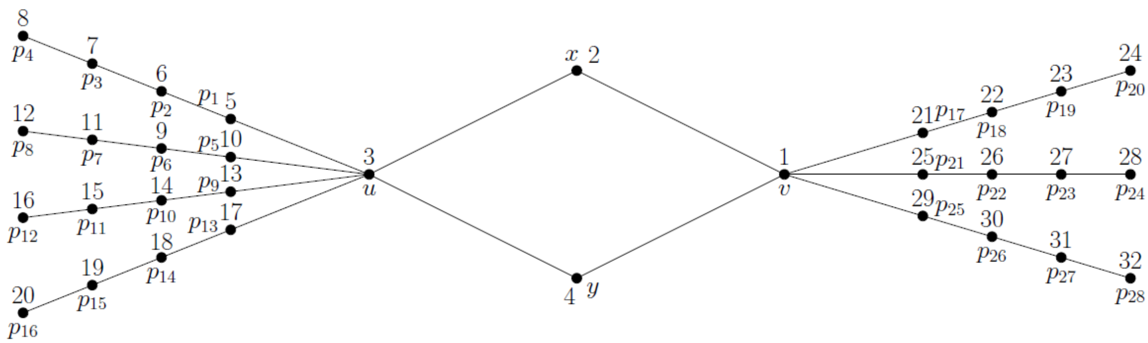

Fig. 3 A prime labeling for the graph constructed by expanding every pendant vertex of $J^*_{4,3}$ into a path P_4

Theorem 2.3. The zigzag graph ZZ_n admits a prime labeling for all $n \in \mathbb{N}$.
Proof. For $n \in \mathbb{N}$, the zigzag graph ZZ_n is defined by the vertex set $V(ZZ_n) = \{v_i, u_i : i = 1,2,\dots,n\}$ and edge set $E(ZZ_n) = \{v_iv_{i+1}, u_iu_{i+1} : i = 1,2,\dots,n-1\} \cup \{v_iu_{i+1} : i = 1,2,\dots,n \text{ and } i \text{ is odd}\} \cup \{u_iv_{i+1} : i = 1,2,\dots,n \text{ and } i \text{ is even}\}$. From this construction, it follows that $|V(ZZ_n)| = 2n$ and $|E(ZZ_n)| = 3n-3$.

Define $S_1 = \{6k : k \not\equiv 0 \ (mod\ 5)\}$, $S_2 = \{6k-4 : k \not\equiv 3 \ (mod\ 5)\}$, $S_3 = \{6k-3 : k \not\equiv 3 \ (mod\ 5)\}$, and $S_4 = \{6k-1 : k \not\equiv 0 \ (mod\ 5)\}$. The sets $S_1 \cup S_2$ and $S_3 \cup S_4$ contain exclusively even and odd indices, respectively.

For $1 \le i \le n$, define a function $f: V(ZZ_n) \to \{1,2,\dots,|V(ZZ_n)|\}$ as follows:

$$f(v_i) = \begin{cases} 2i-2 & i \in S_1 \\ 2i+2 & i \in S_2 \\ 2i-1 & i\ odd, i \notin S_1 \cup S_2 \\ 2i & i\ even, i \notin S_1 \cup S_2 \end{cases}, \text{ and } f(u_i) = \begin{cases} 2i-2 & i \in S_3 \\ 2i+2 & i \in S_4 \\ 2i & i\ odd, i \notin S_3 \cup S_4 \\ 2i-1 & i\ even, i \notin S_3 \cup S_4 \end{cases}.$$

For zigzag edges: By definition, v_i and u_{i+1} are adjacent for any odd $i \notin S_1 \cup S_2$. Therefore, $gcd\big(f(v_i), f(u_{i+1})\big) = gcd(2i-1, 2(i+1)-1) = \gcd(2i-1, 2i+1) = \gcd(2i-1, 2) = 1$, since any integer of the form $2i-1$ is odd. Similarly, by definition, u_i and v_{i+1} are adjacent for any even $i \notin S_3 \cup S_4$. Therefore, $gcd\big(f(u_i), f(v_{i+1})\big) = \gcd(2i-1, 2(i+1)-1) = \gcd(2i-1, 2i+1) = \gcd(2i-1, 2) = 1$, since any integer of the form $2i-1$ is odd.

