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Original Article

Maximum Degree Energy and Minimum Degree Energy in the Context of Some Graph Operations

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Abstract *- Let G be a k-regular graph on n vertices. In this paper, we determine the maximum and minimum degree energies of two specific graph operations: the extended m-splitting graph. $Spl_m^*(G)$ and the m-semi shadow graph $SD_m(G)$. By applying block matrix decompositions and unitary similarity transformations, we express the maximum and minimum degree spectra of these constructs explicitly in terms of the ordinary adjacency spectrum of the base graph G. Furthermore, we derive exact, closed-form expressions for their respective maximum and minimum degree energies. As applications of our main theorems, the corresponding energies for well-known families of regular graphs—including cycle, complete, and complete bipartite graphs—are explicitly established.*

Keywords *- Extended m-Splitting, m-Semi Shadow, Maximum Degree Energy, Minimum Degree Energy, Regular Graphs.*

AMS Subject Classification*: 05C50, 05C76.*

1. Introduction

The concept of graph energy, originally introduced by Gutman [1, 2], is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix. This spectral invariant has deep roots in chemical graph theory, where it is utilized to estimate the total π-electron energy of molecules and predict molecular stability [3, 4]. Driven by the theoretical and practical utility of adjacency energy, researchers have defined analogous energies for various degree-based matrix representations of graphs. Adiga and Smitha [5] introduced the maximum degree energy of a graph, and subsequently, Adiga and Swamy [6] formalized bounds for minimum degree eigenvalues.

Understanding the spectral properties of larger graphs constructed via operations on simpler base graphs is a fundamental problem in algebraic graph theory [7, 8]. Standard graph operations, such as splitting and shadow graphs, have been extensively analyzed for their ordinary energetic properties. More specifically related to degree-based invariants, Rao et al. [10] successfully established the maximum and minimum degree energies for p-splitting and p -shadow graphs, paving the way for analogous studies on generalized structural operations.

Recently, the catalog of such network operations was expanded: Vaidya and Rathod [11] introduced the extended m -splitting graph, while Patel et al. [9] conceptualized the m -semi shadow graph. While the ordinary energies of these newly constructed graphs have been explored, their degree-based matrix energies remain an open inquiry. In this paper, extending the foundational methodology of Rao et al. [10], we compute the exact maximum and minimum degree spectra and energies for the extended m -splitting and m -semi shadow graphs when the base graph is regular.

2. Preliminaries and Notation

Let $G = (V, E)$ be a simple, finite, and undirected graph with vertex set $V(G)$ of order n. The adjacency matrix of G is the $n \times n$ matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. The eigenvalues of $A(G)$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, form the ordinary spectrum of G, and the ordinary graph energy is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

For a vertex $v_i \in V(G)$, d_i denotes its degree. The maximum degree matrix $M(G) = [d_{ij}]$ and the minimum degree matrix $M_{min}(G) = [d'_{ij}]$ are formally defined as:

$$d_{ij}=\begin{cases}\max\{d_i,d_j\} & if\ \ v_i\sim v_j,\\ 0 & otherwise,\end{cases}\ \text{and}\ \ d'_{ij}=\begin{cases}\min\{d_i,d_j\} & if\ \ v_i\sim v_j,\\ 0 & otherwise,\end{cases}$$

The maximum degree of energy $E_M(G)$ and minimum degree energy $E_{min}(G)$ are the sums of the absolute values of the eigenvalues of $M(G)$ and $M_{min}(G)$, respectively. A key property utilized throughout this structural analysis is that if G is a k -regular graph, all vertices have uniform degree k. Consequently, for an edge between any two adjacent vertices, the maximum and minimum degree values coincide perfectly, reducing both matrices to scalar multiples of the adjacency matrix:

$$M(G)=M_{min}(G)=k\cdot A(G)$$

To evaluate the partitioned matrices that arise from the graph operations, we rely on standard properties of the Kronecker product $\otimes$. Specifically, if λ is an eigenvalue of matrix A with eigenvector x, and μ is an eigenvalue of matrix B with eigenvector y, then $\lambda\mu$ is an eigenvalue of $A\otimes B$ with corresponding eigenvector $x\otimes y$.

3. Extended m-Splitting Graph

Definition 3.1. (Extended m-splitting graph [11]). Let G be a graph with vertex set $V(G)=\{v_1,v_2,\dots,v_n\}$. The extended m-splitting graph of G, denoted $Spl_m^*(G)$, is constructed by adding m new vertices $v_i^{(1)},v_i^{(2)},\cdots,v_i^{(m)}$ for each original vertex $v_i\in V(G)$. The edges are defined such that for each $1\le j\le m$, the new vertex. $v_i^{(j)}$ is adjacent to:

- Every vertex that is adjacent to v_i in G,
- The original parent vertex v_i,
- Every other new vertex $v_i^{(l)}$ associated with the same parent v_i (for $l\neq j$).

More concisely, the closed neighborhood of each new vertex $v_i^{(j)}$ in $\mathrm{Spl}_m^*(G\ )$ satisfies $N_{Spl_m^*(G)}\left[v_i^{(j)}\right]=\{v_i^{(1)},v_i^{(2)},\cdots,v_i^{(m)}\}\cup N_G[v_i]$.

