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Original Article

A New Framework of Neutrosophic Picture Fuzzy Sets and Topological Space

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Abstract - *In this paper, the concept of the Neutrosophic Picture Fuzzy Sets (NPFS) is introduced by combining the ideas of the neutrosophic sets, fuzzy sets, and picture fuzzy sets, which can be applied in more complex uncertain situations than the fuzzy set and picture fuzzy set. The mathematical model of NPFS and their topological structure, namely, Neutrosophic Picture Fuzzy Topological Spaces (NPFTS), are presented. The study highlights the main properties of NPFS and NPFTS and provides the basic operations with NPFSs. The operations have been applied for solving decision-making problems with incomplete, consistent and conflicting information as well as for other purposes connected with NPFSs and NPFTSs. The paper includes illustrative examples of NPFS, NPFTS and their operations.*

Keywords - *Neutrosophic Picture Fuzzy Sets, Neutrosophic Picture Fuzzy Topological Spaces, Neutrosophic Picture Fuzzy Pre, Semi, Regular, b,α and β Open Sets.*

1. Introduction

Many real-world situations are ill-defined, vague, and involve some degree of uncertainty in decision-making processes, pattern recognition, medical diagnosis, data mining, information systems, etc. Sets, however, are not adequate to express such situations. Zadeh [10] (1965) proposed fuzzy sets to handle these problems. A membership value in the interval [0,1] is associated with each element. Fuzzy sets provide a flexible mathematical framework, but they handle only the degree of membership and do not explicitly account for the degree of non-membership. Let $\mathcal{X}_{\mathcal{P}}$ be a non-empty set. A fuzzy set $\mathcal{J}_1^*$ $\mathcal{X}_{\mathcal{P}}$ can be described in the form $\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x) >/x \in \mathcal{X}_{\mathcal{P}}\}$ where the function $\mu_{\mathcal{J}_1^*}$: $\mathcal{X}_{\mathcal{P}} \rightarrow [0,1]$ is called the membership function and $\mu_{\mathcal{J}_1^*}(x)$ denotes the degree to which $x \in \mathcal{J}_1^*$ and $0 \le \mu_{\mathcal{J}_1^*}(x) \le 1$ for each $x \in \mathcal{X}_{\mathcal{P}}$ C.L.Chang [3] introduced the structure of fuzzy topology as an application of fuzzy sets to general topology.

To overcome this restriction, Atanassov [1] (1986) developed Intuitionistic Fuzzy Sets (IFS), which assign both membership and non-membership degrees to each element, subject to a natural constraint. This advancement allowed a more comprehensive representation of incomplete and hesitant information. An Intuitionistic Fuzzy Set (IFS) $\mathcal{J}_1^*$ on $\mathcal{X}_{\mathcal{P}}$ can be represented as $\mathcal{J}_1^* = \{\langle x, \mu_{\mathcal{J}_1^*}(x), \nu_{\mathcal{J}_1^*}(x)\rangle : x \in \mathcal{X}_{\mathcal{P}}\}$ where $\mu_{\mathcal{J}_1^*}(x)$: $\mathcal{X}_{\mathcal{P}} \rightarrow [0,1]$ and $\nu_{\mathcal{J}_1^*}(x)$: $\mathcal{X}_{\mathcal{P}} \rightarrow [0,1]$ represent the membership (namely μA(x)) and the non-membership (namely $\nu_{\mathcal{J}_1^*}(x)$) degree of each element $x \in \mathcal{J}_1^*$. For every $x \in \mathcal{X}_{\mathcal{P}}$ These values satisfy $0 \le \mu_{\mathcal{J}_1^*}(x) + \nu_{\mathcal{J}_1^*}(x) \le 1$, and the collection of all intuitionistic fuzzy sets defined on $\mathcal{X}_{\mathcal{P}}$ is denoted IFS($\mathcal{X}_{\mathcal{P}}$). Intuitionistic fuzzy topology, introduced by Coker [4] (1997), generalizes the notion of open sets under an intuitionistic fuzzy environment.

