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Original Article

Even-Odd Cordial Labeling on Join of Zero Divisor Graphs

S. Naresh kumar[1], D. Bharathi[2]

[1,2]Department of Mathematics, Sri Venkateswara University, Tirupati, Andhra Pradesh, India.

[1]Corresponding Author : nareshkbm@gmail.com

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Abstract *- Let G(V,E) be a graph, and let f: V(G)* $\rightarrow \{1,2,\ldots.|V(G)|\}$ *be a bijective vertex labeling. The induced edge labeling* f^**: E(G)* $\rightarrow \{0,1\}$ *is defined by assigning the label 1 to an edge uv whenever the labels f(u) and f(v) have same parity, and the label 0 whenever they have different parities. If the difference between the number of edges labeled 0 and the number of edges labeled 1 is at most one, then f is called an even – odd cordial labeling (EOCL). A graph that admits such a labeling is known as an even – odd cordial Graph (EOCG). In this work, we study the existence of even – odd cordial labeling for the join of zero – divisor graphs, namely* $\Gamma(Z_{np}) + \Gamma(Z_4)$ *,* $\Gamma(Z_{np}) + \Gamma(Z_6)$ *, and* $\Gamma(Z_{np}) + \Gamma(Z_9)$ *(for n = 2,3,5). The results establish conditions under which these graph families satisfy the requirements of even – odd cordial labeling, thereby extending the study of cordial labeling s to joins of zero – divisor graphs.*

Note: Two integers have the same parity when they are both even or both odd.

Keywords and Phrases *- Cordial labeling, Even-Odd cordial labeling, Zero divisors,Zero - divisor graphs.*

1. Introduction

Graph labeling has attracted significant attention in the field of graph theory because of its rich mathematical structure and numerous applications in computer science, communication networks, coding theory, and combinatorial optimization. In a graph labeling, integers are assigned to the vertices, edges, or both of a graph in accordance with the objective of obtaining desirable structural properties. Since the introduction of graceful and harmonious labeling, many other labeling techniques have been developed, Among the numerous graph labeling techniques, cordial labeling and its various extensions have received considerable attention because of their inherent balance conditions imposed on vertex and edge labels.

Zero – divisor graphs form a fundamental link between commutative algebra and graph theory byt associating the zero – divisors of a ring with the vertices of a graph, where adjacency is determined by the zero- product conditionws.. The concept originated from the work of Beck [4], who introduced graphs associated with commutative rings in the context of graph coloring. Later, Anderson and Livigston [8,10], established the widely accepted definition of the zero – divisor graph by considering only the non zero zero – divisors of a commutative ring. Since then, zero – divisor graphs have become an active area of research, with significan contribution by Redmond [6,7], Mulay [11], Lucas[12]. These studies have explored numberous structural properties of zero – divisor graphs, including connectivity, diameter, girth, clique number, chromatic number, domination, Hamiltonicity, and planarity, there by revealing strong relationships between algebraic and graph – theoretic properties.

Graph labeling has emerged as another important direction in the study of zero – divisor graphs. Following the introduction of graceful labeling by Rosa[5] and cordial labeling by Chait [3], a variety of labeling schemes have been investigated on algebraic graphs. The comprehensive survey by Gallian [13] highlights the rapid development of graph labeling theory and its numerous applications. Motivated by these developments, researchers have studied graceful, harmonious, cordial, product cordial, divisor cordial, edge cordial, and other labeling techniques on zero – divisor graphs of finite commutative rings. Significant contributions in this direction have been made by Mooney [14], Sundaram and Ponaraj [16], Vaidya and Prajapati [16], and several others, demonstrating that graph labeling provides valuable insight into the structural characterisitics of zeo – divisor graphs.

More recently, parity – based labeling techniques have attracted increasing attention due to their mathematical significance and broad applicability. Even – odd cordial labeling has been investigated for several standard graph families; however, its application to zero – divisor graphs remains relatively unexplored. Although graph operations such as union, product, corona, and join have been extensively studied under various labeling schemes, only limited work has been reported on even- odd cordial labeling of join graphs involving zero – divisor graphs. This gap in the existing literature motivates the present work, which focuses on establishing even – odd cordial labeling for join graphs associated with zero – divisor graphs of finite commutative rings, thereby extending the current body of knowledge in the both graph labeling and algebraic graph theory.

The present work focuses on the existence of EOC labeling for the join graphs $\Gamma(Z_{nk})$ + $\Gamma(Z_4)$, $\Gamma(Z_{nm})$ + $\Gamma(Z_6)$, $\Gamma(Z_{nh})$ + $\Gamma(Z_9)$, where n = 2,3, or 5, k, m, and h are prime numbers , and $\Gamma(Z_n)$ represents the zero – divisor graph associated with the ring Z_n. Suitable vertex labeling functions are constructed for each family of graphs, and the corresponding induced edge labeling are analyzed. It is shown that these join graphs satisfy the required balance condition for edge labels, thereby establishing that they are even – odd cordial graphs. These results extend the existing literature on parity – based graph labeling and contribute to the study of cordial labeling on algebraically defined graphs.

2. Preliminaries

Definition 2.1. Cordial Labeling

A cordial labeling of a graph assigns 0 or 1 to each vertex, such that the difference between the number of vertices labeled 0 and 1 is at most 1, and the same holds for the number of edges labeled 0 and 1 based on the labels of their endpoints

Definition 2.2. Even –Odd Cordial Labeling

Let G (V,E) be a graph. Let f be a map from V(G) to $\{1,2,\ldots,|V(G)|\}$ is bijective such that the induced edge mapping f* : E(G) → {0,1 } an edge uv is assigned the label 0 if f(u) and f(v) have different parities and label 1 if f(u) and f(v) have same parities. The number of edges labeled with 0 and the number of edges labeled with 1 differ by atmost 1. It is called even-odd cordial labeling(EOCL) and a graph admits even-odd cordial labeling(EOCL) is called even-odd Cordial Graph(EOCG).

Note: Two integers have the same parity when they are both even or both odd.

Example 2.3. The following is a simple example of an even-odd Cordial labeling(EOCL).

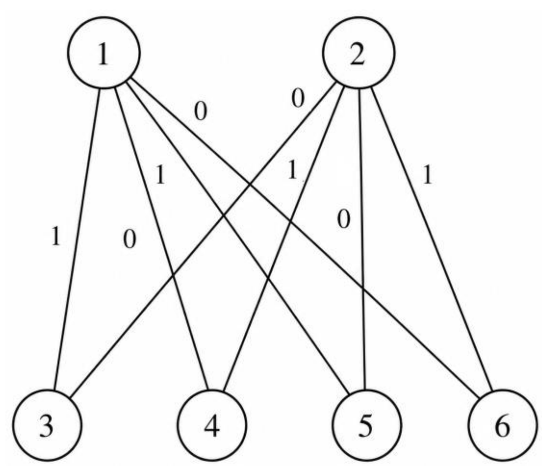

Fig. 1 EOC Graph $k_{2,4}$

Definition 2.4. Divisor Graph

A divisor graph is a graph where vertices represent elements of a set of positive integers, and two vertices are connected by an edge if one integer divides the other.