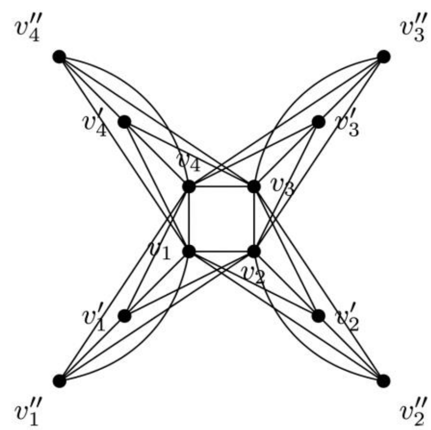

Fig. 1 Extended 2-splitting graph of C_4

The original vertices are denoted by v_i, while the corresponding

New vertices are denoted by v_i' and v_i'' (representing $v_i^{(1)}$ and $v_i^{(2)}$).

3.1. Maximum Degree Energy of the Extended m -splitting of a Regular Graph

Following [1], the maximum degree energy $E_M(G)$ of a simple graph G is the sum of the absolute values of the eigenvalues of its maximum degree matrix $M(G)$, where

$$M(G)=[d_{ij}],\quad d_{ij}=\begin{cases}\max\{d_i,d_j\} & if\ \ v_i\sim v_j,\\ 0 & otherwise,\end{cases}$$

and d_i denotes the degree of v_i in G. If G is k -regular, then $M(G)=k\cdot A(G)$, where $A(G)$ is the adjacency matrix of G.

Let G be a k-regular graph. In $\mathrm{Spl}_{\mathrm{m}}^{*}(G)$, We denote the set of original vertices by $V_0 = V(G)$ and the set of new vertices by V_n. For any original vertex $v \in V_0$Its degree is $\deg(v) = k + m + km = k(m+1) + m =: a$. For any new vertex $v \in V_n$Its degree is $\deg(u) = 1 + k + (m-1) = k + m =: c$.

Theorem 3.1.1. Let G be a k -regular graph on n vertices with adjacency eigenvalues $\{\mu_1 = k, \mu_2, \dots, \mu_n\}$. Set

$$t := \frac{(m+1)k + m}{k+m} = 1 + \frac{mk}{k+m}$$

Then the maximum degree spectrum of the extended m -splitting graph $\mathrm{Spl}_{\mathrm{m}}^{*}(G)$ is

$$\mathrm{Spec}(\mathrm{M}(\mathrm{Spl}_{\mathrm{m}}^{*}(G))) = \{c\lambda_{i_+}^{(M)}, c\lambda_{i_-}^{(M)} : i = 1, \dots, n\} \cup \{-c(with\ multiplicity\ n(m-1)\},$$

Where $c = k + m$, and for each i,

$$\lambda_{i_\pm}^{(M)} = \frac{t\mu_i + (m-1)}{2} \pm \frac{1}{2}\sqrt{\alpha\mu_i^2 + \beta\mu_i + \gamma}\ ,$$

With constants $\alpha = t^2(4m+1)$, $\beta = 8t^2m - 2t(m-1)$, $\gamma = (m-1)^2 + 4t^2m$.

Consequently, the maximum degree of energy of $\mathrm{Spl}_m^*(G)$ is

$$E_M(\mathrm{Spl}_m^*(G)) = (k+m)\left(n(m-1) + \sum_{i=1}^{n}\sqrt{\alpha\mu_i^2 + \beta\mu_i + \gamma}\right).$$

Proof. Order the vertices of $\mathrm{Spl}_m^*(G)$ such that the n original vertices in V_o appear first, followed by the mn new vertices in V_n grouped by their original parent vertex. By the definition of the maximum degree matrix, the block M_{11} arises from the edges between original vertices V_o, each carrying a maximum weight$(a, a) = a$. The off-diagonal block M_{12} corresponds to edges between V_o and V_n (both original-new adjacencies and parent-child adjacencies), carrying weight $\max(a, c) = a$. The block M_{22} corresponds to edges within each m-clique of new vertices, carrying weight $\max(c, c) = c$. Thus, M has the block form.

$$M = \begin{bmatrix} a\,A(G) & a\,(A(G) + I_n) \otimes 1_m^{\top} \\ a\,(A(G) + I_n) \otimes 1_m & c\,I_n \otimes (J_m - I_m) \end{bmatrix}$$

Let $\{u_1, \dots, u_n\}$ be an orthonormal eigenbasis of $A(G)$ with $A(G)u_i = \mu_i u_i$. The matrix M acts on $\mathbb{R}^n \oplus \mathbb{R}^{mn}$. Define the vectors:

$$x_i := \begin{pmatrix} u_i \\ 0 \end{pmatrix}, \qquad y_i := \begin{pmatrix} 0 \\ u_i \otimes 1_m \end{pmatrix}$$

For each fixed i, we define the subspace. $S_{u_i} := \mathrm{span}\{x_i, y_i\} \oplus (0 \oplus (u_i \otimes 1_m^{\perp}))$. This subspace is invariant under M. On the $(m-1)$-dimensional slice $0 \oplus (u_i \otimes 1_m^{\perp})$, we have $(J_m - I_m)w = -w$ for $w \perp 1_m$. Applying M to any vector $\begin{pmatrix} 0 \\ u_i \otimes w \end{pmatrix}$ where $w \perp 1_m$ yields $-c$, which means $-c = -(k+m)$ is an eigenvalue with multiplicity $n(m-1)$.

