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Original Article

Parity, Pairs, and the Power of Small Differences: Even/Odd Symmetry, Decimal Definiteness, and Infinitesimal Imbalance

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Abstract *- We present a comprehensive construct wherein nature organizes by pairs (parity, bipartition, entangled duals) while macroscopic asymmetry emerges from small differences (infinitesimal deviations, symmetry breaking, representational obstructions). The paper unifies (i) a parity and pair-based worldview in which evenness encodes constraint and definiteness, and oddness encodes obstruction and imbalance, (ii) a rigorous account of decimal expansions, including termination criteria, multiplicative order, and the identity 0.999... = 1, (iii) a mathematically precise treatment of nonzero but negligible differences via hyperreals and the standard-part map, and (iv) multiple worked physical models demonstrating how minute biases accumulate into macroscopic drift, namely pair-constrained dynamics, baryogenesis toy kinetics with freeze-out, entanglement stabilization under noise, diffusion-abrasion with cumulative loss, quantization error propagation, graph-theoretic frustration, and Ising parity constraints. We correct the literal arithmetic claim that division by even denominators is definite while odd denominators are indefinite, reframing it as a symmetry metaphor grounded in base-dependent factor compatibility, while preserving the philosophical thesis that pairs govern balance and small differences power becoming. We close by situating the framework within broader discussions of identity under symmetry, impermanence, and scale-relative equivalence.*

Keywords *- Baryogenesis, Hyperreals, Infinitesimals, Parity Symmetry, Symmetry Breaking.*

1. Introduction

We elaborate on two interlocking principles.

Pair-construct (parity). Nature is deeply structured by pairwise complements, including particle and antiparticle, spin up and spin down, left and right chirality, even and odd parity subspaces, and correlated and anticorrelated quantum states. Evenness (invariance under a $\mathbb{Z}_2$ action) imposes selection rules and constraints, whereas oddness (antisymmetry) relaxes constraints and opens new channels. [1, 2]

Small-differences principle. Tiny deviations, such as infinitesimal gaps, base-dependent representational obstructions, microscopic CP biases, and low-variance noise, can accumulate into macroscopic consequences. [3, 4] Decimal phenomena, such as $7/9 = 0.\overline{7}$, $8/9 = 0.\overline{8}$, and $0.\overline{9} = 1$, motivate a theme, namely that what is neglected at one layer can be causal at another.

We synthesise these by using parity as the macroscopic grammar of balance and infinitesimals as the microscopic engine of historical drift. We also revisit division and decimals with full rigour. Although the literal parity claim about denominators is false in base 10, its metaphorical resonance is accurately formalised via symmetry compatibility (prime factors of the base) and multiplicative order. [7]

2. Decimal Anatomy, Termination, Repetition, and Pattern Induction

2.1. Prime-Factor Compatibility and Termination

Proposition 2.1 *(Termination criterion in base b).* Let $p/q \in \mathbb{Q}$ be in lowest terms and let $b \geq 2$ be an integer base. The expansion of p/q in base b terminates if and only if every prime factor of q divides b.

Proof. A base-b expansion terminates after k places if and only if $p/q = m/b^k$ for some $m \in \mathbb{Z}$, that is, $b^k p = mq$. Since $\gcd(p,q) = 1$, every prime divisor of q must divide b^k and hence b. Conversely, if $q = \prod_i p_i^{\alpha_i}$ and each $p_i \mid b$, then for sufficiently large k, we have $q \mid b^k$, so $p/q = m/b^k$ for some m. ▫

Remark 2.2. In base 10, termination occurs if and only if q has only the prime factors 2 and 5. Thus, the parity of q alone is not decisive. For example, $1/6 = 0.1\overline{6}$ repeats, owing to the prime 3, while $1/5 = 0.2$ terminates despite 5 being odd.

2.2. Repeating Period and Multiplicative Order

Theorem 2.3 *(Period length via multiplicative order).* Let p/q be in lowest terms with gcd. $(q,b) = 1$. The base-b expansion of p/q is purely repeating with period length equal to $\mathrm{ord}_q(b)$, the least $k \geq 1$ such that $b^k \equiv 1 \pmod{q}$.