For edges v_iv_{i+1} where $1 \le i < n$ and $i \not\equiv 29 \ (mod\ 30)$:

Case 1: **If i is odd, then $i+1$ is even.**

Since i is odd, $f(v_i) = 2i-1$. If $i+1 \in S_1$, then $f(v_{i+1}) = 2(i+1)-2 = 2i$. Thus, $\gcd\big(f(v_i), f(v_{i+1})\big) = \gcd(2i-1, 2i) = 1$, as the labels are consecutive integers. If $i+1 \in S_2$, then $f(v_{i+1}) = 2(i+1)+2 = 2i+4$, which yields $\gcd(2i-1, 2i+4) = \gcd(2i-1, 5)$. Since $i+1 \in S_2 = \{6k-4 : k \not\equiv 3 \ (mod\ 5)\}$, substituting $i = 6k-5$ into the condition $2i-1 \equiv 0 \ (mod\ 5)$ yields $12k-11 \equiv 0 \ (mod\ 5)$. This simplifies to $2k \equiv 1 \ (mod\ 5)$, giving $k \equiv 3 \ (mod\ 5)$. This contradicts the definition of S_2, where $k \not\equiv 3 \ (mod\ 5)$. Therefore, $2i-1$ is not divisible by 5, and hence $\gcd(2i-1, 5) = 1$.

Finally, if $i+1 \notin S_1 \cup S_2$, then $f(v_{i+1}) = 2i+2$, and $\gcd\big(f(v_i), f(v_{i+1})\big) = \gcd(2i-1, 2i+2) = \gcd(2i-1, 3)$. For this to exceed 1, we must have $2i-1 \equiv 0 \ (mod\ 3)$, which simplifies to $i \equiv 2 \ (mod\ 3)$. Combined with i being odd ($i \equiv 1 \ (mod\ 2)$), this requires $i \equiv 5 \ (mod\ 6)$, or $i+1 = 6k$. For $i+1$ to remain in this case, it must avoid $S_1 = \{6m : m \not\equiv 0 \ (mod\ 5)\}$, implying $k \equiv 0 \ (mod\ 5)$. Setting $k = 5q$ yields $i = 30q-1$, which means $\gcd(2i-1, 3) = 3$ only if $i \equiv 29 \ (mod\ 30)$. As this condition is not met, $\gcd\big(f(v_i), f(v_{i+1})\big) = \gcd(2i-1, 3) = 1$ for all odd $1 \le i < n$.

Case 2: **If i is even, then $i+1$ is odd.**

Since $i+1$ is odd, $f(v_{i+1}) = 2(i+1)-1 = 2i+1$. If $i \in S_1$, then $f(v_i) = 2i-2$. Thus, $\gcd\big(f(v_i), f(v_{i+1})\big) =$

$\gcd(2i-2,2i+1) = \gcd(2i-2,3)$. By the definition of $S_1 = \{6k : k \not\equiv 0 \ (mod\ 5)\}$, substituting $i = 6k$ yields $2i-2 = 12k-2 = 3(4k-1)+1$. Since $2i-2 \not\equiv 0 \ (mod\ 3)$, it follows that $\gcd(2i-2,3) = 1$. If $i \in S_2$, then $f(v_i) = 2i+2$, and if $i \notin S_1 \cup S_2$, then $f(v_i) = 2i$. In both cases, $f(v_i)$ (being $2i+2$ or $2i$) and $f(v_{i+1}) = 2i+1$ are consecutive integers, implying $gcd\big(f(v_i), f(v_{i+1})\big) = 1$ for all even $1 \le i < n$.