On the 2-dimensional slice $\mathrm{span}\{x_i, y_i\}$, we evaluate the action of M:

$$Mx_i = \begin{pmatrix} aA(G)u_i \\ a(A(G) + I_n)u_i \otimes 1_m \end{pmatrix} = \begin{pmatrix} a\mu_i u_i \\ a(\mu_i + 1)u_i \otimes 1_m \end{pmatrix} = a\mu_i x_i + a(\mu_i + 1)y_i,$$

$$My_i = \begin{pmatrix} a(A(G) + I_n)u_i \otimes 1_m^{\top}1_m \\ cI_n u_i \otimes (J_m - I_m)1_m \end{pmatrix} = \begin{pmatrix} am(\mu_i + 1)u_i \\ c(m-1)u_i \otimes 1_m \end{pmatrix} = am(\mu_i + 1)x_i + c(m-1)y_i.$$

Therefore, the restriction of M to $\mathrm{span}\{x_i, y_i\}$ is represented by the 2×2 matrix:

$$M|_{\mathrm{span}\{x_i, y_i\}} = \begin{bmatrix} a\,\mu_i & a\,m(\mu_i + 1) \\ a\,(\mu_i + 1) & c\,(m-1) \end{bmatrix}$$

Dividing the matrix by c leads to the normalized 2×2 matrix:

$$M' = \begin{bmatrix} t\,\mu_i & t\,m(\mu_i+1) \\ t\,(\mu_i+1) & m-1 \end{bmatrix}$$

The characteristic polynomial is given by $\lambda^2 - \mathrm{Tr}(M')\lambda + \mathrm{Det}(M') = 0$, where $\mathrm{Tr}(M') = t\mu_i + (m-1)$ and $\mathrm{Det}(M') = t(m-1)\mu_i - t^2 m(\mu_i+1)^2$. The discriminant yields $\Delta = \left(\mathrm{Tr}(M')\right)^2 - 4\mathrm{Det}(M') = t^2(4m+1)\mu_i^2 + [8t^2 m - 2t(m-1)]\mu_i + [(m-1)^2 + 4t^2 m]$, matching our definitions for α, β, and γ. Summing the absolute values of all eigenvalues yields (2).

Corollary 3.1.1. (Cycle). Let C_n be the cycle on n vertices. The maximum degree of energy of $\mathrm{Spl}_m^*(C_n)$ is

$$E_M(\mathrm{Spl}_m^*(C_n)) = (m+2)\left(n(m-1) + \sum_{j=0}^{n-1}\sqrt{4\alpha\cos^2(\frac{2\pi j}{n}) + 2\beta\cos(\frac{2\pi j}{n}) + \gamma}\right)$$

with constants $\alpha = t^2(4m+1), \beta = 8t^2 m - 2t(m-1), \gamma = (m-1)^2 + 4t^2 m$, where $t = 1 + \frac{2m}{m+2}$.

Proof. The base cycle graph C_n is 2-regular ($k = 2$) with an adjacency spectrum given by $\mu_j = 2\cos(2\pi j/n)$ for $j = 0,1,\dots,n-1$. Substituting these parameters directly into the master maximum degree energy expression in (2) yields the result.

Corollary 3.1.2. (Complete Graph). Let K_n be the complete graph on n vertices. The maximum degree of energy of $\mathrm{Spl}_m^*(K_n)$ is

$$E_M(\mathrm{Spl}_m^*(K_n)) = (m+n-1)\left(n(m-1) + \sqrt{\alpha(n-1)^2 + \beta(n-1) + \gamma} + (n-1)\sqrt{\alpha - \beta + \gamma}\right)$$

with constants $\alpha = t^2(4m+1), \beta = 8t^2 m - 2t(m-1), \gamma = (m-1)^2 + 4t^2 m$, where $t = 1 + \frac{(n-1)m}{m+n-1}$.

Proof. The complete graph K_n is $(n-1)$-regular ($k = n-1$) with an adjacency spectrum consisting of $\mu_1 = n-1$ and a repeating eigenvalue $\mu_i = -1$ of multiplicity $n-1$. Distributing these values across the summation indices of (2) produces the formula.

Corollary 3.1.3. (Complete Bipartite Graph). Let $K_{k,k}$ be the complete bipartite graph on $n = 2k$ vertices. The maximum degree of energy of $\mathrm{Spl}_m^*(K_{k,k})$ is

$$E_M(\mathrm{Spl}_m^*(K_{k,k})) = (m+k)\left(2k(m-1) + \sqrt{\alpha k^2 + \beta k + \gamma} + \sqrt{\alpha k^2 - \beta k + \gamma} + (2k-2)\sqrt{\gamma}\right)$$

with constants $\alpha = t^2(4m+1), \beta = 8t^2 m - 2t(m-1), \gamma = (m-1)^2 + 4t^2 m$ where $t = 1 + \frac{mk}{m+k}$.

