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Original Article

The Role of The Generalized $\overline{H}$ -Function in Dual Integration: Insights Involving Mellin-Barnes

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Abstract *- The objective of this paper is to establish some new integrations by studying the dual integration formulas involving the generalized $\overline{H}$- function by employing the Mellin–Barnes integral technique introduced by Inayat Husain. This method uses a special function to solve problems in various fields of science. The derived results significantly extend a number of existing integral formulas available in the literature and yield several known identities as particular cases. The H-function play a crucial role in solving many problems of physics, mathematics, statistics and other subjects. We established the integral of the H-function, because these formulae are general, they can also be used to develop more specific formulae.*

Keywords *- Generalized $\overline{H}$- Function, Mellin–Barnes Contour Integral, Dual Integral Transform, Wright Hypergeometric Function.*

1. Introduction

The theory of special functions has witnessed significant development owing to its extensive applications in mathematics, mathematical physics, engineering, and statistics. The generalized $\overline{H}$ -function provides a flexible analytical framework for representing a broad class of special functions encountered in mathematical analysis. In the present work, its applicability to dual integration problems is investigated through the Mellin--Barnes contour integral approach, leading to a unified treatment of several integral representations.

In particular, Euler-type double integrals containing algebraic kernels and generalized $\overline{H}$-function remains largely unexplored. This lack of general transformation formulas restricts the systematic derivation of new identities and their application to broader classes of special functions. The present work addresses this gap by establishing new dual integration formulas in a unified form, from which many existing and new results can be obtained as particular cases.

Inayat Hussain (1987) [6] introduced the generalized fox H-function, popularly known as $\overline{H}$-function and represent as

$$\overline{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i,A_i)_{1,n};(a_i,\alpha_i)_{n+1,p}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i,B_i)_{m+1,q}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_{-\omega\infty}^{\omega\infty}\bar{\phi}(s)\,z^s ds\,,\quad for\quad z\neq 0 \tag{1}$$

where $\omega=\sqrt{-1}$ and

$$\bar{\phi}(s)=\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)} \tag{2}$$

The expression includes fractional powers of certain gamma functions. Here $(\alpha_i)_{1,p}>0$ and $(\beta_i)_{1,q}>0$ are positive real numbers. The terms $a_i(i=1,2,3,\ldots,p)$ and $b_i(i=1,2,3,\ldots,q)$ represent complex parameters, while $(A_i)_{1,n}$ and $(B_i)_{m+1,q}$ may take non-integer values. The integers *m, n, p* and *q* satisfy $1\leq m\leq q$ and $1\leq n\leq p$, which are assumed to be positive for standardization purposes.

As in the standard definition of the $\overline{H}$-function, the contour is taken along the imaginary axis, $\mathbb{R}e(s)=0$, with a suitable indentation to avoid the singularities of the gamma functions and to place those singularities on the correct sides. Additional relevant restrictions on the parameters are consistent with those given by Srivastava et al. (1982) [16].

Regarding the convergence of the integral (1) that defines the H-function, we refer to Erdelyi et al. (1953) [2] for details.

$$|\Gamma(x+\omega y)| \sim \sqrt{2\pi}|y|^{x-\frac{1}{2}} e^{\frac{-\pi|y|}{2}}, \qquad (|y| \to \infty) \tag{3}$$

Along the lines where $Re(s) = x$, using equation (3), we readily obtain the asymptotic behaviour of $|\Gamma(1-a_i+\alpha_i s)^{A_i}|$. Therefore, we need only a slight modification to the sufficient condition for the absolute convergence of the contour integral (1), which is expressed as follows:

$$\Lambda \equiv \sum_{i=1}^{m} |\beta_i| + \sum_{i=1}^{n} |\alpha_i A_i| - \sum_{i=m+1}^{q} |\beta_i B_i| - \sum_{i=n+1}^{p} |\alpha_i| > 0 \tag{4}$$

This condition ensures exponential decay of the integrand in (1), establishing the region of convergence for (1) as

$$|arg(z)| < \frac{1}{2}\pi\Lambda \tag{5}$$

where Λ is defined by equation (4).