Considering the results across all cases, we conclude that $\gcd\big(f(v_i), f(v_{i+1})\big) = 1$ for all $1 \le i < n$.

For edges $u_i u_{i+1}$ where $1 \le i < n$ and $i \not\equiv 14 \ (mod\ 30)$:

***Case A:* If i is odd, then $i+1$ is even.**

Since $i+1 \notin S_3 \cup S_4$, we have $f(u_{i+1}) = 2i+1$. The label $f(u_i)$ takes the form $2i-2$, $2i+2$ or $2i$ depending on its membership in S_3, S_4, or $(S_3 \cup S_4)^c$, respectively. For $i \in S_4$ or $i \notin S_3 \cup S_4$, the labels $f(u_i)$ and $f(u_{i+1})$ are consecutive, thus $\gcd(f(u_i), f(u_{i+1})) = 1$. If $i \in S_3$ then $\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd(2i-2,2i+1) = \gcd(2i-2,3)$. Given the definition $S_3 = \{6k-3 : k \not\equiv 3 \ (mod\ 5)\}$, we have $2i-2 = 2(6k-3)-2 = 12k-8 = 3(4k-3)+1$, which is never a multiple of 3. Hence, $\gcd(f(u_i), f(u_{i+1})) = 1$ for all odd $1 \le i < n$.

***Case B:* If i is even, then $i+1$ is odd.**

Since $i \notin S_3 \cup S_4$, we have $f(u_i) = 2i-1$. The label $f(u_{i+1})$ takes the form $2i$, $2i+4$, or $2i+2$, depending on whether $i+1$ is a member of S_3, S_4, or $(S_3 \cup S_4)^c$, respectively. Following the Euclidean algorithm, these labels yield potential gcds of 1, 5, or 3. In this case, where $i \in S_4$ and $\gcd\big(f(u_i), f(u_{i+1})\big) = 5$, we require $i+1 \in S_4 = \{6k-1 : k \not\equiv 0 \ (mod\ 5)\}$. The condition $i+1 = 6k-1$ implies that $i = 6k-2$, and thus $2i-1 = 2(6k-2)-1 = 12k-5$. However, $2i-1 \equiv 0 \ (mod\ 5)$ implies $k \equiv 0 \ (mod\ 5)$, which contradicts the definition of S_4. In this case, where $i \notin S_3 \cup S_4$ and $\gcd\big(f(u_i), f(u_{i+1})\big) = 3$, the condition $2i-1 \equiv 0 \ (mod\ 3)$, implies $i \equiv 2 \ (mod\ 3)$. Combined with $i \equiv 0 \ (mod\ 2)$, we have $i \equiv 2 \ (mod\ 6)$, or $i = 6k-4$, which means $i+1 = 6k-3$. For $i+1$ to avoid membership in $S_3 = \{6k-3 : k \not\equiv 3 \ (mod\ 5)\}$, we must force $k \equiv 3 \ (mod\ 5)$. Let $k = 5q+3$. Thus, $i = 6(5q+3)-4 = 30q+14$. Thus, $gcd(2i-1,3) = 3$ iff $i \equiv 14 \ (mod\ 30)$. Since we restrict $i \not\equiv 14 \ (mod\ 30)$ in our premise, the gcd is never 3. Thus, $\gcd\big(f(u_i), f(u_{i+1})\big) = 1$ for all even $1 \le i < n$.

Observe that for $i \equiv 14 \ (mod\ 30)$, $\gcd\big(f(u_i), f(u_{i+1})\big) = \gcd\big(2i-1, 2(i+1)\big) = \gcd(2i-1,3) = \gcd(2(30k+14)-1,3) = \gcd(60k+27,3) = 3$. Similarly, for $i \equiv 29 \ (mod\ 30)$, $\gcd\big(f(v_i), f(v_{i+1})\big) = \gcd\big(2i-1, 2(i+1)\big) = \gcd(2i-1,3) = \gcd(2(30k+29)-1,3) = \gcd(60k+57,3) = 3$. The function f must be modified to satisfy the required constraints.