Proof. The complete bipartite graph $K_{k,k}$ is k-regular on $2k$ vertices. Its standard adjacency spectrum is given by $\{k, -k, 0^{(2k-2)}\}$. Evaluating the main radical sum in (2) over these three isolated spectral components yields the stated expression.

3.2. Minimum Degree Energy of the Extended m-splitting of a Regular Graph

Following the block-spectral method developed in the previous section, analogous results are now derived for the minimum degree matrix. The minimum degree of energy $E_{\min}(G)$ of a simple graph G is the sum of the absolute values of the eigenvalues of its minimum degree matrix $M_{\min}(G)$:

$$M_{\min}(G) = [d_{ij}], \qquad d_{ij} = \begin{cases} \min\{d_i, d_j\}, & \text{if } v_i \sim v_j \\ 0, & \text{otherwise} \end{cases}$$

where d_i denotes the degree of v_i in G. Let G be k-regular and let $\mathrm{Spl}_m^*(G)$ be the extended m-splitting graph of G. The degrees in $\mathrm{Spl}_m^*(G)$ are:

$$\deg(v) = a := k(m+1) + m \quad \text{for } v \in V_o, \qquad \deg(u) = c := k + m \quad \text{for } u \in V_n$$

Theorem 3.2.1. Let G be a k-regular graph on n vertices with adjacency eigenvalues $\{\mu_1 = k, \mu_2, \dots, \mu_n\}$. In $\mathrm{Spl}_m^*(G)$, the minimum degree matrix $M_{\min}(\mathrm{Spl}_m^*(G))$ has spectrum:

$$\mathrm{Spec}(M_{\min}(\mathrm{Spl}_m^*(G))) = \{\lambda_{i+}^{(m)}, \lambda_{i-}^{(m)} : i = 1, \dots, n\} \cup \{-c \text{ (with multiplicity } n(m-1))\},$$

where, for each i,

$$\lambda_{i\pm}^{(m)} = \frac{a\,\mu_i + c(m-1) \pm \sqrt{\alpha\mu_i^2 + \beta\mu_i + \gamma}}{2},$$

with constants $\alpha = a^2 + 4mc^2$, $\beta = 8c^2m - 2ac(m-1)$, and $\gamma = c^2(m+1)^2$.

Consequently, the minimum degree energy is:

$$E_{\min}(\mathrm{Spl}_m^*(G)) = n(m-1)c + \sum_{i=1}^{n} \sqrt{\alpha\mu_i^2 + \beta\mu_i + \gamma}.$$

Proof. Order the vertices as all n original. V_o followed by the mn new vertices V_n grouped by parent. Along an edge of $\mathrm{Spl}_m^*(G)$, the minimum of endpoint degrees is: - $\min(a, a) = a$ on original–original edges, - $\min(a, c) = c$ on original–new edges, - $\min(c, c) = c$ on new–new edges within each m-clique.

Hence $M_{\min}(\mathrm{Spl}_m^*(G))$ has the block form:

$$M_{\min} = \begin{bmatrix} a\,A(G) & c\,(A(G) + I_n) \otimes 1_m^{\top} \\ c\,(A(G) + I_n) \otimes 1_m & c\,I_n \otimes (J_m - I_m) \end{bmatrix}$$

Let $\{u_1, \dots, u_n\}$ be an orthonormal eigenbasis of $A(G)$ with $A(G)u_i = \mu_i u_i$. For each fixed i, define the invariant subspace:

$$S_{u_i} := \mathrm{span}\{x_i, y_i\} \;\oplus\; (0 \oplus (u_i \otimes 1_m^{\perp})), \quad \text{where } x_i = \begin{pmatrix} u_i \\ 0 \end{pmatrix},\ y_i = \begin{pmatrix} 0 \\ u_i \otimes 1_m \end{pmatrix}.$$

On the slice $0 \oplus (u_i \otimes 1_m^{\perp})$, we have $(J_m - I_m)w = -w$ for $w \perp 1_m$, so $-c$ appears with multiplicity $m-1$ for each i, yielding $n(m-1)$ in total.

Restricting to $\mathrm{span}\{x_i, y_i\}$, applying $M_{\min}$ Similarly to Theorem 3.2, it yields the 2×2 block:

$$M_{\min}|_{\mathrm{span}\{x_i, y_i\}} = \begin{bmatrix} a\,\mu_i & c\,m(\mu_i + 1) \\ c\,(\mu_i + 1) & c\,(m-1) \end{bmatrix}$$

The eigenvalues of this 2×2 matrix are exactly. $\lambda_{i\pm}^{(m)}$ displayed in the statement (the discriminant simplifies to $\alpha\mu_i^2 + \beta\mu_i + \gamma$). Summing the absolute values gives the energy formula.