Proof. Long division in base b yields remainders. $r_n \in \{1,2,\ldots,q-1\}$ satisfying $r_{n+1} \equiv b\, r_n \pmod{q}$. The minimal k with $r_k = r_0$ gives the repeat period. Since $r_n \equiv b^n r_0 \pmod{q}$ and $r_0 \neq 0$, the smallest k with $b^k \equiv 1 \pmod{q}$ is the period. ▫

Example 2.4. In base 10, $1/7$ has a period $\mathrm{ord}_7(10) = 6$, and $1/9$ has period $\mathrm{ord}_9(10) = 1$, hence $0.\overline{1}$. It follows that $7/9 = 0.\overline{7}$ and $8/9 = 0.\overline{8}$.

2.3. The Identity 0.999… = 1 and the Nonexistence of "0.000…1"

Proposition 2.5. In $\mathbb{R}$, one has 0.999 ... = 1.

Proof. Let $x_k = 1 - 10^{-k}$. Then $x_k \to 1$. By the definition of an infinite decimal as a limit, $0.999 \ldots = \lim_{k\to\infty} x_k = 1$. ▫

Remark 2.6. The string "0.000 ... 1" is not a well-formed real number in positional notation, because an ellipsis denotes an infinite tail and hence there is no terminal 1. There is no real number strictly between $0.\overline{9}$ and 1.

2.4. Induction Pitfalls and Pattern Extension

Heuristics such as "$7/9 \to 0.\overline{7}$ and $8/9 \to 0.\overline{8}$, thus $9/9 \to 0.\overline{9}$" are mnemonic rather than proofs. The correct reconciliation is that in $\mathbb{R}$ we have $0.\overline{9} = 1$. Nevertheless, the philosophical intuition that vanishing gaps matter can be formalised via infinitesimals, as set out in Section 3, preserving meaning without contradicting Proposition 2.5.

3. Infinitesimals, A Rigorous Home for "Small but Nonzero"

3.1. Hyperreals and the Standard-Part Map

Assumption 3.1 *(Hyperreal field).* There exists an ordered field. ${}^*\mathbb{R} \supset \mathbb{R}$ containing nonzero infinitesimals, that is, elements $\varepsilon \neq 0$ with $|\varepsilon| < 1/n$ for every standard $n \in \mathbb{N}$. The standard-part map st sends each finite hyperreal to its unique nearest real number. [5, 6]

Definition 3.2 *(Resolution equivalence).* Let $\{d_\lambda\}_{\lambda>0}$ be a family of resolution-monotone pseudometrics, so that a smaller λ detects finer differences. Define $x \equiv_\lambda y$ whenever $d_\lambda(x,y) \leq \lambda$.

Example 3.3 *(Standard part as coarse-graining).* On the finite hyperreals, define $d_\lambda(x,y) = \min\{1, |x-y|/\lambda\}$. If $x - y$ is infinitesimal, then for any fixed $\lambda > 0$, we have $d_\lambda(x,y) \approx 0$, and $\mathrm{st}(x) = \mathrm{st}(y)$ acts as a coarse-graining map that identifies x and y at the real level.

4. Parity and Pairs, Symmetry, Constraint, and Obstruction

4.1. Parity Operator and Even/Odd Decomposition

Let Π be the parity operator, defined by $(\Pi\psi)(\mathbf{x}) = \psi(-\mathbf{x})$ on $L^2(\mathbb{R}^3)$. If $[H,\Pi] = 0$, we may choose eigenstates with definite parity.

Proposition 4.1 *(Projectors and decomposition).* Define $P_\pm = 1/2\,(I \pm \Pi)$. For any ψ,

$$\psi = P_+\psi + P_-\psi, \tag{1}$$

with $P_+\psi$ even and $P_-\psi$ odd. If $[H,\Pi] = 0$, then $\langle P_+\psi, P_-\phi\rangle = 0$ for eigenstates.

