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Original Article

New Congruences for (2,5)-Biregular Overpartition

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Abstract - *Let $\overline{R}_{l,\mu}$ denote the number of overpartitions of an integer $n \geq 0$ such that no parts of n is a multiple of l and μ, with l and μ being coprime. In this paper, we have established a number of infinite families of congruences and many Ramanujan-type congruences modulo powers of 2 for $\overline{R}_{2,5}(n)$.*

Keywords - *Overpartition, Congruences, Dissection, Regular Partition, Biregular Partition.*

1. Introduction

A partition of a positive integer n is a non-increasing sequence. $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots\ldots \geq \lambda_m$ of positive integers whose sum is n, where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_m$ They are called the parts of the partition.

The generating function for $p(n)$ is defined as

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{f_1}$$

with $|q| < 1$ and the standard $q-$Pochhammer symbol $(a;q)_\infty$ is defined as

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$$

$$\text{and} \quad f_k = (q^k;q^k)_\infty = \prod_{m=0}^{\infty} (1 - q^{km}) \quad \text{for} \quad integers \quad k \geq 0$$

The study of the arithmetic properties of the partition function $p(n)$ was initiated by Ramanujan [21, 22]. He established the following celebrated congruences, valid for all $n \geq 0$:

$$\begin{aligned} p(5n+4) &\equiv 0 \ (mod\ 5) \\ p(7n+5) &\equiv 0 \ (mod\ 7) \\ p(11n+6) &\equiv 0 \ (mod\ 11) \end{aligned}$$

Let $l \geq 1$ be an integer. An $l-regular$ partition of an integer $n \geq 0$ is denoted by $b_l(n)$ Moreover, it is defined as a partition of n in which no part of n is a multiple of l. The generating function of $l-regular$ partition is given by

$$\sum_{n=0}^{\infty} b_l(n)q^n \equiv \frac{f_l}{f_1}$$

The arithmetic properties of $b_l(n)$ have been studied by many mathematicians. In [4], Cui and Gu have investigated $b_l(n)$ for $l = 2,4,5,8,13,16$ and found many infinite families of congruence modulo 2. After that, many researchers studied $b_l(n)$. For references one can see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11].

An overpartition of n, denoted by $\overline{p}(n)$, is a partition of n such that the first occurrence of a part may be overlined. Corteel and Lovejoy [12] defined the generating function for the overpartition function as follows:

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{f_2}{f_1^2}$$

For $n = 3$, we have $\overline{p}(n) = 8$ given by

$$3, \quad \overline{3}, \quad 2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1}, \quad 1+1+1, \quad \overline{1}+1+1$$

Nadji, Ahmia and Ramirez [14] studied the congruence properties of Biregular Overpartition. If $\lambda, \mu \geq 1$ be positive integers with $gcd(\lambda, \mu) = 1$, then an $(\lambda, \mu) - biregular$ overpartition of a positive integer n is a partition of n so that no parts are multiples of λ or μ. It is denoted by the symbol. $\overline{R}_{\lambda,\mu}(n)$. The generating function of $\overline{R}_{\lambda,\mu}(n)$ is defined as

$$\sum_{n=0}^{\infty} \overline{R}_{\lambda,\mu}(n)q^n = \frac{f_2 f_\lambda^2 f_\mu^2 f_{2\lambda\mu}}{f_1^2 f_{2\lambda} f_{2\mu} f_{\lambda\mu}^2} \tag{1.1}$$

Thus, $\overline{R}_{2,5}(5) = 6$, given by $3+1+1,\ \overline{3}+1+1,\ 3+\overline{1}+1,\ \overline{3}+\overline{1}+1,\ 1+1+1+1+1,\ \overline{1}+1+1+1+1.$

The authors established many Ramanujan-type congruences and several infinite families of congruences modulo 3 and powers of 2 $\overline{R}_{\lambda,\mu}(n)$ for the pairs $(\lambda, \mu) \in \{(4,3), (4,9), (8,3), (8,9)$. Among their notable findings were congruences of the form

$$\overline{R}_{4,3}(4n+2) \equiv 0 \ (mod\ 4)$$
$$\overline{R}_{4,3}(8n+4) \equiv 0 \ (mod\ 8)$$
$$\overline{R}_{4,3}(8n+6) \equiv 0 \ (mod\ 16)$$
$$\overline{R}_{4,9}(4n+3) \equiv 0 \ (mod\ 8)$$