Corollary 3.2.1. (Cycle). Let C_n be the cycle on n vertices. The minimum degree energy of $\mathrm{Spl}_m^*(C_n)$ is:

$$E_{\min}(\mathrm{Spl}_m^*(C_n)) = n(m-1)(m+2) + \sum_{j=0}^{n-1} \sqrt{4\alpha\cos^2(\frac{2\pi j}{n}) + 2\beta\cos(\frac{2\pi j}{n}) + \gamma}$$

where $\alpha = (3m+2)^2 + 4m(m+2)^2$, $\beta = 2(m+2)(4m(m+2) - (m-1)(3m+2))$, and $\gamma = (m+2)^2(m+1)^2$.

Proof. For the cycle C_n, the structural invariants are regularity $k = 2$, vertex degree bound $a = 3m+2$, and child degree limit $c = m+2$. Mapping its adjacency roots $\mu_j = 2\cos(2\pi j/n)$ directly into the minimum degree master equation (3) yields the result.

Corollary 3.2.2. (Complete Graph). Let K_n be the complete graph on n vertices. The minimum degree energy of $\mathrm{Spl}_m^*(K_n)$ is:

$$E_{\min}(\mathrm{Spl}_m^*(K_n)) = n(m-1)(m+n-1) + \sqrt{\alpha(n-1)^2 + \beta(n-1) + \gamma} + (n-1)\sqrt{\alpha - \beta + \gamma}$$

where $\alpha = (nm + n - 1)^2 + 4m(m + n - 1)^2$, $\beta = 2(m + n - 1)(4(m + n - 1)m - (m - 1)(mn + n - 1))$, and $\gamma = (m + n - 1)^2(m + 1)^2$.

Proof. The complete graph K_n has regularity $k = n - 1$, which sets the matrix weights to $a = nm + n - 1$ and $c = m + n - 1$. Applying its discrete adjacency spectrum $\{n - 1, -1^{(n-1)}\}$ to the structural parameters of equation (3), derive the formulation.

Corollary 3.2.3. (Complete Bipartite Graph). Let $K_{k,k}$ be the complete bipartite graph on $n = 2k$ vertices. The minimum degree energy of $\mathrm{Spl}_m^*(K_{k,k})$ is:

$$E_{\min}(\mathrm{Spl}_m^*(K_{k,k})) = 2k(m - 1)(m + k) + \sqrt{\alpha k^2 + \beta k + \gamma} + \sqrt{\alpha k^2 - \beta k + \gamma} + (2k - 2)\sqrt{\gamma}$$

where $\alpha = (km + k + m)^2 + 4m(m + k)^2$, $\beta = 2(m + k)(4(k + m)m - (m - 1)(mk + k + m))$, and $\gamma = (m + k)^2(m + 1)^2$.

Proof. The base graph $K_{k,k}$ is k-regular on $2k$ vertices, establishing matrix scales $a = km + k + m$ and $c = m + k$. Distributing its concentrated spectrum $\{k, -k, 0^{(2k-2)}\}$ across the partitioned sum of Theorem 3.6 maps directly to this format.

4. m-Semi Shadow Graph

Definition 4.1 ([9]). Let G be a graph with vertices $v_1, v_2, \dots, v_n$. The m-semi shadow graph $SD_m(G)$ of graph G is constructed by taking graph G and m copies of G say $G^1, G^2, \dots, G^m$ with vertices $v_i^1, v_i^2 \dots, v_i^m$, where $1 \le i \le n$, and joining each vertex v_i^j to all vertices that are adjacent to v_i in G, where $1 \le i \le n$ and $1 \le j \le m$.

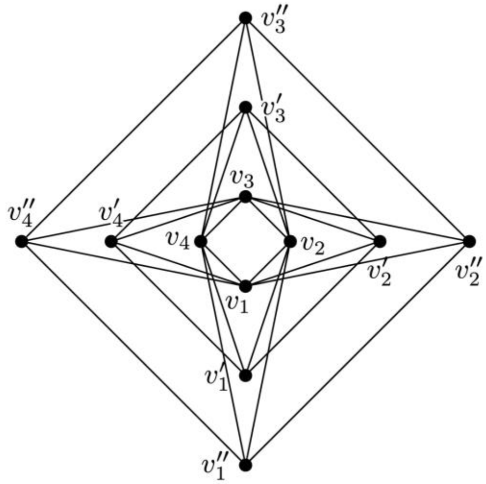

Fig. 2 2-semi shadow graph of C_4.