4.2. Selection Rules and Definiteness

Theorem 4.2 *(Parity selection rule).* If $[O,\Pi] = 0$, then $\langle \psi_f | O | \psi_i \rangle = 0$ whenever ψ_i and ψ_f have opposite parity.

Proof. Let $\Pi\psi_i = \pi_i\psi_i$ and $\Pi\psi_f = \pi_f\psi_f$ with $\pi_i, \pi_f \in \{\pm 1\}$. Then

$$\langle\psi_f|O|\psi_i\rangle = \langle\Pi\psi_f|\Pi O\Pi|\Pi\psi_i\rangle = \pi_f\pi_i\langle\psi_f|O|\psi_i\rangle. \tag{2}$$

If $\pi_f\pi_i = -1$, the matrix element vanishes. ▫

Evenness thus yields definiteness in the form of strong constraints, whereas oddness, in the form of parity violation, relaxes these constraints. [1, 2]

4.3. Even and Odd Across Mathematics

In Fourier analysis, even components correspond to cosines and odd components to sines, and many symmetric operators act diagonally in this basis. In graph theory, bipartite graphs, which are 2-colourable, contain no odd cycles, and an odd cycle obstructs pair-colouring, introducing frustration and complex phase behaviour in spin systems. [14] In representation theory, a $\mathbb{Z}_2$ action splits spaces into invariant even and odd subspaces, and operators commuting with the action are block-diagonal.

5. Rehabilitating the Decimal Intuition via Symmetry

The literal claim that even denominators imply terminating decimals is false in base 10. The correct reframing is as follows.

- *Symmetry compatibility.* Termination, understood as definiteness, occurs when the prime set of the denominator is compatible with the prime set of the base, which for base 10 is $\{2,5\}$.
- *Symmetry obstruction.* The presence of primes not dividing the base obstructs termination, yielding periodic repetition with period given by the multiplicative order.
- *Parity metaphor.* Treat evenness as compatibility with a two-prime, and therefore pair-structured, base. Although metaphorical with respect to arithmetic parity, this captures the structural point that a conserved or compatible symmetry produces tight constraints, whereas an obstruction introduces complexity.

6. Small Differences Accumulate, Deterministic and Stochastic

6.1. Deterministic Accumulation

Let $x: [0,T] \to X$ be absolutely continuous with metric speed

$$v(t) = \lim_{h\to 0}\frac{d\big(x(t+h),x(t)\big)}{h} \tag{3}$$

when the limit exists. Then

$$d\big(x(T),x(0)\big) \le \int_0^T v\,(t)\,dt. \tag{4}$$

Hence, sub-resolution drifts with $v(t) \le \delta$ grows linearly, giving $d\big(x(T),x(0)\big) \le \delta T$.

6.2. Stochastic Accumulation

Let $S_N = \sum_{n=1}^{N}\Delta_n$, where the Δ_n are independent with $\mathbb{E}[\Delta_n] = \mu$ and $\mathrm{Var}(\Delta_n) = \sigma_n^2$. Then $\mathbb{E}[S_N] = \mu N$ and $\mathrm{Var}(S_N) = \sum_{n=1}^{N}\sigma_n^2$. Unbiased small errors accumulate dispersion of order. $\sqrt{\sum_n \sigma_n^2}$, whereas biased microsteps drift linearly.

7. Worked Models I, Pair Constraints Versus Symmetry-Breaking Sources

7.1. Model A, Pair-Constrained Linear System with a $\mathbb{Z}_2$ Involution

Let J be an involution on $\mathbb{R}^n$, so that $J^2 = I$. Decompose $x = x_+ + x_-$ with $x_\pm = 1/2\,(I \pm J)x$. Consider

$$\dot{x}(t) = Ax(t) + Sx(t) + \xi(t), \quad JAJ = A, \quad JSJ = -S, \quad J\xi \overset{d}{=} \xi. \tag{5}$$

Proposition 7.1 *(Sector equations and imbalance).* Let $\xi_\pm = 1/2\,(I \pm J)\xi$. Then

$$\dot{x}_+ = Ax_+ + Sx_- + \xi_+, \quad \dot{x}_- = Ax_- + Sx_+ + \xi_-. \tag{6}$$

If $S = 0$, the sectors decouple and parity statistics are conserved. If $S \ne 0$, cross-coupling induces transfer and possible secular growth of x_-, depending on the spectra of $A \pm S$.