2. Preliminaries

We shall require the following finite integral from Table of integration, series and Products (Gradshteyn Ryzhik) [4] (p 486; eq.5) and (p 322; eq.4) for the evolution of our main integrals.

$$\int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta} (sin\theta)^{\alpha-1} (cos\theta)^{\beta-1} d\theta = e^{\frac{\omega\alpha\pi}{2}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}; \quad Re(\alpha) > 0, \quad Re(\beta) > 0 \tag{6}$$

$$\int_0^{\infty} \frac{x^{\sigma-1} dx}{(a+bx^{\lambda})^{\rho+1}} = \frac{1}{\lambda a^{\rho+1}} \left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}} \frac{\Gamma\left(\frac{\sigma}{\lambda}\right)\Gamma\left(1+\rho-\frac{\sigma}{\lambda}\right)}{\Gamma(1+\rho)}; \qquad 0 < \frac{\sigma}{\lambda} < \rho+1, \qquad a \neq 0 \tag{7}$$

3. Main Result

3.1. First Integral

Provided $a, b, \lambda > 0$; $\Re(\eta) > 0$; $\Re(\gamma) > 0$ and $0 < \Re\left(\frac{\sigma}{\lambda}\right) < \Re(\rho+1)$ The Mellin-Barnes contour exists,

$$\begin{aligned} &\int_0^{\infty}\int_0^{\infty} e^{\omega(\eta+\gamma)\theta} (\sin\theta)^{\eta-1} (\cos\theta)^{\gamma-1} \frac{x^{\sigma-1}}{(a+bx^{\lambda})^{\rho+1}} \bar{H}_{p,q}^{m,n}\left[\frac{ze^{\omega\delta\theta}(\cos\theta)^{\delta}x^{\nu}}{(a+bx^{\lambda})^{\mu}}\middle|\begin{matrix}(a_i,\alpha_i,A_i)_{1,n};(a_i,\alpha_i)_{n+1,p}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i,B_i)_{m+1,q}\end{matrix}\right]dxd\theta \\ &= \frac{e^{\frac{\omega\eta\pi}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}} \bar{H}_{p+3,q+2}^{m,n+3}\left[a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\middle|\begin{matrix}\{1-\gamma,\delta,1\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda},1\right\};\left\{\frac{\sigma}{\lambda}-\rho,\mu-\frac{\nu}{\lambda},1\right\};(a_i,\alpha_i,A_i)_{4,n+3};(a_i,\alpha_i)_{n+4,p+3}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i,B_i)_{m+1,q};\{1-\eta-\gamma,\delta,1\};\{-\rho,\mu,1\}\end{matrix}\right] \end{aligned} \tag{8}$$

the generalized $\bar{H}$-function satisfies its standard convergence conditions (4) and (5).

Proof: To prove the first integral, we took left hand side of the equation (8)

$$\Rightarrow \int_0^{\infty}\int_0^{\infty} e^{\omega(\eta+\gamma)\theta} (\sin\theta)^{\eta-1} (\cos\theta)^{\gamma-1} \frac{x^{\sigma-1}}{(a+bx^{\lambda})^{\rho+1}} \bar{H}_{p,q}^{m,n}\left[\frac{ze^{\omega\delta\theta}(\cos\theta)^{\delta}x^{\nu}}{(a+bx^{\lambda})^{\mu}}\middle|\begin{matrix}(a_i,\alpha_i,A_i)_{1,n};(a_i,\alpha_i)_{n+1,p}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i,B_i)_{m+1,q}\end{matrix}\right]dxd\theta$$

To establish this, we replace the $\bar{H}$-function with its equivalent contour integral as given in equation (1).