4.1. Maximum Degree Energy of the m-semi Shadow Graph of a Regular Graph

Theorem 4.1.1. Let G be a k-regular graph on n vertices with ordinary adjacency eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Let $SD_m(G)$ denote the m-semi shadow graph of G, and let $E_M(G)$ denote the maximum degree energy of G. Then:

1. The spectrum of the maximum degree matrix $M(SD_m(G))$ consists of the eigenvalue 0 with multiplicity $n(m - 1)$, along with the pairs:

$$\mu_i^{(1),(2)} = k(m + 1)\left(\frac{1 \pm \sqrt{1 + 4m}}{2}\right)\lambda_i \quad \text{for } i = 1,2, \dots, n.$$

2. The maximum degree energy of $SD_m(G)$ is given by:

$$E_M(SD_m(G)) = (m + 1)\sqrt{1 + 4m} \cdot E_M(G).$$

Proof. 1. *Matrix Construction:* Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of the base graph G, and let $V(G^j) = \{v_1^j, v_2^j, \dots, v_n^j\}$ for $1 \le j \le m$ be the vertices of the m copies. By definition of the m-semi shadow graph $SD_m(G)$, the degrees of the vertices are: - $d_{SD_m(G)}(v_i^j) = k$ for all $1 \le j \le m$. - $d_{SD_m(G)}(v_i) = k + mk = k(m+1)$.

The entries of the maximum degree matrix $M(SD_m(G)) = [d_{ij}]$ are defined as $\max\{d(u), d(v)\}$ if $u \sim v$, and 0 otherwise. - For an edge $v_i \sim v_l$ within the base graph G, $\max\{d(v_i), d(v_l)\} = k(m+1)$. - For an edge $v_i \sim v_l^j$ connecting the base graph to a copy, $\max\{d(v_i), d(v_l^j)\} = \max\{k(m+1), k\} = k(m+1)$. - There are no edges within or between the copy sets G^j.

Ordering the vertices as $(V, V^1, V^2, \dots, V^m)$, the maximum degree matrix can be partitioned into the following $(m+1) \times (m+1)$ block matrix:

$$M(SD_m(G)) = k(m+1) \begin{bmatrix} A & A & A & \dots & A \\ A & 0 & 0 & \dots & 0 \\ A & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A & 0 & 0 & \dots & 0 \end{bmatrix}_{(m+1)n \times (m+1)n}$$

where A is the ordinary adjacency matrix of G.

2. *Spectral Derivation:* Since the blocks of $M(SD_m(G))$ are isotropic polynomial functions of A, its eigenvalues can be obtained by replacing the matrix block A with its scalar eigenvalues λ_i. For each $\lambda_i \in \mathrm{Spec}(A)$We examine the characteristic equation of the resulting $(m+1) \times (m+1)$ matrix block:

$$\det \left[xI_{m+1} - k(m+1)\lambda_i \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \beta & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix} \right] = 0$$

Let $y = \frac{x}{k(m+1)\lambda_i}$. The determinant equation reduces to:

$$\det \begin{bmatrix} y-1 & -1 & -1 & \dots & -1 \\ -1 & y & 0 & \dots & 0 \\ -1 & 0 & y & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & y \end{bmatrix} = 0$$

Using the Schur complement for block determinants, this evaluates directly to:

$$y^{m-1}(y(y-1) - m) = 0 \Rightarrow y^{m-1}(y^2 - y - m) = 0.$$

Solving for y yields $y = 0$ with multiplicity $m - 1$, and the quadratic roots $y = \frac{1 \pm \sqrt{1+4m}}{2}$. Substituting back $x = y \cdot k(m+1)\lambda_i$, and accounting for all n eigenvalues of A, we obtain the spectrum stated in Part 1.

3. *Energy Formulation:* The maximum degree energy is the absolute sum of these eigenvalues:

$$E_M(SD_m(G)) = \sum_{i=1}^{n} \left| k(m+1)\left(\frac{1+\sqrt{1+4m}}{2}\right)\lambda_i \right| + \sum_{i=1}^{n} \left| k(m+1)\left(\frac{1-\sqrt{1+4m}}{2}\right)\lambda_i \right|$$

Factoring out the scalar components and noting that $\sqrt{1+4m} > 1$ for all $m \ge 1$:

$$E_M(SD_m(G)) = k(m+1)\left(\frac{1+\sqrt{1+4m}}{2} + \frac{\sqrt{1+4m}-1}{2}\right)\sum_{i=1}^{n} |\lambda_i|$$

$$E_M(SD_m(G)) = k(m+1)\sqrt{1+4m} \cdot E(G)$$

Since G is k-regular, its maximum degree matrix satisfies $M(G) = k \cdot A(G)$, meaning its maximum degree of energy is $E_M(G) = k \cdot E(G)$. Substituting $E_M(G)$ yields:

$$E_M\big(SD_m(G)\big) = (m+1)\sqrt{1+4m} \cdot E_M(G).$$

This completes the proof.

Corollary 4.1.1. (Cycle Graph). Let C_n be the cycle graph on n vertices. The maximum degree energy of the m-semi shadow graph $SD_m(C_n)$ is given by:

$$E_M\big(SD_m(C_n)\big) = 4(m+1)\sqrt{1+4m}\sum_{j=0}^{n-1}\left|\cos\left(\frac{2\pi j}{n}\right)\right|.$$

Proof. The cycle graph C_n is 2-regular ($k = 2$), and its ordinary adjacency spectrum consists of $\lambda_j = 2\cos\left(\frac{2\pi j}{n}\right)$ for $j = 0,1,\dots,n-1$. Its ordinary graph energy is $E(C_n) = \sum_{j=0}^{n-1}\left|2\cos\left(\frac{2\pi j}{n}\right)\right|$. The maximum degree energy of the base graph is $E_M(C_n) = k \cdot E(C_n) = 2 \cdot E(C_n)$. Substituting this into the energy relation of the preceding theorem yields the result.