Proof. Apply $P_\pm = 1/2\,(I \pm J)$ to Equation 5 and use $JAJ = A$ together with $JSJ = -S$. ▫

Remark 7.2. This encodes the thesis of the paper. Pair-symmetric dynamics, representing evenness, impose constraints, whereas small symmetry-breaking terms, representing oddness, produce a net bias over time.

7.2. Model B, Baryogenesis Toy Kinetics with Freeze-Out

Let n and $\bar{n}$ denote baryon and antibaryon densities. A minimal toy system is

$$\dot{n} = F(T) - \Gamma n - \lambda n\bar{n} + \delta, \quad \dot{\bar{n}} = F(T) - \Gamma \bar{n} - \lambda n\bar{n} - \delta, \tag{7}$$

with pair production F, decay rate Γ, annihilation coefficient λ, and a small CP-violating source δ. [3, 4] Define $\Delta = n - \bar{n}$ and $\Sigma = n + \bar{n}$, so that

$$\dot{\Delta} = -\Gamma\Delta + 2\delta, \quad \dot{\Sigma} = 2F - \Gamma\Sigma - \frac{\lambda}{2}(\Sigma^2 - \Delta^2). \tag{8}$$

In equilibrium, $\Delta^* = 2\delta/\Gamma$. In an expanding universe, the coefficients are time-dependent, and freeze-out occurs when reactions fall out of equilibrium. Integration yields a residual asymmetry. $\Delta_{\text{freeze}} \approx \int 2\delta(t)\, e^{-\int_t^{t_f} \Gamma(t')\, dt'}\, dt$, illustrating how tiny CP biases produce a macroscopic matter excess.

7.3. Model C, Entanglement Stabilisation and Parity-Protected Codes

Consider the Bell state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, which has even computational parity. Subject to local noise $\mathcal{E}_{\Delta t}$ and a parity-preserving stabiliser $\mathcal{S}$ applied periodically,

$$\rho_t = (\mathcal{S} \circ \mathcal{E}_{\Delta t})^{\lfloor t/\Delta t \rfloor}(\rho_0). \tag{9}$$

Entanglement monotones, such as the concurrence $C(\rho_t)$ Moreover, the negativity, decay more slowly than they would without $\mathcal{S}$. [8, 9, 10] This demonstrates that pair-conserving symmetries stabilise relational definiteness, whereas symmetry-breaking noise accelerates decay and effective randomness.

8. Worked Models II, Decimal Symmetry, Graph Frustration, and Ising Parity

8.1. Model D, Decimal Repetition as Symmetry Obstruction

For p/q in lowest terms, base-b termination holds if and only if the prime factors of q divide b. Otherwise, the expansion is repeating with a period $\text{ord}_q(b)$. The definiteness of a terminating expansion is thus a compatibility of the denominator with the prime structure of the base, which reframes the slogan "even is definite, odd is indefinite" as "pair-compatible is definite, pair-obstructed is indefinite".

8.2. Model E, Graph-Theoretic Frustration and Odd Cycles

A graph G is bipartite if and only if it contains no odd cycle. [14] Consider an Ising model on G with Hamiltonian

$$H(\sigma) = -\sum_{(i,j)\in E} J_{ij}\, \sigma_i \sigma_j. \tag{10}$$

If $J_{ij} < 0$ favours anti-alignment, a bipartite graph admits a ground state with all edges satisfied by assigning opposite spins to the two partitions. The presence of an odd cycle forces frustration, since not all edges can be simultaneously satisfied. [11] This is an odd obstruction to pair structure that induces complex energy landscapes.