Recently, in the paper [13], Alanazi et al. studied the properties of $\overline{R}_{\lambda,\mu}(n)$. The authors established seven-way combinatorial identity which are related to $\overline{R}_{\lambda,\mu}(n)$. The authors also establish many new congruences for $\overline{R}_{\lambda,\mu}(n)$. For example, they showed that

$$\overline{R}_{2,3}(9n+6) \equiv 0 \ (mod\ 6)$$
$$\overline{R}_{4,3}(9n+3) \equiv 0 \ (mod\ 6)$$
$$\overline{R}_{4,9}(12n+4) \equiv 0 \ (mod\ 12)$$

Meher [15] established many families of new congruences modulo powers of 2 and 3 with $(\lambda, \mu) \in \{(2,9), (5,2), (5,4), (8,3)\}$ and in general for $(5, 2^t)$ for all $t \geq 3$ and for $(4, 3^t), (3, 2^t)$ for all $t \geq 1$.

Inspired by the aforementioned studies, we concentrate on the (2,5)-biregular overpartition function to explore its arithmetic characteristics. Utilizing generating function methods and q-series dissections, we derive multiple new congruences modulo powers of 4 and 8. These congruences are distinct from the known results in the literature. Our findings offer additional proof of the complex arithmetic nature of biregular overpartition functions and broaden the current set of known congruence relations in this field.

2. Preliminaries

Ramanujan's general theta function $f(a, b)$ [17, pp. 34, 18.1] is given by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab| < 1 \tag{2.1}$$

Three special cases of $f(a, b)$ are defined for $|q| < 1$, by [17, p.36, Entry 22]

$$\phi(q) = f(q,q) = \sum_{k=0}^{\infty} q^{k^2} = \frac{f_2^5}{f_1^2 f_4^2} \tag{2.2}$$

$$\psi(q) = f(q, q^3) = \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} = \frac{f_2^2}{f_1}, \tag{2.3}$$

$$f(-q) = f(-q, -q^2) = \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(3k+1)}{2}} = f_1 \tag{2.4}$$

This section consists of some important 2 and 3-dissections. We also present $p -$ dissection of $\psi(q)$ and f_1.

Lemma 2.1 [16, Equation (1.10.1)] *The following 2-dissections hold*

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}} \tag{2.5}$$

Lemma 2.2 [16, Equation(14.8.5)] *We have, the following 3-dissection*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3} - 3q f_9^3 \tag{2.6}$$

Lemma 2.3 [6] *We have the 2-dissections*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \tag{2.7}$$

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2} \tag{2.8}$$

Lemma 2.4 [4, Theorem 2.1] *For any odd prime* p*, we have*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, \qquad q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}).$$

Moreover, for $0 \leq k \leq \frac{p-3}{2}$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod 8$$

Lemma 2.5 [4, Theorem 2.2] *For any prime* $p \geq 5$*, we have*

$$f_1 = \sum_{\substack{k=-(p-1)/2 \\ k \neq \pm(p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2}$$

where

$$\frac{\pm p-1}{6} = \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-(p-1)}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, if

$$-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}, \text{and } k \neq \frac{\pm p-1}{6},$$

then

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} (\text{mod} p)$$

Lemma 2.6 [16, p. 212] *We have the following* 5*-dissection formula:*

$$f_1 = f_{25}\left(a(q^5) - q - \frac{q^2}{a(q^5)}\right),$$

where

$$a := a(q) := \frac{(q^2, q^3, q^5; q^5)_\infty}{(q, q^4, q^5; q^5)_\infty}.$$

Lemma 2.7 [24, Lemma 1.2] *We have the binomial theorem*

$$f_{pm}^{p^{k-1}j} \equiv f_m^{p^k j} (\text{mod} p^k)$$

where p, a prime and k, m, j are positive integers. In particular, the following congruences will be used frequently; consequently, explicit reference to this lemma may be omitted in many instances.

$$f_m^{2^k j} \equiv f_{2m}^{2^{k-1}j} (\text{mod} 2^k)$$

For an odd prime p and a any integer coprime to p, the $Legendre\ \ symbol$ $\left(\frac{a}{p}\right)$ is defined as

$$\left(\frac{\mathrm{a}}{\mathrm{p}}\right) = \begin{cases} 1, & \text{if a is a quadratic residue of p,} \\ -1, & \text{if a is a quadratic non} - \text{residue of p.} \end{cases}$$