Definition 2.5. Zero Divisors

Let R be a ring , an element a ∈ R,a $\neq 0$ is called a zero divisor if there exists b ∈ R,b $\neq 0$ such that a . b = 0 or b . a = 0

Definition 2.6. Zero Divisor Graph $\Gamma(Z_n)$

The zero divisor of a commutative ring Z_n with unity is an undirected graph where vertices are the non zero divisors of Z_n, and two vertices are adjacent if their product is zero.

Definition 2.8 Join Zero Divisor Graphs

The join of two zero-divisor graphs $\Gamma(\mathrm{R}_1)$ and $\Gamma(\mathrm{R}_2)$ is a graph where the vertices are the union of the vertices of $\Gamma(\mathrm{R}_1)$ and $\Gamma(\mathrm{R}_2)$, edges exist between

i) vertices within $\Gamma(\mathrm{R}_1)$ if they are adjacent in $\Gamma(\mathrm{R}_1)$
ii) vertices within $\Gamma(\mathrm{R}_2)$ if they are adjacent in $\Gamma(\mathrm{R}_2)$
iii) Every vertex of $\Gamma(\mathrm{R}_1)$ and every vertex of $\Gamma(\mathrm{R}_2)$

3. Main Section

In this section, we study even – odd cordial (EOC) labeling of several families of join zero – divisor graphs.

Theorem 3.1. Let k > 2 be a prime number, and let G denote the join graph $\Gamma(Z_{2k})$ + $\Gamma(Z_4)$. Then G admits an EOC labeling.

Proof. Let G = $\Gamma(Z_{2k})$ + $\Gamma(Z_4)$

The graph G = (V(G), E(G)) is characterized by the following vertex and edge sets :

V(G) = $\{w_1, w_2, \ldots\ldots w_k, r\}$

E(G) = $\{w_\alpha w_k,\ w_\alpha x,\ w_k r\ for\ \ 1 \le \alpha \le k-1\}$

Accordingly, the graph G contains k +1 vertices and 2k – 1 edges; that is,

Define a bijective labeling f : V(G) → $\{1,2, \ldots\ldots k+1\}$ by f $(w_\alpha) = \alpha$, $1 \le \alpha \le k$,

and f(r) = k+1.

The vertex assignment f gives rise to an edge – labeling function f^*: E(G) → {0,1}, defined according to the following rule:

$$f^*(w_\alpha w_k) = \begin{cases} 0, & \alpha \text{ is even}, \\ 1, & \alpha \text{ is odd}, \end{cases} \quad 1 \le \alpha \le k-1,$$

$$f^*(w_\alpha r) = \begin{cases} 1, & \alpha \text{ is even} \\ 0, & \alpha \text{ is odd} \end{cases} \quad 1 \le \alpha \le k.$$

We note that $e_{f^*}(0) = k$ and $e_{f^*}(1) = k-1$

Thus, the edge labels satisfy the required cordiality condition $\left|e_{f^*}(1) - e_{f^*}(0)\right| \le 1$, then the join graph $\Gamma(Z_{2k})$ + $\Gamma(Z_4)$ is EOC for every prime number k > 2.

Example 3.2.

A EOC labeling for G = $\Gamma(Z_{2k})$ + $\Gamma(Z_4)$, is given in figure 2 where k = 7

V(G) = $\{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6,\ w_7,\ x\}$,

E(G) = $\{w_\alpha w_7,\ w_\alpha x,\ w_7 x \mid 1 \le \alpha \le 6\}$.

Therefore $|V(G)|$ = 8 & $|E(G)|$ = 13

To facilitate the proposed construction, assign pariwise distinct integers from {1,2, ... ,8} to the vertices of G through the bijective mapping f : V(G) → {1,2, ... ,8}, defined as follows:

f$(w_\alpha) = \alpha$, $1 \le \alpha \le 7$, *and assign f(x) = 8.*

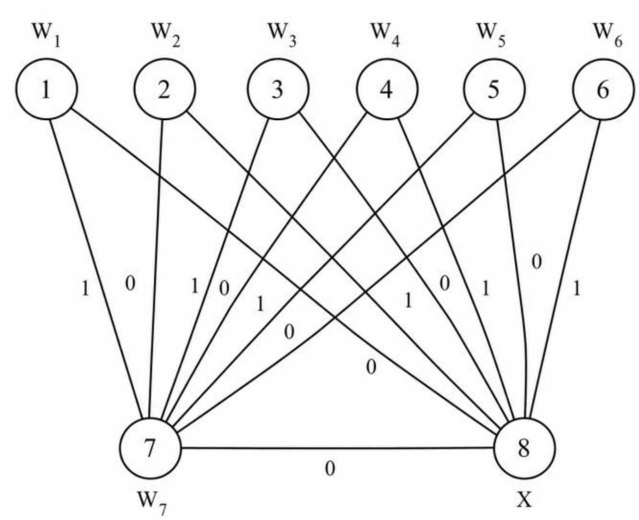

Fig. 2 E.O.C.G of G = $\Gamma(Z_{14}) + \Gamma(Z_4)$

We note that $e_{f^*}(1)$ = 6 and $e_{f^*}(0)$ = 7

therefore, the edge labeling satisfies the inequality $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$, then $\Gamma(Z_{14}) + \Gamma(Z_4)$ is EOCL graph.

Theorem 3.3. Let G = $\Gamma(Z_{2k}) + \Gamma(Z_6)$, where k is a prime number satisfying k > 2. Then G admits and even – odd cordial labeling.

Proof: Consider the graph G = $\Gamma(Z_{2k}) + \Gamma(Z_6)$, where k is prime.

The graph G = (V(G), E(G)) is formally defined by specifying its vertex set V(G) and edge set E(G) as follows:

V(G) = $\{w_1, w_2, \ldots.. w_k, r, s, t\}$

E(G) = $\{w_\alpha w_k,\ w_\alpha r,\ w_\alpha s,\ w_\alpha t,\ w_k r,\ w_k s,\ w_k t,\ rs,\ st\ ,1 \leq \alpha \leq k-1\}$

It follows that

$|V(G)|$ = k+3 & $|E(G)|$ = 4k+1

Define a bijection f : V(G) $\rightarrow \{1,2, \ldots., k+3\}$ by

f $(w_\alpha) = \alpha$, $1 \leq \alpha \leq k$, f(x) = k+1, f(z) = k+2, f(y) = k+3.