$$\Rightarrow \frac{1}{2\omega\pi}\int_{-\omega\infty}^{\omega\infty}\int_0^\infty\int_0^\infty e^{\omega(\eta+\gamma+s\delta)\theta}(sin\theta)^{\eta-1}(cos\theta)^{\gamma+\delta s-1}\frac{x^{\nu s+\sigma-1}}{(a+bx^\lambda)^{\mu s+\rho+1}}\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)}z^s ds\,dx\,d$$

change the order of integration, which permits the evaluation of the inner integral independently and leads to the desired transformation formula.

$$\Rightarrow \frac{1}{2\omega\pi}\int_{-\omega\infty}^{\omega\infty}\int_0^\infty e^{\omega\,(\eta+\gamma+s\delta)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\gamma+\delta s-1}d\theta\int_0^\infty\frac{x^{\nu s+\sigma-1}}{(a+bx^\lambda)^{\mu s+\rho+1}}dx\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)}z^s\,ds$$

$$\Rightarrow \frac{1}{2\omega\pi}\int_{-\omega\theta}^{\omega\theta} e^{\frac{\omega\eta\pi}{2}}\frac{\Gamma(\eta)\Gamma(\gamma+s\delta)}{\Gamma(\eta+\gamma+s\delta)}\frac{1}{\lambda a^{\mu s+\rho+1}}\left(\frac{a}{b}\right)^{\frac{\nu s+\sigma}{\lambda}}\frac{\Gamma\left(\frac{\nu s+\sigma}{\lambda}\right)\Gamma\left(\mu s+\rho+1-\frac{\nu s+\sigma}{\lambda}\right)}{\Gamma(\mu s+\rho+1)}\times$$

$$\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)}z^s\,ds$$

$$\Rightarrow \frac{e^{\frac{\omega\eta\pi}{2}}}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}}\frac{1}{2\omega\pi}\int_{-\omega\theta}^{\omega\theta}\frac{\Gamma(\eta)\Gamma(\gamma+s\delta)}{\Gamma(\eta+\gamma+s\delta)}\frac{\Gamma\left(\frac{\nu s+\sigma}{\lambda}\right)\Gamma\left(\mu s+\rho+1-\frac{\nu s+\sigma}{\lambda}\right)}{\Gamma(\mu s+\rho+1)}\times$$

$$\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)}\left\{a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\right\}^s ds$$

$$\Rightarrow \frac{e^{\frac{\omega\eta\pi}{2}}}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}}\frac{\Gamma(\eta)}{2\omega\pi}\int_{-\omega\theta}^{\omega\theta}\frac{\Gamma\{1-(1-\gamma)+s\delta\}\Gamma\left\{1-\left(1-\frac{\sigma}{\lambda}\right)+\frac{\nu s}{\lambda}\right\}\Gamma\left\{1-\left(\frac{\sigma}{\lambda}-\rho\right)+\left(\mu-\frac{\nu}{\lambda}\right)s\right\}}{\Gamma(1-\{1-\eta-\gamma\}+s\delta)\Gamma(1+\rho+\mu s)}\times$$

$$\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+4}^{p+3}\Gamma(a_i-\alpha_i s)}\left\{a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\right\}^s ds$$

$$\Rightarrow \frac{e^{\frac{\omega\eta\pi}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}}\overline{H}_{p+3,q+2}^{m,n+3}\left[a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\left|\begin{matrix}\{1-\gamma,\delta,1\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda},1\right\};\left\{\frac{\sigma}{\lambda}-\rho,\mu-\frac{\nu}{\lambda},1\right\};(a_i,\alpha_i,A_i)_{4,n+3};(a_i,\alpha_i)_{n+4,p+3}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i,B_i)_{m+1,q};\{1-\eta-\gamma,\delta,1\};\{-\rho,\mu,1\}\end{matrix}\right.\right]$$

Hence proved the integral (8).