3. Results and Discussion

Theorem 3.1 *Suppose* p*, a prime with* $\left(\frac{-5}{p}\right) = -1$ *and* $1 \le u \le p-1$. *For all* $n \ge 0$, $\beta \ge 0$ *and* $n_1 = 13,17$*, we have*

$$\overline{R}_{2,5}(4n+3) \equiv 0 (\text{mod} 4) \tag{3.1}$$

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^\beta \cdot p^{2\delta} \cdot (4n+1)\right) q^n \equiv 2 f_1 f_5 (\text{mod} 4) \tag{3.2}$$

$$\overline{R}_{2,5}\left(4 \cdot 5^{\beta+1} \cdot p^{2\delta} \cdot n + n_1 \cdot 5^\beta \cdot p^{2\delta}\right) \equiv 0 (\text{mod} 4) \tag{3.3}$$

$$\overline{R}_{2,5}\left(5^\beta \cdot p^{2\delta+1}\left((4n+1)p + u\right)\right) \equiv 0 (\text{mod} 4) \tag{3.4}$$

Proof. Setting $\lambda = 2$ and $\mu = 5$ in (1.1) gives

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(n) q^n = \frac{f_2^3 f_5^2 f_{20}}{f_1^2 f_4 f_{10}^3} \tag{3.5}$$

Employing (2.7) in (3.5) yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(n) q^n = \frac{f_2^3 f_{20}}{f_4 f_{10}^3}\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}\right)^2 \tag{3.6}$$

Extracting the odd-power terms from (3.6) yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(2n+1) q^n \equiv 2\frac{f_{10}^2 f_2^2}{f_1^2 f_5^2} \tag{3.7}$$

By Lemma 2.7, (3.7) yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(2n+1) q^n \equiv 2 f_2 f_{10} (\text{mod} 4) \tag{3.8}$$

Extracting the coefficient of q^{2n+1} from (3.8) yields (3.1).
Upon extracting the coefficient of q2nq^{2n}q2n from (3.8), the following expression is obtained:

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(4n+1)q^n \equiv 2f_1f_5 (\mathrm{mod}4)$$

Thus (3.2) is true for $\beta = \delta = 0$. Assume (3.2) is true for some $\beta \geq 0$ with $\delta = 0$. Employing Lemma 2.6 in (3.2) gives

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(5^{\beta}(4n+1))q^n \equiv 2f_5f_{25}\left(a(q^5) - q - \frac{q^2}{a(q^5)}\right)(\mathrm{mod}4)$$

Extracting the terms containing q5n+1q^{5n+1}q5n+1 from the last congruence yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{\beta+1}(4n+1)\right)q^n \equiv 2f_1f_5 (\mathrm{mod}4) \tag{3.9}$$

(3.9) is the $\beta + 1$ case of (3.2) with $\delta = 0$. So, (3.2) is true for all $\beta \geq 0$ with $\delta = 0$. Assume (3.2) is true for some $\delta \geq 0$. Employing lemma (2.5) in (3.2), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{\beta}p^{2\delta}(4n+1)\right)q^n \\ \equiv 2\Bigg(\sum_{\substack{x=-\frac{p-1}{2} \\ x\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^x q^{\frac{3x^2+x}{2}} f\left(-q^{\frac{3p^2+(6x+1)p}{2}}, -q^{\frac{3p^2-(6x+1)p}{2}}\right) \\ + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}\Bigg) X \Bigg(\sum_{\substack{y=-\frac{p-1}{2} \\ y\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^y q^{\frac{5(3y^2+y)}{2}} f\left(-q^{\frac{5(3p^2+(6y+1)p)}{2}}, -q^{\frac{5(3p^2-(6y+1)p)}{2}}\right) \\ + (-1)^{\frac{\pm p-1}{6}} q^{\frac{5(p^2-1)}{24}} f_{5p^2}\Bigg)(\mathrm{mod}4)\end{aligned}$$

Consider the congruence

$$\frac{3x^2+x}{2} + 5\frac{3y^2+y}{2} \equiv \frac{p^2-1}{4} \tag{3.11}$$

(3.11) is equivalent to

$$(6x+1)^2 \ + \ 5(6y+1)^2 \ \equiv 0(\mathrm{mod}4)$$

Since $(\frac{-5}{p}) = -1$, $x = y = \frac{\pm p-1}{6}$ is the only solution of the above Equation. Extracting the terms containing $q^{pn+\frac{p^2-1}{4}}$ from (3.10) yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{\beta}p^{2\delta}(4pn+p^2)\right)q^n \equiv 2f_pf_{5p}(\mathrm{mod}4) \tag{3.12}$$

From (3.12), extraction of the terms containing q^{pn} gives

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{\beta}p^{2\delta+2}(4n+1)\right)q^n \equiv 2f_1f_5(\text{mod}4) \tag{3.13}$$

Therefore, (3.2) is true for $\delta+1$. Thus, by induction, (3.2) holds for all $\beta \geq 0$ and $\delta \geq 0$. Utilizing Lemma 2.6 in (3.2) and then extracting the terms containing q^{5n+j} where $j \in \{3,4\}$, yields (3.3)

Finally, the extraction of the terms containingq^{pn+u}, $1 \leq u \leq p-1$, from (3.12) yields (3.4).