The vertex assignment f gives rise to an edge – labeling function f*: E(G) → $\{0,1\}$, defined according to the following rule:

$$f^*(w_\alpha w_k) = \begin{cases} 0, & \alpha \text{ is even}, \\ 1, & \alpha \text{ is odd}, \end{cases} \quad 1 \leq \alpha \leq k-1,$$

$$f^*(w_\alpha x) = \begin{cases} 1, & \alpha \text{ is even}, \\ 0, & \alpha \text{ is odd}, \end{cases} \quad 1 \leq \alpha \leq k,$$

$$f^*(w_\alpha y) = \begin{cases} 1, & \alpha \text{ is even}, \\ 0, & \alpha \text{ is odd}, \end{cases} \quad 1 \leq \alpha \leq k,$$

$$f^*(w_\alpha z) = \begin{cases} 0, & \alpha \text{ is even}, \\ 1, & \alpha \text{ is odd}, \end{cases} \quad for\ 1 \leq \alpha \leq k.$$

$f^*(rs)$ = 0 and $f^*(st)$ = 1

We note that $e_{f^*}(0) = 2k+1$ and $e_{f^*}(1) = 2k$

Accordingly, the induced edge labels satisfy the conditon $\left|e_{f^*}(1) - e_{f^*}(0)\right| \leq 1$, then $\Gamma(Z_{2k}) + \Gamma(Z_6)$ is EOCL graph for all prime numbers k > 2.

Example 3.4.
A EOC labeling for G = $\Gamma(Z_{2k}) + \Gamma(Z_6)$, is given in figure 3 where k = 5

V(G) = $\{w_1, w_2, w_3, w_4, w_5, r, s, t\}$

E(G) = $\{w_\alpha w_5,\ w_\alpha r,\ w_\alpha s,\ w_\alpha t,\ w_k r,\ w_5 s,\ w_5 t,\ rs,\ st\ ,1 \leq \alpha \leq 4\}$

Here $|V(G)|$ = 8 & $|E(G)|$ = 21

Assign the labels to the vertices according to the mapping

Now f : V(G) $\rightarrow \{1,2,\ldots.,8\}$, where f($w_\alpha$) = α, $1 \leq \alpha \leq 5$, and

f(r) = 6, f(s) = 8, f(t) = 7.

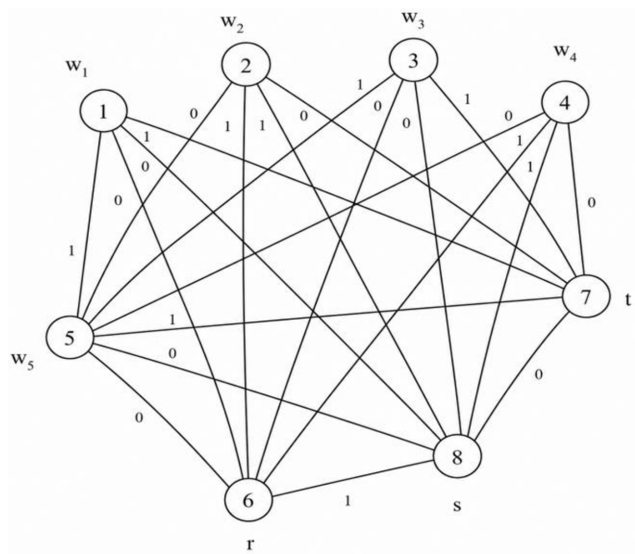

Fig. 3 E.O.C.G of G = $\Gamma(Z_{10}) + \Gamma(Z_6)$

We note that $e_{f^*}(0) = 11$ and $e_{f^*}(1) = 10$.

Therefore, the constructed labeling fulfills the even – odd cordiality criterion, since $\left|e_{f^*}(1) - e_{f^*}(0)\right| \leq 1$, then $\Gamma(Z_{10}) + \Gamma(Z_6)$ is EOCL graph.

Theorem 3.5. Let G = $\Gamma(Z_{2k}) + \Gamma(Z_9)$, where k is a prime number satisfying k > 2. Then G admits an EOC labeling.
Proof.
Let G = (V (G), E (G)) = Γ $(Z_{2k}) + \Gamma(Z_9)$. Then V(G) and E(G) are defined by

V(G) = $\{w_1, w_2, \ldots\ldots w_k, r, s\}$

E(G) = $\{w_\alpha w_k, w_\alpha r, w_\alpha s, w_k r, w_k s, rs\ for\ 1 \leq \alpha \leq k-1\}$

Accordingly, the graph G consists of k + 2 vertices and 3k edges.

Thus, $|V(G)|$ = k+2 and $|E(G)|$ = 3k.

To construct an even – odd cordial labeling, define a bijective vetex labeling

f : V(G) $to\ \{1,2,\ldots.,k+2\}$ by assigning the labels f (w_α) = α , $1 \leq \alpha \leq k$, f(x) = k+1, f(y) = k+2

The vertex assignment f gives rise to an edge – labeling function f^*: E(G) → {0,1}, defined according to the following rule:

$f^*(w_\alpha w_k) = \begin{cases} 0, & \alpha \text{ is even}, \\ 1, & \alpha \text{ is odd}, \end{cases} \quad 1 \le \alpha \le k-1,$

$f^*(w_\alpha r) = \begin{cases} 1, & \alpha \text{ is even}, \\ 0, & \alpha \text{ is odd}, \end{cases} \quad 1 \le \alpha \le k,$

$f^*(w_\alpha s) = \begin{cases} 1, & \alpha \text{ is even}, \\ 0, & \alpha \text{ is odd}, \end{cases} \quad 1 \le \alpha \le k.$

$f^*(rs) = 1$

We note that $e_{f^*}(0) = \frac{3k+1}{2}$ and $e_{f^*}(1) = \frac{3k-1}{2}$

The obtained edge labeling fulfils the condition $\left|e_{f^*}(1) - e_{f^*}(0)\right| \le 1$, then it is EOCL graph for all prime numbers k > 2.

Example 3.6:

A EOC labeling for G = $\Gamma(Z_{2k}) + \Gamma(Z_9)$, is given in figure 4 where k = 7

V(G) = $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7, r, s\}$

E(G) = $\{w_\alpha w_7, w_\alpha r, w_\alpha s, w_7 r, w_7 s, rs, 1 \le \alpha \le 6\}$,

Here $|V(G)|$ = 9, & $|E(G)|$ = 21

Consider the vertex assignment f : V(G) → {1,2, … . ,9},

where f(w_α) = α, $\quad 1 \le \alpha \le 7$,

and f(r) = 8, f(s) = 9

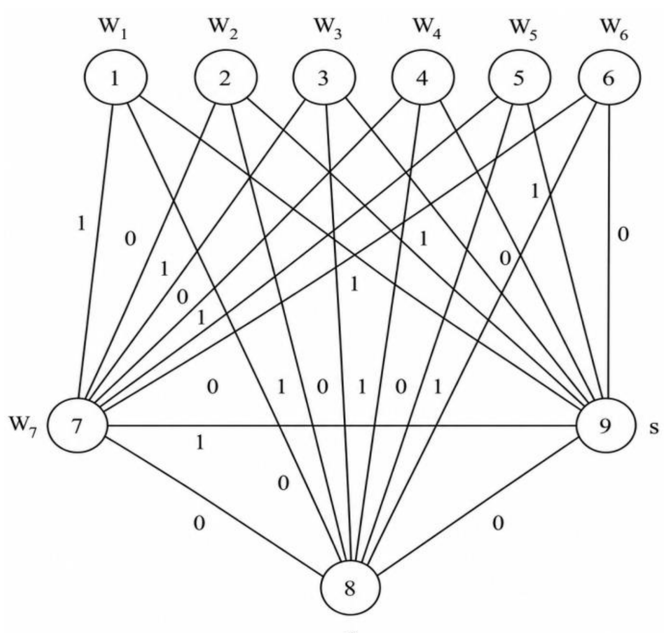

Fig. 4 E.O.C.G of G = Γ(Z_{14}) + Γ(Z_9)

We note that $e_{f^*}(1) = 10$ and $e_{f^*}(0) = 11$.