8.3. Model F, Ising Parity Constraints and Selection

Define a global $\mathbb{Z}_2$ symmetry $\sigma \mapsto -\sigma$. External fields h_i break this symmetry, giving

$$H(\sigma) = -\sum_{(i,j)} J_{ij}\, \sigma_i \sigma_j - \sum_i h_i\, \sigma_i. \tag{11}$$

For small h_iSymmetry breaking selects a preferred magnetisation sign, which is a macroscopic imbalance produced by a tiny field in the thermodynamic limit. [12] Near criticality, the susceptibility amplifies small biases, so small differences matter.

9. Worked Models III, Diffusion-Abrasion and Quantisation Accumulation

9.1. Model G, Surface Diffusion-Abrasion Under Handling

Let Ω be a thin shell, such as an orange peel. The moisture $u(x, t)$ satisfies

$$\partial_t u = D\Delta u - \kappa(u - u_{\text{amb}}) \text{ in } \Omega, \quad -D\nabla u \cdot n = h(u - u_h)\mathbf{1}_{\Gamma_h(t)} \text{ on } \partial\Omega. \tag{12}$$

The total moisture $U(t) = \int_\Omega u$ obeys

$$\frac{dU}{dt} = -\kappa \int_\Omega (u - u_{\text{amb}})\, dx - \int_{\Gamma_h(t)} h\,(u - u_h)\, dS. \tag{13}$$

For each handling event m, let ΔU_m be the net change, with $|\Delta U_m| \le \delta$. After M events, $\sum_{m=1}^{M} \Delta\, U_m = O(M\delta)$, which shows linear accumulation.

The surface micro-geometry $h_s(x, t)$ may evolve as

$$\partial_t h_s = -\alpha\phi(x, t) + \eta(x, t), \tag{14}$$

where ϕ encodes frictional contact and η encodes microscopic noise. A seminorm $\| h_s \|_{\mathcal{H}}$ then drifts under repeated events, producing measurable microstate divergence from the initial state.

9.2. Model H, Quantisation Error Accumulation in Digital Signal Processing

A signal $s[n]$ quantised to $\hat{s}[n] = Q(s[n])$ yields the error $e[n] = \hat{s}[n] - s[n]$, with $\mathbb{E}\big[e[n]\big] = 0$ and $\mathrm{Var}(e[n]) = \sigma_q^2$ under standard assumptions. [13] For a linear time-invariant system with impulse response $h[k]$, the output error variance is

$$\mathrm{Var}(y_e[n]) = \sigma_q^2 \sum_k h\,[k]^2. \tag{15}$$

In deep pipelines or feedback systems, small per-sample errors accumulate into significant degradation of the signal-to-noise ratio, which is a discrete-time instance of infinitesimal accumulation.

10. Linking Macro Pairs and Micro Infinitesimals

At the macro level, parity and pair symmetries impose constraints and definiteness, and selection rules forbid transitions. At the micro level, small symmetry-breaking terms such as the CP violation δ, sub-resolution losses such as ΔU_m, or quantisation errors such as $e[n]$Slowly bypass constraints and accumulate. The synthesis is that the world can be pair-structured at the level of laws and symmetries, yet asymmetric in historical outcome, owing to integrated micro-biases.

11. Philosophical Integration, Identity, Symmetry, and Impermanence

11.1. Identity as Equivalence Under Symmetry and Scale

Let identity-for-a-purpose be equivalence under a coarse-graining that respects symmetries, such as parity, and resolutions, such as instrument limits. Then the following holds.

- At coarse scale, parity-invariant macrostates are the same.
- At the fine scale, infinitesimals register persistent change.
- Over time, integrated micro-perturbations transform macro-identity through decay, selection, and asymmetry.

11.2. Interpretive Corollary

Evenness provides grammar, and oddness provides narrative. What the standard part collapses at an instant, history integrates over epochs. Thus impermanence is the integral of infinitesimals, and the arrow of time is written in differences too small to matter locally yet decisive globally.