Remark: In view of (3.7), we have $\overline{R}_{2,5}(2n+1) \equiv 0(mod\ 2)$.

Theorem 3.2 *For any* $n \geq o$, $\alpha \geq 0$ *and* $r_1 = 11, 19$*, we have*

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{\alpha}(4n+3)\right)q^n \equiv 4f_1^3f_5^3(\text{mod}8) \tag{3.14}$$

$$\overline{R}_{2,5}(4\cdot 5^{\alpha+1}\cdot n + r_1\cdot 5^{\alpha}) \equiv 0(\text{mod}8) \tag{3.15}$$

Proof. Equation (3.7) implies that.

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(2n+1)q^n \equiv 2f_{10}^2f_2^2\left(\frac{1}{f_1^4}\right)\left(\frac{f_1}{f_5}\right)^2 \tag{3.16}$$

Utilizing (2.5) and (2.8) in (3.16), followed by the extraction of the terms containing q^{2n+1}, yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(4n+3)q^n \equiv 4\frac{f_2^{15}f_{10}^3}{f_5^3f_1^{11}f_4^4}(\text{mod}8) \tag{3.17}$$

An application of Lemma 2.7 yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(4n+3)q^n \equiv 4f_1^3f_5^3(\text{mod}8) \tag{3.18}$$

$$\equiv 4f_{25}^3f_5^3\left(\mathrm{a}(\mathrm{q}^5) - \mathrm{q} - \frac{\mathrm{q}^2}{\mathrm{a}(\mathrm{q}^5)}\right)^3(\text{mod}8) \tag{3.19}$$

Extracting the terms containing q^{5n+3} from (3.19), we obtain

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(4(5n+3)+3))q^n \equiv 4f_1^3f_5^3(\text{mod}8)$$

which yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(5(4n+3))q^n \equiv 4f_1^3f_5^3(\text{mod}8)$$

Repeated application of the above relation, together with induction on α, gives

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{\alpha}(4n+3)\right)q^n \equiv 4f_1^3f_5^3(\text{mod}8) \tag{3.20}$$

$$\equiv 4f_{25}^3f_5^3\left(a(q^5) - q - \frac{q^2}{a(q^5)}\right)^3 \tag{3.21}$$

Equation (3.15) follows by extracting the terms containing. q^{5n+2} and q^{5n+4}, from the above congruence.

Theorem 3.3 *For all* $n,\ k \geq 0$*, we have*

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^k \cdot (12n+3)\right)q^n \equiv \sum_{n=0}^{\infty} \overline{R}_{2,5}(12n+3)q^n (\text{mod} 8) \tag{3.22}$$

$$\overline{R}_{2,5}\left(5^{k+1}.\,12n \ + \ 5^k.(12j+3)\right) \equiv 0 (\text{mod} 8) \tag{3.23}$$

Proof. Employing (2.6) in (3.18) yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(4n+3)q^n \equiv 4(f_3 - 3qf_9^3)(f_{15} - 3q^5 f_{45}^3) \tag{3.24}$$

Extraction of the terms containing $q^{3n}$from (3.24) yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(12n+3)q^n \equiv 4f_1 f_5 + 4q^2 f_3^3 f_{15}^3 (\text{mod} 8) \tag{3.25}$$

Which, upon application of Lemma 2.6, implies

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(12n+3)q^n \equiv 4f_5 f_{25}\left(a(q^5) - q - \frac{q^2}{a(q^5)}\right) + 4q^2 f_{15}^3 f_{75}^3 \left(a(q^{15}) - q^3 - \frac{q^6}{a(q^{15})}\right)^3 \tag{3.26}$$

Extraction of the terms containing q^{5n+1} from (3.26), gives

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(12(5n+1)+3)q^n \equiv 4f_1 f_{15} + 4q^2 f_3^3 f_{15}^3 (\text{mod} 8) \tag{3.27}$$

which yields

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5(12n+3)\right)q^n \equiv \sum_{n=0}^{\infty} \overline{R}_{2,5}(4n+3)q^n (\text{mod} 8) \tag{3.28}$$

From the aforementioned relation and an induction argument on α, we obtain (3.22). Equation (3.23) is obtained by employing Lemma 2.6 in (3.22) and then extracting the terms containing. q^{5n+j}, where $j = 3,\ 4$.