The above labeling scheme yields $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$, then $\Gamma(Z_{10}) + \Gamma(Z_6)$ is EOCL graph.

Theorem 3.7. Let G = $\Gamma(Z_{3m}) + \Gamma(Z_4)$, where m is a prime number satisfying m >3. Then G admits an EOC labeling.

Proof. Let G = $\Gamma(Z_{3m}) + \Gamma(Z_4)$

The combinatorial structure of the graph G is represented by the following vertex and edge sets: V(G) = $\{w_1, w_2, \dots.. w_m, w_{2m}, x\}$

E(G) = $\{w_\alpha w_m,\ w_\alpha w_{2m},\ w_\alpha x,\ w_m x,\ w_{2m} x\ \ for\ 1 \leq \alpha \leq m-1\}$

Consequently, the graph G has m +2 vertices and 3m-1 edges; that is,

$|V(G)|$ = m+2 and $|E(G)|$ = 3m-1.

Now construct a bijection f : V(G) $\rightarrow \{1,2, \dots., m+2\}$ by defining

f $(w_\alpha) = \alpha$, $1 \leq \alpha \leq m$, together with f(w_{2m}) = m+1, f(x) = m+2.

The vertex assignment f gives rise to an edge – labeling function f*: E(G) → {0,1}, defined according to the following rule:

$$f^*(w_\alpha w_m) = \begin{cases} 0, & when\ \ \alpha\ is\ even, \\ 1, & when\ \ \alpha\ is\ odd, \end{cases} \qquad 1 \leq \alpha \leq m-1,$$

$$f^*(w_\alpha w_{2m}) = \begin{cases} 1, & when\ \ \alpha\ is\ even, \\ 0, & when\ \ \alpha\ is\ odd, \end{cases} \qquad 1 \leq \alpha \leq m-1,$$

$$f^*(w_\alpha x) = \begin{cases} 1, & when\ \alpha\ is\ even, \\ 0, & when\ \alpha\ is\ odd, \end{cases} \qquad 1 \leq \alpha \leq m.$$

$f^*(w_{2m} x) = 0$

We note that $e_{f^*}(1) = \frac{3m-1}{2}$ and $e_{f^*}(0) = \frac{3m-1}{2}$

The foregoing labeling construction verifies the inequlality $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$, then it is EOC graph for all prime numbers m > 3.

Example 3.8:

A EOC labeling for G = $\Gamma(Z_{3m}) + \Gamma(Z_4)$, is given in figure 5 where m = 5

For the graph G = $\Gamma(Z_{15}) + \Gamma(Z_4)$, the vertex and edge sets are defined as follows:

V(G) = $\{w_1, w_2, w_3, w_4, w_5, w_{10}, x\}$

E(G) = $\{w_\alpha w_5, w_\alpha w_{10}, w_\alpha x, w_5 x, w_{10} x, 1 \leq \alpha \leq 4\}$

Accordingly, the graph G contains 7 vertices and 14 edges.

Hence, $|V(G)|$ = 7 and $|E(G)|$ = 14.

Define a bijective vertext labeling f : V(G) $\rightarrow \{1,2, \dots.,7\}$ as follows

f$(w_\alpha) = \alpha$, for $1 \leq \alpha \leq 4$ and f(w_{10}) = 6, f(x) = 7

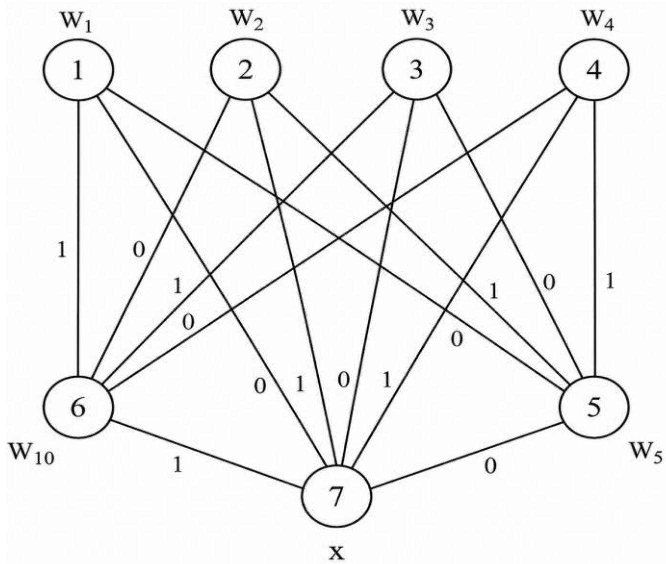

Fig. 5 E.O.C.G of G = $\Gamma(Z_{15}) + \Gamma(Z_4)$

We note that $e_{f^*}(1) = 7$ and $e_{f^*}(0) = 7$

The preceding labeling construction establishes that $\left|e_{f^*}(1) - e_{f^*}(0)\right| \leq 1$, then $\Gamma(Z_{10}) + \Gamma(Z_6)$ is EOCL graph.\

Theorem 3.9. Let $\Gamma(Z_{3m}) + \Gamma(Z_6)$, where m is a prime number satisfying m > 3. Then G admits an EOC labeling.

Proof. Say G = $\Gamma(Z_{3m}) + \Gamma(Z_6)$,

We begin by describing the structure of the graph G.

The graph G is specified through its vertex set as follows:

$$V(G) = \{w_1, w_2, \ldots\ldots w_m, w_{2m}, r, s, t\}$$

E(G) $=\left\{w_\alpha w_m,\ w_\alpha w_{2m},\ w_\alpha r,\ w_\alpha s,\ w_\alpha t,\ w_m r,\ w_m s,\ w_m t,\ w_{2m} r,\ w_{2m} s,\ w_{2p} t, rs,\ st\ for\ 1 \leq \alpha \leq m-1\right\}$

Consequently, the graph G has m +4 vertices and 5m+3 edges.

Hence $|V(G)|$ = m+4 and $|E(G)|$ = 5m+3.

To obtain the desire even – odd cordial labeling, define a bijective vertex mapping

f : V(G) $\rightarrow \{1,2, \ldots\ldots, m+4\}$ by assigning f $(w_\alpha) = \alpha$, $1 \leq \alpha \leq m$,

f(w_{2m}) = m+1, f(r) = m+2, f(s) = m+4, f(t) = m+3.