Corollary 4.1.2. (Complete Graph). Let K_n be the complete graph on n vertices. The maximum degree energy of the m-semi shadow graph $SD_m(K_n)$ is given by:

$$E_M\big(SD_m(K_n)\big) = 2(n-1)^2(m+1)\sqrt{1+4m}.$$

Proof. The complete graph K_n is $(n-1)$-regular ($k = n-1$). Its ordinary adjacency spectrum is $\{n-1, -1^{(n-1)}\}$, which gives an ordinary graph energy of $E(K_n) = (n-1) + |-1|(n-1) = 2(n-1)$. The maximum degree energy of the base graph is $E_M(K_n) = k \cdot E(K_n) = (n-1) \cdot 2(n-1) = 2(n-1)^2$. Applying the preceding theorem yields the stated formula.

Corollary 4.1.3. (Complete Bipartite Graph). Let $K_{n,n}$ be the complete bipartite graph on $2n$ vertices. The maximum degree energy of the m-semi shadow graph $SD_m\big(K_{n,n}\big)$ is given by:

$$E_M\left(SD_m\big(K_{n,n}\big)\right) = 2n^2(m+1)\sqrt{1+4m}.$$

Proof. The complete bipartite graph $K_{n,n}$ is n-regular ($k = n$) on $2n$ vertices. Its ordinary adjacency spectrum is $\{n, -n, 0^{(2n-2)}\}$. The ordinary graph energy is $E\big(K_{n,n}\big) = |n| + |-n| = 2n$. Therefore, its maximum degree energy is $E_M\big(K_{n,n}\big) = k \cdot E\big(K_{n,n}\big) = n(2n) = 2n^2$. Substituting this into the theorem gives the final expression.

4.2. Minimum Degree Energy of the m-semi Shadow of a Regular Graph

Theorem 4.2.1. Let G be a k-regular graph on n vertices with adjacency spectrum $\{\lambda_1,\dots,\lambda_n\}$ and minimum degree energy $E_m(G)$. The minimum degree spectrum and energy of its m-semi shadow graph $SD_m(G)$ are given by:

1. $\text{Spec}(m\big(SD_m(G)\big))$ consists of the eigenvalues $2k\lambda_i$ with multiplicity $m-1$ for each $i = 1,\dots,n$ along with the pairs:
$$\mu_i^{(1),(2)} = \frac{k}{2}\left[(m+3) \pm \sqrt{m^2+14m+1}\right]\lambda_i \quad \text{for } i = 1,2,\dots,n.$$
2. The minimum degree energy of $SD_m(G)$ is given by:
$$E_m\big(SD_m(G)\big) = \left(2(m-1) + \sqrt{m^2+14m+1}\right)E_m(G).$$

Proof. Following the degree profile from the previous theorem, the endpoints have degrees $k(m+1)$ and $2k$. Because $k(m+1) \geq 2k$ for $m \geq 1$, the minimum degree evaluations yield $k(m+1)$ for internal base graph edges, and $2k$ for both cross-edges and internal copy edges. The global minimum degree matrix decomposes into:

$$m\big(SD_m(G)\big) = \begin{bmatrix} k(m+1)A & 2kA & 2kA & \dots & 2kA \\ 2kA & 2kA & 0 & \dots & 0 \\ 2kA & 0 & 2kA & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2kA & 0 & 0 & \dots & 2kA \end{bmatrix}_{(m+1)n\times(m+1)n}$$

Applying a standard block matrix unitary transformation matrix $U \otimes I_n$, where U acts on the layer blocks to isolate the copy variations, the structure is transformed via block similarity into:

$$m\big(SD_m(G)\big) \cong \begin{bmatrix} k(m+1)A & 2\sqrt{m}kA & 0 \\ 2\sqrt{m}kA & 2kA & 0 \\ 0 & 0 & 2I_{n(m-1)} \otimes kA \end{bmatrix}$$

From this orthogonal decomposition, the copy variations directly provide the eigenvalue $2k\lambda_i$ with a base multiplicity of $m-1$ for every spectral component of A. The remaining components are governed by the localized 2×2 block coefficient matrix:

$$\begin{bmatrix} k(m+1) & 2\sqrt{m}k \\ 2\sqrt{m}k & 2k \end{bmatrix} \otimes A$$

For each individual eigenvalue λ_i of A, the characteristic polynomial of this scalar block system reduces to:

$$\theta^2 - k(m+3)\theta + 2k^2(m+1) - 4mk^2 = 0 \Rightarrow \theta^2 - k(m+3)\theta + 2k^2(1-m) = 0$$

Solving for θ via the quadratic formula yields the exact roots:

$$\theta = \frac{k(m+3) \pm \sqrt{k^2(m+3)^2 - 8k^2(1-m)}}{2} = \frac{k}{2}\Big[(m+3) \pm \sqrt{m^2+14m+1}\Big]$$

Scaling these roots by the structural spectrum λ_i yields the pairs $\mu_i^{(1),(2)}$ stated in Part 1.