Theorem 3.4 *For all* $n \geq 0,\ \alpha \geq 0$*,* $u = 1{,}2$ *and* $v = 0\ {,}4$*, we have*

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{2\alpha}(12n+7)\right)q^n \ \equiv 4f_3^3 f_5 (\text{mod} 8) \tag{3.29}$$

$$\overline{R}_{2,5}\left(5^{2\alpha}(12(5n+u)+7)\right) \equiv 0 (\text{mod} 8) \tag{3.30}$$

$$\overline{R}_{2,5}\left(5^{2\alpha+1}(12(5n+v)+11)\right) \equiv 0 (\text{mod} 8) \tag{3.31}$$

Proof. Upon extraction of the terms containing q^{3n+1} from (3.26), gives

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(12n+7))q^n \equiv 4f_3^3 f_5 (\text{mod} 8)$$

which is the $\alpha = 0$ case of (3.29). Assume (3.29) is true for some $\alpha \geq 0$. Employing Lemma 2.6 in the above congruence

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{2\alpha}(12n+7)\right)q^n \ \equiv 4f_5 f_{75}^3 \left(a(q^{15}) - q^3 - \frac{q^6}{a(q^{15})}\right)^3 \tag{3.32}$$

Extracting the terms containing q^{5n+4} from (3.32), we obtain

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{2\alpha}(60n+55)\right)q^n \ \equiv 4qf_1 f_{15}^3 (\text{mod} 8) \tag{3.33}$$

Utilizing Lemma 2.6 in (3.33) and extracting the terms containing q^{5n+2}, we obtain

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}\left(5^{2\alpha+2}(12n+7)\right)q^n \equiv 4f_3^3 f_5 (\text{mod} 8) \tag{3.34}$$

Thus (3.29) holds for $\alpha + 1$. Hence, by induction, (3.29) always holds.

Extracting the terms containing q^{5n+1} and q^{5n+2} from (3.32), we obtain (3.30).

Finally, utilizing lemma 2.6 in (3.33) and extracting the terms containing. q^{5n} and q^{5n+4}, we obtain (3.31).

Theorem 3.5.. For all $n \geq 0$ and $u = 2,4,7,9,12,14,17,19,22,23,24$, we have

$$\overline{R}_{2,5}\left(100n + (4u+3)\right) = 0 \ (mod\ 8) \tag{3.35}$$

Proof. Recall Jecobi's Identity[22, Theorem 1.3.9]

$$f_1^3 = \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(k+1)}{2}}. \tag{3.36}$$

Utilizing (3.23) into (3.18), we obtain

$$\sum_{n=0}^{\infty} \overline{R}_{2,5}(4n+3)q^n \equiv 4 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (-1)^{k+l} q^{\frac{k(k+1)}{2}+5\frac{l(l+1)}{2}} (\text{mod}8) \tag{3.37}$$

It follows that the possible residues of $\frac{k(k+1)}{2}$ When using modulo 25 are 0, 1, 3, 6, 10, 11, 13, 15, 18, 20, and 21. Additionally, the possible residues of $5\frac{l(l+1)}{2}$ For modulo 25, include 0, 5, and 15. Consequently,

$$q^{\frac{k(k+1)}{2}+5\frac{l(l+1)}{2}} \not\equiv \{2,4,7,9,12,14,17,19,22,23,24\}\ (mod\ 25).$$

By extracting the terms of the form q^{25n+u}, where $u \in \{2,4,7,9,12,14,17,19,22,23,24\}$ from (3.37), Equation (3.22) is obtained.

4. Conclusion

In this study, we discovered several novel Ramanujan-type congruences and infinite families of congruences for $\overline{R}_{2,5}$ with respect to modulo (4) and (8). The derivations were achieved through basic (q)-series manipulations and techniques involving generating functions. These findings offer fresh perspectives on the arithmetic characteristics of $\overline{R}_{2,5}$ Moreover, enhance the understanding of congruences related to partition functions. Exploring congruences modulo higher powers and for associated overpartition functions presents a promising avenue for future research.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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