The vertex assignment f gives rise to an edge – labeling function f^*: E(G) → {0,1}, defined according to the following rule:

$f^*(w_\alpha w_m) = \begin{cases} 0, & \alpha\ is\ even, \\ 1, & \alpha\ is\ odd, \end{cases}$ $\quad 1 \leq \alpha \leq m-1,$

$f^*(w_\alpha w_{2m}) = \begin{cases} 1, & \alpha\ is\ even, \\ 0, & \alpha\ is\ odd, \end{cases}$ $\quad 1 \leq \alpha \leq m-1,$

$f^*(w_\alpha r) = \begin{cases} 1, & \alpha\ is\ even, \\ 0, & \alpha\ is\ odd, \end{cases}$ $\quad 1 \leq \alpha \leq m,$

$f^*(w_\alpha s) = \begin{cases} 0, & \alpha \text{ is even}, \\ 1, & \alpha \text{ is odd}, \end{cases} \qquad 1 \le \alpha \le m,$

$f^*(w_\alpha t) = \begin{cases} 1, & \alpha \text{ is even}, \\ 0, & \alpha \text{ is odd}, \end{cases} \qquad 1 \le \alpha \le m.$

$f^*(w_{2m} r) = 0$, $f^*(w_{2m} s) = 0$, $f^*(w_{2m} t) = 1$

$f^*(rs) = 1$, $f^*(st) = 0$.

We note that $e_{f^*}(1) = \frac{5m+3}{2}$ and $e_{f^*}(0) = \frac{5m+3}{2}$

The resulting edge labeling preserves the required parity balance, that is $|e_{f^*}(1) - e_{f^*}(0)| \le 1$. Then $\Gamma(Z_{3m}) + \Gamma(Z_6)$ is EOCL graph for all prime numbers m > 3.

Example 3.10:

A EOCL labeling for G = $\Gamma(Z_{3m}) + \Gamma(Z_6)$, is given in figure 6 where m = 7

Consider the graph G = $\Gamma(Z_{21}) + \Gamma(Z_6)$. Its structure is completely determined by the following vertex and edge sets:

V(G) = $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_{14}, r, s, t\}$

E(G) = $\{w_\alpha w_7, w_\alpha w_{14}, w_\alpha r, w_\alpha s, w_\alpha t, w_7 r, w_7 s, w_7 t, w_{14} r, w_{14} s, w_{14} t, rs, st \text{ for } 1 \le \alpha \le 6\}$

Here $|V(G)| = 11$, & $|E(G)| = 38$

Define the one – to – one vertex labeling f : V(G) $\rightarrow \{1,2,\dots,11\}$ by

f(w_α) = α, $1 \le \alpha \le 7$,

f(w_{14}) = 8, f(r) =9 , f(s) = 11 , f(t) = 10

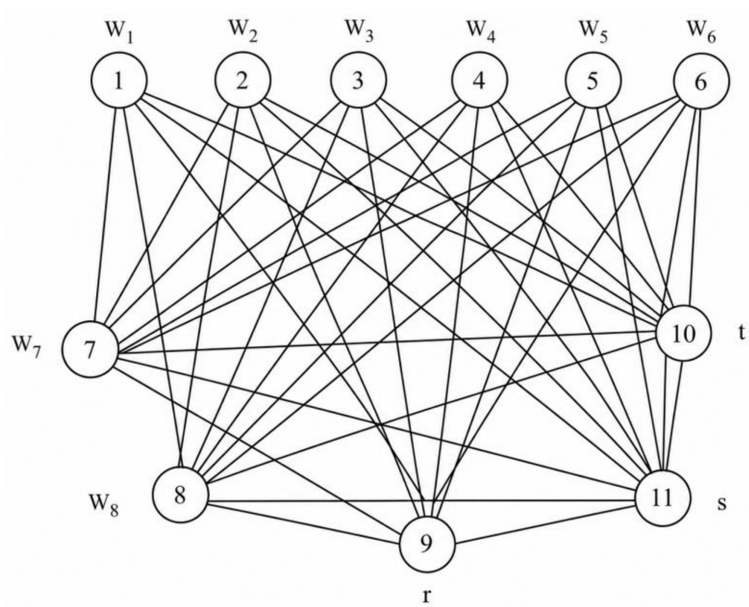

Fig. 6 E.O.C.G of G = $\Gamma(Z_{21}) + \Gamma(Z_6)$

We note that $e_{f^*}(1) = 19$ and $e_{f^*}(0) = 19$

The proposed labeling achieves the required balance between the two edge – label classes, namely $\left|e_{f^*}(1) - e_{f^*}(0)\right| \leq 1$, then $\Gamma(Z_{21}) + \Gamma(Z_6)$ is EOCL graph.

Theorem 3.11. Let G denote the join graph $\Gamma(Z_{3m}) + \Gamma(Z_9)$, where m is a prime number greater than 3. Then G is an EOC graph.

Proof. Say $G = \Gamma(Z_{3m}) + \Gamma(Z_9)$,

We begin by describing the structure of the graph G.

The graph G is specified through its vertex set as follows:

$V(G) = \{w_1,\ w_2, \ldots\ldots, w_m,\ w_{2m},\ r,\ s\}$

$E(G) = \{w_\alpha w_m,\ w_\alpha w_{2m},\ w_\alpha r,\ w_\alpha s,\ w_m r,\ w_m s,\ w_{2m} r,\ w_{2m} s,\ rs\ for\ 1 \leq \alpha \leq m-1\}$

Hence $|V(G)|$ = m+3, and $|E(G)|$ = 4m+1,

The vertex labeling is f: V(G) $\rightarrow \{1,2, \ldots., m+3\}$ given by

$f(w_\alpha) = \alpha$, for $1 \leq \alpha \leq m$,

$f(w_{2m})$ = m+1, f(r) = m+2, f(s) = m+3.

The vertex assignment f gives rise to an edge – labeling function f^*: E(G) → {0,1}, defined according to the following rule:

$$f^*(w_\alpha w_m) = \begin{cases} 0, & \alpha\ is\ even, \\ 1, & \alpha\ is\ odd, \end{cases} \qquad 1 \leq \alpha \leq m-1,$$

$$f^*(w_\alpha w_{2m}) = \begin{cases} 1, & \alpha\ is\ even, \\ 0, & \alpha\ is\ odd, \end{cases} \qquad 1 \leq \alpha \leq m-1,$$

$$f^*(w_\alpha r) = \begin{cases} 1, & \alpha\ is\ even, \\ 0, & \alpha\ is\ odd, \end{cases} \qquad 1 \leq \alpha \leq m,$$

$$f^*(w_\alpha s) = \begin{cases} 0, & \alpha\ is\ even, \\ 1, & \alpha\ is\ odd, \end{cases} \qquad 1 \leq \alpha \leq m.$$

$f^*(w_{2m} r) = 0,\ f^*(w_{2m} s) = 1,\ f^*(rs) = 0.$

We note that $e_{f^*}(0) = 2m+1$ and $e_{f^*}(1) = 2m$

The resulting edge labeling preserves the required parity balance, that is $\left|e_{f^*}(1) - e_{f^*}(0)\right| \leq 1$. Then $\Gamma(Z_{3m}) + \Gamma(Z_9)$ is EOC graph for all prime numbers m > 3.