Summing the absolute values across all sections to compute the energy yields:

$$E_m\big(SD_m(G)\big) = n(m-1)|2k\lambda_i| + \sum_{i=1}^{n}\left|\mu_i^{(1)}\right| + \sum_{i=1}^{n}\left|\mu_i^{(2)}\right|$$

Since $\sqrt{m^2+14m+1} > m+3$, the radical components strictly dominate the trace boundary, meaning $\mu_i^{(1)}$ is positive and $\mu_i^{(2)}$ is completely negative for positive λ_i. This perfectly maps the absolute sum of these pairs to:

$$\left|\mu_i^{(1)}\right| + \left|\mu_i^{(2)}\right| = \sqrt{m^2+14m+1} \cdot k|\lambda_i|$$

Combining this with the layer variations component completes the proof:

$$E_m\big(SD_m(G)\big) = \left(2(m-1) + \sqrt{m^2+14m+1}\right) \cdot k\sum_{i=1}^{n}|\lambda_i| = \left(2(m-1) + \sqrt{m^2+14m+1}\right)E_m(G).$$

Corollary 4.2.1. (Cycle Graph). Let C_n be the cycle graph on n vertices. The minimum degree energy of the m-semi shadow graph $SD_m(C_n)$ is given by:

$$E_m\big(SD_m(C_n)\big) = 4\left(2(m-1) + \sqrt{m^2+14m+1}\right)\sum_{j=0}^{n-1}\left|\cos\left(\frac{2\pi j}{n}\right)\right|.$$

Proof. For the 2-regular cycle graph C_n, the adjacency eigenvalues are $\lambda_j = 2cos\left(\frac{2\pi j}{n}\right)$ for$j = 0,1,\dots,n-1$.

The base minimum degree energy is dynamically equivalent to its maximum degree energy, giving. $E_m(C_n) = 2E(C_n) = 4\sum_{j=0}^{n-1}\left|\cos\left(\frac{2\pi j}{n}\right)\right|$. Factoring this base energy into the structural multiplier derived in the

The minimum degree theorem completes the proof.

Corollary 4.2.2. (Complete Graph). Let K_n be the complete graph on n vertices. The minimum degree energy of the m-semi shadow graph $SD_m(K_n)$ is given by:

$$E_m\big(SD_m(K_n)\big) = 2(n-1)^2\left(2(m-1) + \sqrt{m^2 + 14m + 1}\right).$$

Proof. As established, the complete graph K_n is regular with degree $k = n - 1$ and ordinary energy$E(K_n) = 2(n-1)$. Because K_n is regular, its base minimum degree energy satisfies$E_m(K_n) = k\ E(K_n) = 2(n-1)^2$. Applying the algebraic scaling factor for the minimum degree transformation of the m-semi shadow graph yields the closed form.

Corollary 4.2.3. (Complete Bipartite Graph). Let $K_{n,n}$ be the complete bipartite graph on $2n$ vertices. The minimum degree energy of the m-semi shadow graph $SD_m(K_{n,n})$ is given by:

$$E_m\left(SD_m(K_{n,n})\right) = 2n^2\left(2(m-1) + \sqrt{m^2 + 14m + 1}\right).$$

Proof. The n-regular complete bipartite graph $K_{n,n}$ has an ordinary graph energy of $2n$, generated by its

non-zero spectral elements $\pm n$. The minimum degree energy of the base graph evaluates to $E_m(K_{n,n}) = kE(K_{n,n}) = n(2n) = 2n^2$. Mapping this into the general minimum degree formula isolates the stated energy expression.

5. Conclusion

In this work, we completely characterized the maximum and minimum degree spectra for the extended m-splitting $\mathrm{Spl}_m^*(G)$ and the m-semi shadow graph SD_m(G) of a k -regular graph G. By leveraging block matrix structural properties and subspace decompositions, we derived closed-form expressions for the maximum and minimum degree energies of these operations purely as continuous functions of the ordinary spectrum of the base graph. These exact formulations expand the known spectral properties of degree-based invariants under advanced graph constructions. A natural direction for future research involves investigating the network-theoretic significance of these spectral states, specifically analyzing how iterative expansion (copy) operations impact network resilience, vulnerability, and structural entropy in large-scale complex networks. Furthermore, extending this matrix-block methodology to derive degree-based centrality measures and Laplacian spectra (such as algebraic connectivity) presents a promising avenue for evaluating information diffusion and clustering efficiency in these modified networks.

Data Availability

The reference cited as Gutman [1], "The Energy of a Graph," Reports of the Mathematical-Statistical Section at the Graz Research Centre, vol. 103, pp. 1-22, 1978, is a legitimate and historically significant publication that has been extensively cited in the scientific literature. However, the original report is not available through online sources. In accordance with the journal's reference policy requiring online accessibility of cited sources, this reference is acknowledged as an offline publication. Its inclusion is intended to recognize its historical and scientific significance, and its offline availability is hereby confirmed through this Data Availability Statement.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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