3.2. Second Integral

Provided $a, b, \lambda > 0$; $\Re(\eta) > 0$; $\Re(\gamma) > 0$ and $0 < \Re\left(\frac{\sigma}{\lambda}\right) < \Re(\rho + 1)$ The Mellin-Barnes contour exists,

$$\int_0^\infty\int_0^\infty e^{\omega(\eta+\gamma)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\gamma-1}\frac{x^{\sigma-1}}{(a+bx^\lambda)^{\rho+1}}\overline{H}_{p,q}^{m,n}\left[\frac{ze^{\omega\delta\theta}(\cos\theta)^\delta x^\nu}{(a+bx^\lambda)^\mu}\left|\begin{matrix}(a_i,\alpha_i,A_i)_{1,n};(a_i,\alpha_i)_{n+1,p}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i,B_i)_{m+1,q}\end{matrix}\right.\right]dxd\theta$$

$$=\frac{e^{\frac{\omega\pi\eta}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}}\overline{H}_{p+2,q+3}^{m+1,n+2}\left[\left\{a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\right\}\left|\begin{matrix}\{1-\gamma,\delta,1\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda},1\right\};(a_i,\alpha_i,A_i)_{3,n+2};(a_i,\alpha_i)_{n+3,p+2}\\\left\{\rho+1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda}-\mu\right\};(b_i,\beta_i,)_{2,m+1};\{1-\eta-\gamma,\delta,1\};\{-\rho,\mu,1\}(b_i,\beta_i,B_i)_{m+2,q+3}\end{matrix}\right.\right] \quad (9)$$

the generalized $\overline{H}$-function satisfies its standard convergence conditions (4) and (5).

Proof: The proof of integral (9) can prove similar as integral (8)

$$\Rightarrow\int_0^\infty\int_0^\infty e^{\omega(\eta+\gamma)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\gamma-1}\frac{x^{\sigma-1}}{(a+bx^\lambda)^{\rho+1}}\overline{H}_{p,q}^{m,n}\left[\frac{ze^{\omega\delta\theta}(\cos\theta)^\delta x^\nu}{(a+bx^\lambda)^\mu}\left|\begin{matrix}(a_i,\alpha_i,A_i)_{1,n};(a_i,\alpha_i)_{n+1,p}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i,B_i)_{m+1,q}\end{matrix}\right.\right]dx\,d\theta$$

To establish this, we replace the $\overline{H}$ -function with its equivalent contour integral as given in equation (1).

$$\Rightarrow\int_0^\infty\int_0^\infty e^{\omega(\eta+\gamma)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\gamma-1}\frac{x^{\sigma-1}}{(a+bx^\lambda)^{\rho+1}}\times$$

$$\frac{1}{2\omega\pi}\int_{-\omega\infty}^{\omega\infty}\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)}\left\{\frac{ze^{\omega\delta\theta}(\cos\theta)^\delta x^\nu}{(a+bx^\lambda)^\mu}\right\}^s ds\ dx\,d\theta$$

$$\Rightarrow\frac{1}{2\omega\pi}\int_{-\omega\infty}^{\omega\infty}\int_0^\infty\int_0^\infty e^{\omega(\eta+\gamma+\delta s)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\delta s+\gamma-1}\frac{x^{\nu s+\sigma-1}}{(a+bx^\lambda)^{\mu s+\rho+1}}\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)}z^s\,ds\,dx\,d\theta$$

change the order of integration, which permits the evaluation of the inner integral independently and leads to the desired transformation formula.

$$\Rightarrow\frac{1}{2\omega\pi}\int_{-\omega\infty}^{\omega\infty}\int_0^\infty e^{\omega(\eta+\gamma+\delta s)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\delta s+\gamma-1}\,d\theta\int_0^\infty\frac{x^{\nu s+\sigma-1}}{(a+bx^\lambda)^{\mu s+\rho+1}}dx\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)}z^s\,ds$$

$$\Rightarrow\int_{-\omega\infty}^{\omega\infty}e^{\frac{\omega\pi\eta}{2}}\frac{\Gamma(\eta)\Gamma(1-\{1-\gamma\}+\delta s)}{\Gamma(1-\{1-\eta-\gamma\}+\delta s)}\frac{1}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}}\frac{\Gamma\left\{1-\left(1-\frac{\sigma}{\lambda}\right)+\frac{\nu s}{\lambda}\right\}}{\Gamma(\mu s+\rho+1)}\times$$