Example 3.12:

A EOC labeling for $G = \Gamma(Z_{3m}) + \Gamma(Z_9)$, is given in figure 7 where m = 5

Let $G = \Gamma(Z_{15}) + \Gamma(Z_9)$. The underlying structure of G is determined through its vertex and edge sets, namely:

$V(G) = \{w_1, w_2, w_3, w_4, w_5, w_{10}, r, s\}$

$E(G) = \{w_\alpha w_5,\ w_\alpha w_{10},\ w_\alpha r,\ w_\alpha s,\ w_5 r,\ w_5 s,\ w_{10} r,\ w_{10} s, rs\ for\ 1 \leq \alpha \leq 4\}$

Here total $|V(G)|$ = 8 & $|E(G)|$ = 21

The desired vertex assignment is realized via the bijection f : V(G) $\rightarrow \{1,2, \ldots., 8\}$, given by f$(w_\alpha) = \alpha, \quad 1 \leq \alpha \leq 5,$

f$(w_{14}) = 6$, f$(r) = 7$, f$(t) = 8$.

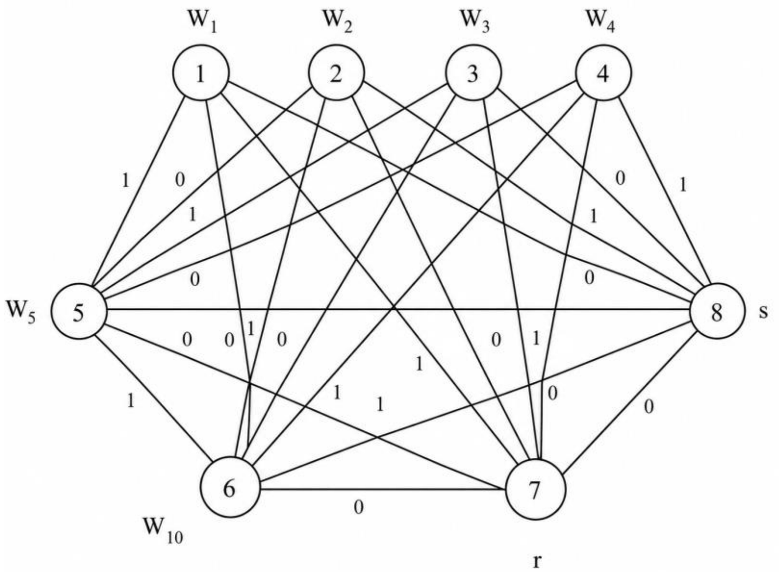

Fig. 7 E.O.C.G of G = Γ(Z_{15}) + Γ(Z_9)

We note that $e_{f^*}(1)$ =10 and $e_{f^*}(0) = 11$

It follows from the above construction that $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$, then Γ($Z_{15}$) + Γ($Z_9$) is EOCG.

Theorem 3.13. Let G = Γ(Z_{5h}) + Γ(Z_4), where h is a prime number satisfying h > 5. Then G admits an EOC labeling.

Proof. Consider G = Γ(Z_{5h}) + Γ(Z_4)

We begin by describing the structure of the graph G. The graph G = (V(G), E(G)) is characterized by the following vertex and edge sets:

V(G) = $\{w_1, w_2, \ldots\ldots, w_h,\ w_{2h},\ w_{3h},\ w_{4h},\ x\}$

E(G) = $\{w_\alpha w_{ih},\ w_\alpha x,\ w_{ih}x,\ 1 \leq \alpha \leq h-1, 1 \leq i \leq 4\}$

Accordingly, the graph G contains h +4 vertices and 5h-1 edges.

Hecnce, $|V(G)|$ = h+4 and $|E(G)|$ = 5h-1.

To obtain the required even – odd cordial labeling, define the bijective vertex mapping

f : V(G) $\rightarrow \{1,2, \ldots\ldots, h+4\}$ by assigning

f $(w_\alpha) = \alpha$, $1 \leq \alpha \leq h$,

f (w_{2h}) = h+1, f (w_{3h}) = h+2, f (w_{4h}) = h+3, f(x) = h+4.

The vertex assignment f gives rise to an edge – labeling function f*: E(G) → $\{0,1\}$, defined according to the following rule:

$$f^*(w_\alpha w_{\theta h}) = \begin{cases} 0 \text{ if } \alpha \text{ is even and } \theta \text{ is odd} \\ 0 \text{ if } \alpha \text{ is odd and } \theta \text{ is even} \\ 1 \text{ if } \alpha \text{ is odd and } \theta \text{ is odd} \\ 1 \text{ if } \alpha \text{ is even and } \theta \text{ is even} \end{cases} \quad for\ 1 \le \alpha \le h-1\ ; 1 \le \theta \le 4$$

$$f^*(w_\alpha x) = \begin{cases} 0 \text{ if } \alpha \text{ is even } for\ 1 \le \alpha \le h-1 \\ 1 \text{ if} \alpha \text{ is odd } for\ 1 \le \alpha \le h-1 \end{cases}$$

$$f^*(w_{\theta p} x) = \begin{cases} 0 \text{ if } \theta \text{ is even } for\ 1 \le \theta \le 4 \\ 1 \text{ if } \theta \text{ is odd } for\ 1 \le \theta \le 4 \end{cases}$$

We note that $e_{f^*}(1) = \frac{5h-1}{2}$ and $e_{f^*}(0) = \frac{5h-1}{2}$

It follows from the foregoing construction that $\left|e_{f^*}(1) - e_{f^*}(0)\right| \le 1$, then $\Gamma(Z_{5h}) + \Gamma(Z_4)$ is EOC graph for all prime numbers h >5.

Example 3.14:
A EOC labeling for G = $\Gamma(Z_{5h}) + \Gamma(Z_4)$, is given in figure 8 where h =7

V(G) = $\{w_1,\ w_2,\ w_3, w_4,\ w_5,\ w_6,\ w_7,\ w_{14},\ w_{21},\ w_{28},\ x\}$,

E(G) = $\{w_\alpha w_{\theta 7},\ w_\alpha x,\ w_{\theta 7} x\ \ 1 \le \alpha \le 4, 1 \le \theta \le 4\}$.

Hence, the graph G comprises 11 vertices and 34 edges,

Accordingly, $|V(G)|$ = 11 and $|E(G)|$ = 34.

Consider the bijective vertex assignment f : V(G) $\rightarrow \{1,2, \dots ,11\}$, where the vertex labels are prescribed as follows:

f(w_α) = α, $\quad 1 \le \alpha \le 7$,

f(w_{14}) = 8, f(w_{21}) = 9, f(w_{28}) = 10, f(x) =11.