For $1 \le i \le n$, define a modified function $g: V(ZZ_n) \to \{1,2,\dots,|V(ZZ_n)|\}$ as follows:

$$g(x) = \begin{cases} f(u_{i+1}) & if\ x = v_i \quad i \equiv 29 \ (mod\ 30) \\ f(v_{i-1}) & if\ x = u_i \quad i \equiv 0 \ (mod\ 30) \\ f(u_{i-1}) & if\ x = v_i \quad i \equiv 15 \ (mod\ 30). \\ f(v_{i+1}) & if\ x = u_i \quad i \equiv 14 \ (mod\ 30) \\ f(x) & otherwise \end{cases}$$

The modified function $g(x)$ ensures coprime labels for all adjacent vertices in ZZ_n by resolving cases where $f(x)$ yielded a gcd of 3. Since $g(x) = f(x)$ for all other vertices, we only need to analyze the specific swapped cases for $i \equiv 29, 14 \ (mod\ 30)$. Using Lemma 1.11, we proceed as follows.

Observe that for $i \equiv 29 \ (mod\ 30)$, the adjacent vertices of v_i are v_{i-1}, v_{i+1}, u_{i-1}, and u_{i+1}, while those of u_{i+1} are u_i, u_{i+2}, v_i, and v_{i+2}. The modified function g interchanges the labels of v_i and u_{i+1}. Since $i = 30k+29$ is odd, the function results in $g(v_i) = f(u_{i+1}) = 2i+1 = 60k+59$ and $g(u_{i+1}) = f(v_i) = 2i-1 = 60k+57$. The neighboring vertices retain their original labels under f, yielding $f(v_{i-1}) = 2i-2 = 60k+56$, $f(u_{i-1}) = 2i-3 = 60k+55$, $f(v_{i+1}) = 2i+2 = 60k+60$, $f(u_i) = 2i = 60k+58$, $f(u_{i+2}) = 2i+4 = 60k+62$, and $f(v_{i+2}) = 2i+3 = 60k+61$. We compute the gcd for all affected incident edges. For the edges incident to v_i, we have $gcd\big(g(v_i), f(v_{i-1})\big) = \gcd(60k+59, 60k+56) = \gcd(3, 60k+56) = 1$ (since $60k+56 \equiv 2 \ (mod\ 3)$), $\gcd\big(g(v_i), f(u_{i-1})\big) = \gcd(60k+59, 60k+55) = \gcd(4, 60k+55) = 1$ (since $60k+55$ is odd), and $\gcd\big(g(v_i), f(v_{i+1})\big) = \gcd(60k+59, 60k+60) = 1$ (as they are consecutive integers). For the edges incident to u_{i+1}, we find $\gcd\big(g(u_{i+1}), f(u_i)\big) = \gcd(60k+57, 60k+58) = 1$, $\gcd\big(g(u_{i+1}), f(u_{i+2})\big) =$

$\gcd(60k+57, 60k+62) = \gcd(60k+57, 5) = 1$ (since $60k+57 \equiv 2 \ (mod\ 5)$), and $\gcd(g(u_{i+1}), f(v_{i+2})) = \gcd(60k+57, 60k+61) = gcd(60k+57, 4) = 1$ (since $60k+57$ is odd). Finally, for the edge connecting the swapped vertices, $gcd(g(v_i), g(u_{i+1})) = \gcd(60k+59, 60k+57) = \gcd(2, 60k+57) = 1$ (as $60k+57$ is odd).