Subsequently, Smarandache introduced neutrosophic sets, a further generalization that incorporates three independent components—truth, indeterminacy, and falsehood—to model highly uncertain, inconsistent, and indeterminate information encountered in real applications. Let $\mathcal{X}_{\mathcal{P}}$ be a fixed set which is non-empty. A Neutrosophic set $\mathcal{J}_1^*$is an object of the form $\mathcal{J}_1^*$= $\{\langle x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x)\rangle : x \in \mathcal{X}_{\mathcal{P}}\}$ where $\mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x)$ denote the degrees of membership, indeterminacy and non-membership function, respectively, of the element x in $\mathcal{J}_1^*$. These values lie in the interval in $]^-0,1^+[$ on $\mathcal{X}_{\mathcal{P}}$. Inspired by the works of Smarandache [5], A.A.Salama et ali[8] developed and investigated the notions of Neutrosophic crisp set and Neutrosophic crisp topological spaces.

Motivated by this gap, we introduce the concept of Neutrosophic Picture Fuzzy Sets (NPFS), which integrates the expressive power of neutrosophic logic with the structural richness of picture fuzzy sets. Neutrosophic Picture Fuzzy sets are characterized by a YES (membership), ABSTAIN (indeterminacy), NO (non-membership) and REFUSAL (refusal membership) function which assigns to each object a value ranging between 0 and 1. After that, using Neutrosophic Picture Fuzzy sets, we introduced Neutrosophic Picture Fuzzy topological spaces, and then some Neutrosophic Picture Fuzzy closed sets, Neutrosophic Picture Fuzzy continuity, maps and Homeomorphism are discussed with suitable examples.

2. Preliminaries

In this section, we introduce concepts of Neutrosophic Picture Fuzzy set and Neutrosophic Picture Fuzzy topological spaces.

Neutrosophic Picture Fuzzy set (i) In a Democracy country voting and elective process happens. For example, a Member of parliament elected in a constituency, that constituency have total of 15,20,728 votes. In the year 2019 total vote polling percentage is 55.26%. The winner in the election(membership) got 4,47,075 votes. The losers(non-membership) in the election got votes 3,07,680, and NOTA(None of the Above) (indeterminacy) votes received 85,747. The number of voters in the election was 8,40,502. This is a situation of the real status of the election. In my observation is winner received vote is less than the voters who did not vote. This is not a great process for electing a leader. Refusal member's great spoiler of the voting and electing process of a leader. The government and the Election commission give more important for raising the polling percentage in elections.

Last seven months, corona virus spreading world level all countries and lakes level people death happened. WHO and the Indian government take necessary steps to prevent the people from corona virus. Stay safe by taking some simple precautions, such as physical distancing, wearing a mask, cleaning your hands, and coughing into a bent elbow or tissue. My concept of view corona negative people are considered as Membership, Corona-positive people are considered as (non-membership). Other people are called (indeterminacy). An asymptomatic person considers as Refusal membership because these people did not have any symptoms of Covid-19. These kinds of people unknowingly spreading virus every day.

3. Neutrosophic Picture Fuzzy set

"Definition 3.1 [Neutrosophic Picture Fuzzy set]:

Let $\mathcal{X}_{\mathcal{P}}$ be a non-empty fixed set. A Neutrosophic Picture Fuzzy set $\mathcal{J}_1^*$ is an object having the form.

$$\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x), \rho_{\mathcal{J}_1^*}(x) >: x \in \mathcal{X}_{\mathcal{P}}\}, (\forall x \in \mathcal{X}_{\mathcal{P}})\ \&\ \left(\mu_{\mathcal{J}_1^*}(x) + \sigma_{\mathcal{J}_1^*}(x) + \gamma_{\mathcal{J}_1^*}(x) + \rho_{\mathcal{J}_1^*}(x) \leq 1\right)$$

$\mu_{\mathcal{J}_1^*}(x) \in [0,1]$ - the degree of membership function, $\sigma_{\mathcal{J}_1^*}(x) \in [0,1]$ - degree indeterminacy, $\gamma_{\mathcal{J}_1^*}(x) \in [0,1]$ - the degree of non-membership function, $\rho_{\mathcal{J}_1^*}(x) \in [0,1]$- the degree of refusal membership function."