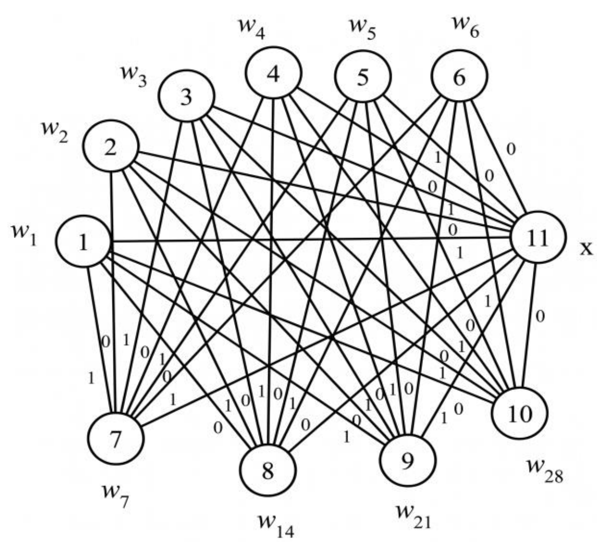

Fig. 8 E.O.C.G of G = $\Gamma(Z_{35}) + \Gamma(Z_4)$

We note that $e_{f^*}(1)$ = 17 and $e_{f^*}(0)$ =17

The above labeling produces a balanced distribution of edges labels, satisfying $\left|e_{f^*}(1) - e_{f^*}(0)\right| \le 1$, then $\Gamma(Z_{35}) + \Gamma(Z_4)$ is EOCL graph.

Theorem 3.15. Let G = $\Gamma(Z_{5h}) + \Gamma(Z_6)$, where h > 5 is a prime number. Then G admits an EOC labeling**.**

Proof. Let G = $\Gamma(Z_{5h}) + \Gamma(Z_6)$,

The underlying structure of G is specified through the following vertex and edge sets:

V(G) = $\{w_1, w_2, \dots\dots w_h, w_{2h}, w_{3h}, w_{4h}, r, s, t\}$

E(G) = $\{w_\alpha w_{\theta h}, w_\alpha r, w_{\theta p} r, w_\alpha s, w_{ih} s, w_\alpha t, w_{\theta h} t, rs, st, 1 \le \alpha \le h-1, 1 \le \theta \le 4\}$

Number of Vertices and Edges are $|V(G)|$ = h+6, and $|E(G)|$ = 7h+7.

To establish the required labeling, define the bijective vertex mapping

f : V(G) to $\{1,2, \dots, h+6\}$, given by

f (w_α) = α , $1 \le \alpha \le h$,

f (w_{2h}) = h+1, f (w_{3h}) = h+2, f (w_{4h}) = h+3, f(r) = h+4, f(s) = h+6, f(t) = h+5.

The vertex assignment f gives rise to an edge – labeling function f*: E(G) → {0,1}, defined according to the following rule:

$$f^*(w_\alpha w_{\theta h}) = \begin{cases} 0 \ if\ \alpha\ is\ even\ and\ \theta\ is\ odd \\ 0 \ if\ \alpha\ is\ odd\ and\ \theta\ is\ even \\ 1 \ if\ \alpha\ is\ odd\ and\ \theta\ is\ odd \\ 1\ if\ \alpha\ is\ even\ \ and\ \theta\ is\ even \end{cases} \quad for\ 1 \le \alpha \le h-1\ ; 1 \le \theta \le 4$$

$$f^*(w_\alpha r)\ \&\ f^*(w_\alpha s) = \begin{cases} 0\ if\ \alpha\ is\ even\ \ for\ 1 \le \alpha \le h-1 \\ 1\ if \alpha\ is\ odd\ \ \ for\ 1 \le \alpha \le h-1 \end{cases}$$

$$f^*(w_\alpha t) = \begin{cases} 1\ if\ \alpha\ is\ even\ \ for\ 1 \le \alpha \le h-1 \\ 0\ if \alpha\ is\ odd\ \ \ for\ 1 \le \alpha \le h-1 \end{cases}$$

$$f^*(w_{\theta h} r)\ \&\ f^*(w_{\theta h} s) = \begin{cases} 0\ if\ \theta\ is\ even\ \ for\ 1 \le \theta \le 4 \\ 1\ if\ \theta\ is\ odd\ \ \ for\ 1 \le \theta \le 4 \end{cases}$$

$$f^*(w_{\theta h} t) = \begin{cases} 1\ if\ \theta\ is\ even\ \ for\ 1 \le \theta \le 4 \\ 0\ if\ \theta\ \ is\ odd\ \ \ for\ 1 \le \theta \le 4 \end{cases}$$

$f^*(rs)$ = 0 and $f^*(st)$ = 1

We note that $e_{f^*}(1) = \frac{7h+7}{2}$ and $e_{f^*}(0) = \frac{7h+7}{2}$

The above labeling yields an admissible edge – label distribution satisfying $|e_{f^*}(1) - e_{f^*}(0)| \le 1$, then $\Gamma(Z_{5h}) + \Gamma(Z_6)$ is EOC graph for all prime numbers h > 5.

Example 3.16:
A EOC labeling for G = $\Gamma(Z_{5h}) + \Gamma(Z_6)$, is given in figure 9 where h =7

V(G) = $\{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6, w_7,\ w_{14},\ w_{21},\ w_{28},\ r, s, t\}$,

E(G) = $\{w_\alpha w_{\theta 7}, w_\alpha x, w_{\theta 7} r, w_\alpha s, w_{\theta 7} s, w_\alpha t, w_{\theta 7} t, rs, st, 1 \le \alpha \le 6, 1 \le \theta \le 4.\}$

Here the total vertices and Edges are $|V(G)|$ = 13, and $|E(G)|$ = 56.

To facilitate the proposed construction, define a bijective labeling function f : V(G) $\rightarrow \{1,2,\ldots,13\}$, by assigning distinct integers to the vertices in the following manner:

f(w_α) = α, $\quad 1 \le \alpha \le 7$,

f(w_{14}) = 8, f(w_{21}) = 9, f(w_{28}) = 10, f(r) =11, f(s) =13, f(t) =12

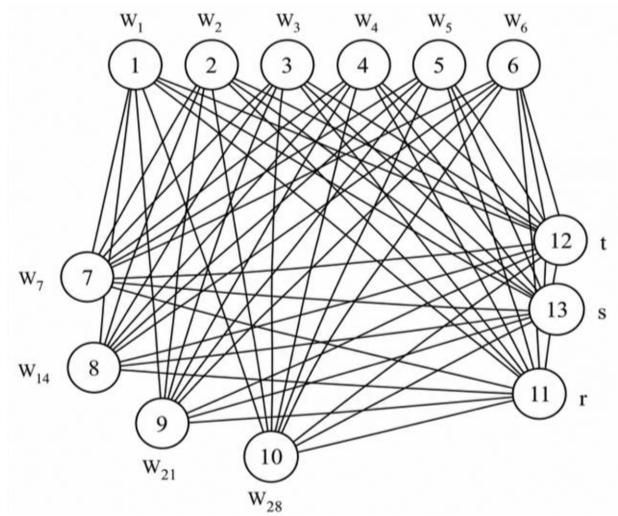

Fig. 9 E.O.C.G of G = Γ(Z_{35}) + Γ(Z_6)

We note that $e_{f^*}(1)$ = 28 and $e_{f^*}(0)$ =28

The above labeling produces a balanced distribution of edges labels, satisfying $\left|e_{f^*}(1) - e_{f^*}(0)\right| \le 1$, then Γ($Z_{35}$) + Γ($Z_6$) is EOCL graph.