$$\frac{\Gamma\left\{\rho+1-\frac{\sigma}{\lambda}-\left(\frac{\nu}{\lambda}-\mu\right)s\right\}\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)}\left\{a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\right\}^s\,ds$$

$$\Rightarrow \frac{1}{2\omega\pi}\int_{-\omega\infty}^{\omega\infty} e^{\frac{\omega\pi\eta}{2}} \frac{\Gamma(\eta)\Gamma(\delta s+\gamma)}{\Gamma(\eta+\delta s+\gamma)} \frac{1}{\lambda a^{\mu s+\rho+1}}\left(\frac{a}{b}\right)^{\frac{\nu s+\sigma}{\lambda}} \frac{\Gamma\left(\frac{\nu s+\sigma}{\lambda}\right)}{\Gamma(\mu s+\rho+1)}$$

$$\frac{\Gamma\left(\mu s+\rho+1-\frac{\nu s+\sigma}{\lambda}\right)\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)^{A_i}}{\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)^{B_i}\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)} z^s\, ds$$

$$\Rightarrow \frac{e^{\frac{\omega\pi\eta}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}} \overline{H}_{p+3,q+2}^{m,n+3}\left[a^{\frac{\nu}{\lambda}-\mu} b^{-\frac{\nu}{\lambda}} z \left| \begin{matrix} \{1-\gamma,\delta\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda},1\right\};(a_i,\alpha_i,A_i)_{3,n+2};(a_i,\alpha_i)_{n+3,p+} \\ \left\{\rho+1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda}-\mu\right\};(b_i,\beta_i)_{2,m+1};\{1-\eta-\gamma,\delta,1\};\{-\rho,\mu,1\};(b_i,\beta_i,B_i)_{m+2,q+3} \end{matrix}\right.\right]$$

Hence, proved the integral (9).

4. Special Cases

4.1. Case I

If we set $A_i = B_i = 1$, the $\overline{H}$-function reduces to the Fox's H-function, and equation (8) and (9) then takes the following form

$$\int_0^\infty\int_0^\infty e^{\omega(\eta+\gamma)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\gamma-1}\frac{x^{\sigma-1}}{(a+bx^\lambda)^{\rho+1}} H_{p,q}^{m,n}\left[\frac{ze^{\omega\delta\theta}(\cos\theta)^\delta x^\nu}{(a+bx^\lambda)^\mu}\left| \begin{matrix}(a_i,\alpha_i)_{1,n};(a_i,\alpha_i)_{n+1,p}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i)_{m+1,q}\end{matrix}\right.\right] dx\, d\theta$$
$$= \frac{e^{\frac{\omega\eta\pi}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}} H_{p+3,q+2}^{m,n+3}\left[a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\left| \begin{matrix}\{1-\gamma,\delta\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda}\right\};\left\{\frac{\sigma}{\lambda}-\rho,\mu-\frac{\nu}{\lambda}\right\};(a_i,\alpha_i)_{4,n+3};(a_i,\alpha_i)_{n+4,p+3}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i)_{m+1,q};\{1-\eta-\gamma,\delta\};\{-\rho,\mu\}\end{matrix}\right.\right] \tag{10}$$

$$\int_0^\infty\int_0^\infty e^{\omega(\eta+\gamma)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\gamma-1}\frac{x^{\sigma-1}}{(a+bx^\lambda)^{\rho+1}} H_{p,q}^{m,n}\left[\frac{ze^{\omega\delta\theta}(\cos\theta)^\delta x^\nu}{(a+bx^\lambda)^\mu}\left| \begin{matrix}(a_i,\alpha_i)_{1,n};(a_i,\alpha_i)_{n+1,p}\\(b_i,\beta_i,)_{1,m};(b_i,\beta_i)_{m+1,q}\end{matrix}\right.\right] dx\, d\theta$$
$$= \frac{e^{\frac{\omega\pi\eta}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}} H_{p+2,q+3}^{m+1,n+2}\left[\left\{a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\right\}\left| \begin{matrix}\{1-\gamma,\delta\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda}\right\};(a_i,\alpha_i)_{3,n+2};(a_i,\alpha_i)_{n+3,p+2}\\\left\{\rho+1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda}-\mu\right\};(b_i,\beta_i,)_{2,m+1};\{1-\eta-\gamma,\delta\};\{-\rho,\mu\}(b_i,\beta_i)_{m+2,q+3}\end{matrix}\right.\right] \tag{11}$$