Similarly, for $i \equiv 14 \ (mod\ 30)$ the adjacent vertices of u_i are u_{i-1}, u_{i+1}, v_{i-1}, and v_{i+1}, while those of v_{i+1} are v_i, v_{i+2}, u_i, and u_{i+2}. The modified function g interchanges the labels of u_i and v_{i+1}. Since $i = 30k+14$ is even, the function results in $g(u_i) = f(v_{i+1}) = 2i+1 = 60k+29$ and $g(v_{i+1}) = f(u_i) = 2i-1 = 60k+27$. The neighboring vertices retain their original labels under f, yielding $f(u_{i-1}) = 2i-2 = 60k+26$, $f(v_{i-1}) = 2i-3 = 60k+25$, $f(u_{i+1}) = 2i+2 = 60k+30$, $f(v_i) = 2i = 60k+28$, $f(v_{i+2}) = 2i+4 = 60k+32$, and $f(u_{i+2}) = 2i+3 = 60k+31$. We compute the gcd for all affected incident edges. For the edges incident to u_i, we have $\gcd(g(u_i), f(u_{i-1})) = gcd(60k+29, 60k+26) = \gcd(3, 60k+26) = 1$ (since $60k+26 \equiv 2 \ (mod\ 3)$); $\gcd(g(u_i), f(v_{i-1})) = \gcd(60k+29, 60k+25) = \gcd(4, 60k+25) = 1$ (since $60k+25$ is odd); and $\gcd(g(u_i), f(u_{i+1})) = \gcd(60k+29, 60k+30) = 1$ (as they are consecutive integers). For the edges incident to v_{i+1}, we find $\gcd(g(v_{i+1}), f(v_i)) = \gcd(60k+27, 60k+28) = 1$ (as they are consecutive integers); $\gcd(g(v_{i+1}), f(v_{i+2})) = \gcd(60k+27, 60k+32) = \gcd(60k+27, 5) = 1$ (since $60k+27 \equiv 2 \ (mod\ 5)$); and $\gcd(g(v_{i+1}), f(u_{i+2})) = \gcd(60k+27, 60k+31) = \gcd(60k+27, 4) = 1$ (since $60k+27$ is odd). Finally, for the edge connecting the swapped vertices, $gcd(g(u_i), g(v_{i+1})) = \gcd(60k+29, 60k+27) = \gcd(2, 60k+27) = 1$ (as $60k+27$ is odd).

Thus, the modified function g successfully resolves all prior constraints and ensures $\gcd(g(x), g(y)) = 1$ for all adjacent $x, y \in V(ZZ_n)$.

Illustration 2.4. Figure 4 illustrates a prime labeling for the zigzag graph ZZ_{16}.

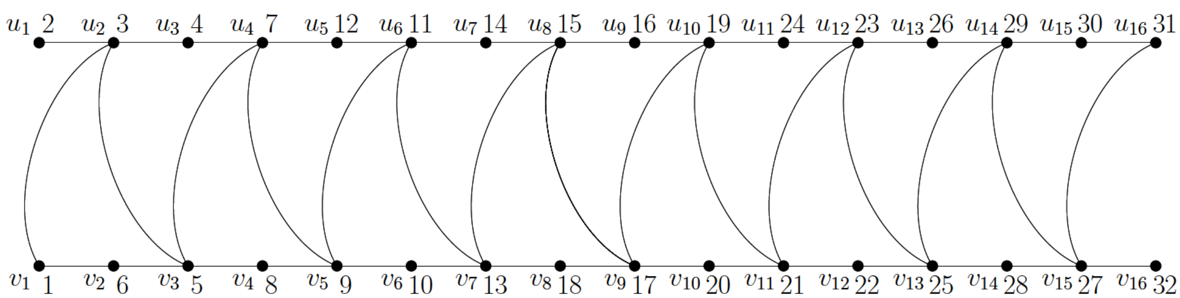

Fig. 4 prime labeling for the zigzag graph ZZ_16.

3. Conclusion

This paper investigates prime labeling in graph networks generated via vertex expansion. We prove that expanding the intermediate vertices of J_n^* or the pendent vertices of $J_{n,m}^*$ into the path P_t preserves primality for all $n, m, t \in \mathbb{N}$. We obtain prime labeling for the zigzag graph. Future research work could apply these expansion techniques to other graph families, investigate alternative labeling schemes, or substitute P_t with different graph structures to benefit network design and secure communications.

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