3.1.1. Operations of Neutrosophic Picture Fuzzy sets

"Definition 3.1.1:

Neutrosophic Picture Fuzzy set $\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x), \rho_{\mathcal{J}_1^*}(x) >: x \in \mathcal{X}_{\mathcal{P}}\}$, on $\mathcal{X}_{\mathcal{P}}$ and $\forall x \in \mathcal{X}_{\mathcal{P}}$ then the complement of $\mathcal{J}_1^*$ is ${\mathcal{J}_1^*}^{C} = \{< x, \gamma_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \mu_{\mathcal{J}_1^*}(x), \rho_{\mathcal{J}_1^*}(x) >: x \in \mathcal{X}_{\mathcal{P}}\}$."

"Definition 3.1.2:

Let $\mathcal{J}_1^*$ and $\mathcal{J}_2^*$ are two Neutrosophic Picture Fuzzy sets, $\forall x \in \mathcal{X}_{\mathcal{P}}$

$\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x), \rho_{\mathcal{J}_1^*}(x) >: x \in \mathcal{X}\}$

$\mathcal{J}_2^* = \{< x, \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_2^*}(x), \gamma_{\mathcal{J}_2^*}(x), \rho_{\mathcal{J}_2^*}(x) >: x \in \mathcal{X}\}$

Then $\mathcal{J}_1^* \subseteq \mathcal{J}_2^* \Leftrightarrow \mu_{\mathcal{J}_1^*}(x) \leq \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_1^*}(x) \leq \sigma_{\mathcal{J}_2^*}(x),\ \gamma_{\mathcal{J}_1^*}(x) \geq \gamma_{\mathcal{J}_2^*}(x)\ \& \rho_{\mathcal{J}_1^*}(x) \geq \rho_{\mathcal{J}_2^*}(x)$"

"Definition 3.1.3:

Let $\mathcal{X}_{\mathcal{P}}$ be a non-empty set, and Let $\mathcal{J}_1^*$ and $\mathcal{J}_2^*$ be two Neutrosophic Picture Fuzzy sets are

$\mathcal{J}_1^* = \{< x, \mu_{\mathcal{J}_1^*}(x), \sigma_{\mathcal{J}_1^*}(x), \gamma_{\mathcal{J}_1^*}(x), \rho_{\mathcal{J}_1^*}(x) >: x \in \mathcal{X}_{\mathcal{P}}\}$,

$\mathcal{J}_2^* = \{< x, \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_2^*}(x), \gamma_{\mathcal{J}_2^*}(x), \rho_{\mathcal{J}_2^*}(x) >: x \in \mathcal{X}_{\mathcal{P}}\}$

Then
1. $\mathcal{J}_1^* \cap \mathcal{J}_2^* = \{< x, \mu_{\mathcal{J}_1^*}(x) \cap \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_1^*}(x) \cap \sigma_{\mathcal{J}_2^*}(x), \gamma_{\mathcal{J}_1^*}(x) \cup \gamma_{\mathcal{J}_2^*}(x), \rho_{\mathcal{J}_1^*}(x) \cup \rho_{\mathcal{J}_2^*}(x) >: x \in \mathcal{X}_{\mathcal{P}}\}$
2. $\mathcal{J}_1^* \cup \mathcal{J}_2^* = \{< x, \mu_{\mathcal{J}_1^*}(x) \cup \mu_{\mathcal{J}_2^*}(x), \sigma_{\mathcal{J}_1^*}(x) \cup \sigma_{\mathcal{J}_2^*}(x), \gamma_{\mathcal{J}_1^*}(x) \cap \gamma_{\mathcal{J}_2^*}(x), \rho_{\mathcal{J}_1^*}(x) \cap \rho_{\mathcal{J}_2^*}(x) >: x \in \mathcal{X}_{\mathcal{P}}\}$"