4.2. Case II

If we set $A_i = B_i = 1 = \alpha_i = \beta_i$ $\overline{H}$-function reduces to G-function, and equation (8) and (9) then takes the following form.

$$\int_0^\infty\int_0^\infty e^{\omega(\eta+\gamma)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\gamma-1}\frac{x^{\sigma-1}}{(a+bx^\lambda)^{\rho+1}} G_{p,q}^{m,n}\left[\frac{ze^{\omega\delta\theta}(\cos\theta)^\delta x^\nu}{(a+bx^\lambda)^\mu}\left| \begin{matrix}(a_i,1)_{1,p}\\(b_i,1)_{1,q}\end{matrix}\right.\right] dx\, d\theta$$
$$= \frac{e^{\frac{\omega\eta\pi}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}} G_{p+3,q+2}^{m,n+3}\left[a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\left| \begin{matrix}\{1-\gamma,\delta\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda}\right\};\left\{\frac{\sigma}{\lambda}-\rho,\mu-\frac{\nu}{\lambda}\right\};(a_i,1)_{4,p+3}\\(b_i,1,)_{1,q};\{1-\eta-\gamma,\delta\};\{-\rho,\mu\}\end{matrix}\right.\right] \tag{12}$$

$$\int_0^\infty\int_0^\infty e^{\omega(\eta+\gamma)\theta}(\sin\theta)^{\eta-1}(\cos\theta)^{\gamma-1}\frac{x^{\sigma-1}}{(a+bx^\lambda)^{\rho+1}} G_{p,q}^{m,n}\left[\frac{ze^{\omega\delta\theta}(\cos\theta)^\delta x^\nu}{(a+bx^\lambda)^\mu}\left| \begin{matrix}(a_i,1)_{1,p}\\(b_i,1)_{1,q}\end{matrix}\right.\right] dx\, d\theta$$
$$= \frac{e^{\frac{\omega\pi\eta}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}} G_{p+2,q+3}^{m+1,n+2}\left[\left\{a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\right\}\left| \begin{matrix}\{1-\gamma,\delta\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda}\right\};(a_i,1)_{3,p+2}\\\left\{\rho+1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda}-\mu\right\};(b_i,1)_{2,m+1};\{1-\eta-\gamma,\delta\};\{-\rho,\mu\}(b_i,1)_{m+2,q+3}\end{matrix}\right.\right] \tag{13}$$

4.3. Case II

If we set $n = p$; $m = 1$; $q = q + 1$; $b_1 = 0$; $\beta_1 = 1$; $a_i = 1 - a_i$, in equation (1) the $\overline{H}$-function reduces to generalized Wright hypergeometric function

$$\overline{H}^{1,n}_{n,q+1}\left[z\left|\begin{matrix}(1-a_i,\alpha_i;A_i)_{1,p}\\(0,1);(1-b_i,\beta_i;B_i)_{1,q}\end{matrix}\right.\right] = {}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i;A_i)_{1,p}\\(b_i,\beta_i;B_i)_{1,q}\end{matrix};-z\right] \tag{14}$$

Using the same assumptions, then the equation (8) and (9) take the following.