12. Related Work and Contextual Notes

Symmetry and selection rules are standard in quantum mechanics and spectroscopy. Parity conservation in electromagnetic and strong interactions, and its violation in weak interactions, are established results. [1, 2] Baryogenesis and the Sakharov conditions are canonical in cosmology, and CP violation within the Standard Model is insufficient to account for the observed asymmetry, which motivates physics beyond the Standard Model. [3, 4] Entanglement, stabiliser codes, and symmetry-protected subspaces underpin modern quantum information. [8, 9, 10] Nonstandard analysis provides a rigorous infinitesimal framework compatible with standard calculus via the transfer principle and the standard-part map. [5, 6] Graph bipartition, frustration, and Ising models connect parity obstructions to complex phase behaviour. [11, 12, 14]

13. Objections and Replies

Objection 1. Even denominators imply definite decimals. This is incorrect in base 10, since termination depends on prime-factor compatibility with the prime set $\{2,5\}$ of the base. The thesis survives as a symmetry metaphor, in which compatibility yields definiteness and incompatibility yields periodic complexity.

Objection 2. The string $0.000\ldots1$ *is a real number.* It is not a real number. One should instead use the hyperreals, in which nonzero infinitesimals exist and are collapsed by the standard-part map, reconciling $0.\overline{9} = 1$ with the intuition that small differences can matter. [5, 6]

Objection 3. Parity always holds. Weak interactions violate parity and also CP. [1, 2] Such controlled violations are precisely how asymmetry, for instance, the excess of matter over antimatter, arises from pair-structured laws.

14. Conclusion

We developed a rigorous and integrative account in which pair symmetries, including parity, bipartitions, and entanglement, structure balance and definiteness, while small differences, including infinitesimals, biases, and noise, accumulate into macroscopic asymmetry. Decimal behaviour, correctly analysed, becomes a vivid metaphor for symmetry compatibility versus obstruction. Multiple models across physics and computation concretely exhibit the small-difference engine of imbalance, reconciling a world of pairs with a history of asymmetry. The framework suggests that quantitative work on parity-protected dynamics and on the integration of sub-resolution bias may be a productive direction for further study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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Appendix 1. Proofs and Technical Lemmas

Lemma 1.1 *(Gronwall-type bound).* If $y'(t) \leq a(t)y(t) + b(t)$ with $a, b \geq 0$, then

$$y(T) \leq y(0)\, e^{\int_0^T a(s)\, ds} + \int_0^T b\,(s)\, e^{\int_s^T a(u)\, du}\, ds. \quad (16)$$

Lemma 1.2 *(Variance accumulation).* If $\{\Delta_n\}$ are independent with $\mathbb{E}[\Delta_n] = 0$ and $\mathrm{Var}(\Delta_n) = \sigma_n^2$, then

$$\mathrm{Var}(\textstyle\sum_{n=1}^{N} \Delta_n) = \sum_{n=1}^{N} \sigma_n^2. \quad (17)$$

Appendix 2. Decimal Period, Multiplicative Order, and the Carmichael Function

Let p/q be in lowest terms with $\gcd.(q, b) = 1$. The period length divides $\lambda(q)$, the Carmichael function, which is the exponent of the group $(\mathbb{Z}/q\mathbb{Z})^{\times}$. In particular, $\mathrm{ord}_q(b) \mid \varphi(q)$ by the Euler theorem, sharpened to $\mathrm{ord}_q(b) \mid \lambda(q)$. [7]

Appendix 3. Hyperreal Snapshot

In ${}^*\mathbb{R}$, an infinitesimal $\varepsilon \neq 0$ satisfies $0 < |\varepsilon| < 1/n$ for every standard n. The standard-part map sends each finite hyperreal to its unique real neighbour. Equality in $\mathbb{R}$ corresponds to equality of standard parts, and infinitesimal differences are small but nonzero micro-variations that are ignored by $\mathbb{R}$. [5, 6]

Appendix 4. Symmetry-Adapted Bases

If $\Pi^2 = I$, then $P_{\pm} = (I \pm \Pi)/2$ are projectors onto the even and odd subspaces. Any operator commuting with Π is block-diagonal in a basis adapted to $P_{\pm}$, which simplifies spectral analysis and clarifies selection rules.