Theorem 3.17. Assume that h is a prime number satisfying h > 5. Under this condition, the join graph Γ(Z_{5h}) + Γ(Z_9) admits an EOC labeling.

Proof. Let G = Γ(Z_{5h}) + Γ(Z_9)

Let the vertex set and edge set of the graph G are

V(G) = $\{w_1, w_2, \ldots\ldots w_h, w_{2h}, w_{3h}, w_{4h}, r, s\}$

E(G) = $\left\{w_\alpha w_{\theta p}, w_\alpha r, w_{\theta p} r, w_\alpha s, w_{\theta p} s, rs, 1 \le \alpha \le h-1, 1 \le \theta \le 4\right\}$

Here the number of vertices and edges are $|V(G)|$ = h+5, and $|E(G)|$ = 6h+3

To establish the required labeling, define the bijective vertex mapping

f: V(G) $\rightarrow \{1,2,\ldots,h+5\}$, by

f (w_α) = α , $\quad 1 \le \alpha \le h$,

f (w_{2h}) = h+1, f (w_{3h}) = h+2, f (w_{4h}) = h+3, f(r) = h+4, f(s) = h+5.

The vertex assignment f gives rise to an edge – labeling function f*: E(G) → {0,1}, defined according to the following rule:

$$f^*(w_\alpha w_{\theta h}) = \begin{cases} 0 \;\; if\ \alpha\ is\ even\ and\ \theta\ is\ odd \\ 0 \;\; if\ \alpha\ is\ odd\ and\ \theta\ is\ even \\ 1 \;\; if\ \alpha\ is\ odd\ and\ \theta\ is\ odd \\ 1\ if\ \alpha\ is\ even\ \ and\ \theta\ is\ even \end{cases} \qquad for\ 1 \le \alpha \le h-1\ ; 1 \le \theta \le 4$$

$f^*(w_\alpha r) = \begin{cases} 0 \ if \ \alpha \ is \ even \ \ for \ 1 \le \alpha \le h-1 \\ 1 \ if \alpha \ is \ odd \ \ \ for \ 1 \le \alpha \le h-1 \end{cases}$

$f^*(w_\alpha s) = \begin{cases} 1 \ if \ \alpha \ is \ even \ \ for \ 1 \le \alpha \le h-1 \\ 0 \ if \alpha \ is \ odd \ \ \ for \ 1 \le \alpha \le h-1 \end{cases}$

$f^*(w_{\theta h} r) = \begin{cases} 0 \ if \theta \ is \ even \ \ for \ 1 \le \theta \le 4 \\ 1 \ if \theta \ is \ odd \ \ \ for \ 1 \le \theta \le 4 \end{cases}$

$f^*(w_{\theta h} s) = \begin{cases} 1 \ if \ \theta \ is \ even \ \ for \ 1 \le \theta \le 4 \\ 0 \ if \ \theta \ is \ odd \ \ \ for \ 1 \le \theta \le 4 \end{cases}$

$f^*(rs)$ = 0 and

We note that $e_{f^*}(0) = 3h+2$ and $e_{f^*}(1) = 3h+1$

The proposed labeling yields a valid edge – label distributin for which $\left|e_{f^*}(1) - e_{f^*}(0)\right| \le 1$, then it is EOC graph for all prime numbers h > 7.

Example 3.18:

A EOC labeling for G = $\Gamma(Z_{5h}) + \Gamma(Z_9)$, is given in figure 10 where h =7

V(G) = $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_{14}, w_{21}, w_{28}, r, s\}$,

E(G) = $\{w_\alpha w_{\theta 7}, w_\alpha r, w_{\theta 7} r, w_\alpha s, w_{\theta 7} s, rs, 1 \le \alpha \le 6, 1 \le \theta \le 4\}$

Here the number of vertices and edges are $|V(G)|$ = 12, and $|E(G)|$ = 45

To establish the desired labeling, define the bijective vertex mapping

f: V(G) $\rightarrow$ {1,2, ….12} by

f(w_α) = α, $1 \le \alpha \le 7$,

f(w_{14}) = 8, f(w_{21}) = 9, f(w_{28}) = 10, f(r) =11, f(s) =12.

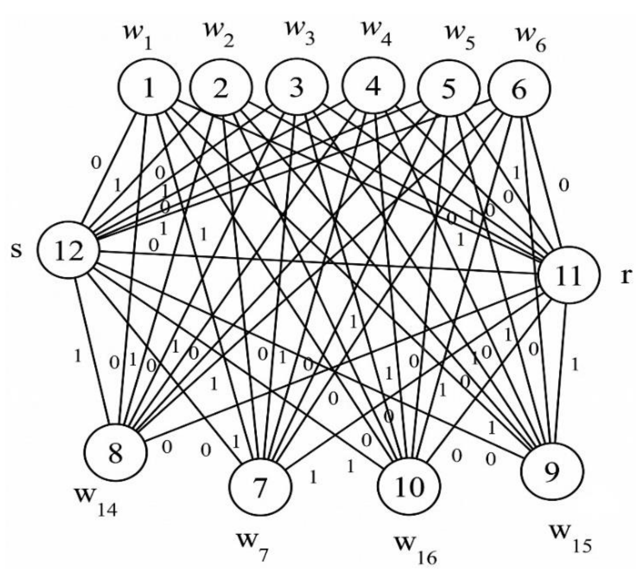

Fig. 10 E.O.C.G of G = $\Gamma(Z_{35}) + \Gamma(Z_9)$

We note that $e_{f^*}(0) = 23$ and $e_{f^*}(1) = 22$.

The proposed vertex assignment induces an admissible edge labeling satisfying $|e_{f^*}(1) - e_{f^*}(0)| \leq 1$, then $\Gamma(Z_{35}) + \Gamma(Z_9)$ is EOCL graph.

4. Conclusion

The present work establishes that the join graphs

- *$\Gamma(Z_{2p}) + \Gamma(Z_m)$, where $m \in \{4,6,9\}$ for every prime $p > 2$ is EOCL graph*
- *$\Gamma(Z_{3p}) + \Gamma(Z_m)$, where, $m \in \{4,6,9\}$ for every prime $P > 3$ is EOCL graph.*
- *$\Gamma(Z_{np}) + \Gamma(Z_m)$, where $m \in \{4,6,9\}$ $for\ every\ prime\ P\ > 5\ is\ EOCL\ graph.$*

These fingings extend the theory of even – odd cordial labeling to a new family of join graphs arising from zero – divisor graphs of finite commutative rings. Consequently, this work contributes to the growing interaction between algebraic graph theory and graph labeling by introducing new classes of even – odd cordial graphs and providing a foundation for further investigations on operations involving zero – divisor graphs.

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