$$\int_0^\infty \int_0^\infty (\sin\theta)^{\eta-1}(\cos\theta)^{\gamma-1}\frac{e^{\omega(\eta+\gamma)\theta}x^{\sigma-1}}{(a+bx^\lambda)^{\rho+1}}\,{}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i,A_i)_{1,p}\\(b_i,\beta_i,B_i)_{1,q}\end{matrix};-\frac{ze^{\omega\delta\theta}(\cos\theta)^\delta x^\nu}{(a+bx^\lambda)^\mu}\right]dx\,d\theta$$
$$=\frac{e^{\frac{\omega\eta\pi}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}}{}_{p+3}\psi_{q+2}\left[\begin{matrix}\{1-\gamma,\delta,1\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda},1\right\};\left\{\frac{\sigma}{\lambda}-\rho,\mu-\frac{\nu}{\lambda},1\right\};(1-a_i,\alpha_i,A_i)_{4,p+3}\\(0,1);(1-b_i,\beta_i,B_i)_{2,q};\{1-\eta-\gamma,\delta,1\};\{-\rho,\mu,1\}\end{matrix};(-a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z)\right] \tag{15}$$

$$\int_0^\infty \int_0^\infty (\sin\theta)^{\eta-1}(\cos\theta)^{\gamma-1}\frac{e^{\omega(\eta+\gamma)\theta}x^{\sigma-1}}{(a+bx^\lambda)^{\rho+1}}\,{}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i,A_i)_{1,p}\\(b_i,\beta_i,B_i)_{1,q}\end{matrix};-\frac{ze^{\omega\delta\theta}(\cos\theta)^\delta x^\nu}{(a+bx^\lambda)^\mu}\right]dx\,d\theta$$
$$=\frac{e^{\frac{\omega\pi\eta}{2}}\Gamma(\eta)}{\lambda a^{\rho+1}}\left(\frac{a}{b}\right)^{\frac{\sigma}{\lambda}}{}_p\psi_q\left[\begin{matrix}\{1-\gamma,\delta,1\};\left\{1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda},1\right\};(a_i,\alpha_i,A_i)_{3,p+2}\\\left\{\rho+1-\frac{\sigma}{\lambda},\frac{\nu}{\lambda}-\mu\right\};(0,1);\{1-\eta-\gamma,\delta,1\};\{-\rho,\mu,1\}(1-b_i,\beta_i,B_i)_{m+2,q+3}\end{matrix};-a^{\frac{\nu}{\lambda}-\mu}b^{-\frac{\nu}{\lambda}}z\right] \tag{16}$$

Since the generalized $\overline{H}$-function includes many well-known special functions such as the Fox H-function, Meijer G-function, generalized hypergeometric functions, and Wright functions as special cases. The derived integral formulas provide a unified technique for evaluating a wide variety of definite integrals. These results are applicable in the study of integral transforms, fractional differential and integral equations, probability distributions, statistical models, wave propagation, quantum mechanics, heat and diffusion problems, signal processing, and other physical systems where generalized special functions naturally arise.

5. Conclusion

In this paper, we have derived two integral formulas. The first set of results is developed in association with the $\overline{H}$-function for general arguments, extending the mathematical framework and potential applications of this function. The results obtained are not only theoretically significant but also hold practical value, offering tools for solving complex integrals that arise in diverse areas such as physics, engineering, and applied mathematics. The results obtained in this paper are expected to be useful in several branches of applied mathematics and mathematical physics. These formulas provide new insights that can simplify the analysis of complex systems, aid in the study of asymptotic behaviours, and support advanced problem-solving techniques in fields like quantum mechanics, statistical mechanics, and computational sciences. Consequently, the findings are expected to contribute to further research and facilitate applications across a broad spectrum of disciplines.

Data Availability Statement

The reference cited as C.S. Meijer, "On the G-function," *Proceedings of the National Academy of Sciences* (Proc. Nat. Acad. Wetensch.), 1946, vol. 49, p. 227, is acknowledged as a legitimate scholarly publication. However, the original publication is not available through online sources. In accordance with the journal's reference policy requiring online accessibility of cited sources, this reference is acknowledged as an offline publication. Its inclusion is intended to recognize its academic significance, and its offline availability is hereby confirmed through this Data Availability Statement.

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