3.1.2. Neutrosophic Picture Fuzzy Topological Spaces

Definition 3.1.4:
Let $\mathcal{X}_{\mathcal{P}}$ be a non-empty set and τ_N be the collection of Neutrosophic Picture Fuzzy subsets of $\mathcal{X}_{\mathcal{P}}$ satisfying the following properties:
1. $0_N, 1_N \in \tau_N$
2. $T_1 \cap T_2 \in \tau_N$ for any $T_1, T_2 \in \tau_N$
3. $\cup T_i \in \tau_N$ for every $\{T_i: i \in j\} \subseteq \tau_N$

Then the space $(\mathcal{X}_{\mathcal{P}}, \tau_N)$ is called a Neutrosophic Picture Fuzzy topological space(N-T-S).
The element of τ_N are called NPFCS (Neutrosophic Picture Fuzzy open set) and its complement is NPFCS(Neutrosophic Picture Fuzzy closed set).

Example 3.1:
Let $\mathcal{X}_{\mathcal{P}}$ =$\{j_1\}$ and $\forall x \in \mathcal{X}_{\mathcal{P}}$

$$\mathcal{J}_1^* = \langle x, \frac{0.3(\mu)}{j_1} + \frac{0.3(\sigma)}{j_1} + \frac{0.25(\gamma)}{j_1} + \frac{0.15(\rho)}{j_1} \rangle$$
$$\mathcal{J}_2^* = \langle x, \frac{0.25(\mu)}{j_1} + \frac{0.3(\sigma)}{j_1} + \frac{0.3(\gamma)}{j_1} + \frac{0.15(\rho)}{j_1} \rangle$$
$$\mathcal{J}_3^* = \langle x, \frac{0.3(\mu)}{j_1} + \frac{0.35(\sigma)}{j_1} + \frac{0.25(\gamma)}{j_1} + \frac{0.1(\rho)}{j_1} \rangle$$
$$\mathcal{J}_4^* = \langle x, \frac{0.25(\mu)}{j_1} + \frac{0.35(\sigma)}{j_1} + \frac{0.3(\gamma)}{j_1} + \frac{0.1(\rho)}{j_1} \rangle$$

Then the collection $\tau_N = \{0_N, \mathcal{J}_1^*, \mathcal{J}_2^*, \mathcal{J}_3^*, \mathcal{J}_4^*, 1_N\}$ iis called a Neutrosophic Picture Fuzzy topological space on $\mathcal{X}_{\mathcal{P}}$.

4. Neutrosophic Picture Fuzzy b-open Sets

In this section, we introduce $NP^{\mathcal{F}}$ b-open sets

Definition 4.1:
A $NP^{\mathcal{F}}$ set $\mathcal{J}_1^*$a $(NP^{\mathcal{F}})TS\mathcal{X}_{\mathcal{P}}$ is called
(i) $NP^{\mathcal{F}}$ Pre-open $(NP^{\mathcal{F}\ Pre\mathcal{OS}})$ iff $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))$.
(ii) $NP^{\mathcal{F}}$ Semi-open. $(NP^{\mathcal{F}Semi-\ \mathcal{OS}})$ iff $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$
(iii) $NP^{\mathcal{F}}$ Regular-open $(NP^{\mathcal{F}\ R\mathcal{OS}})$ iff $\mathcal{J}_1^*$=$NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))$
(iv) $NP^{\mathcal{F}}$ α-open$NP^{\mathcal{F}\ \alpha\mathcal{OS}}$iff $\mathcal{J}_1^* \sqsubseteq (NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)))$
(v) $NP^{\mathcal{F}}$ β-open $NP^{\mathcal{F}\beta\ \mathcal{OS}}$ iff $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)))$

Definition 4.2:
A $NP^{\mathcal{F}}$set $\mathcal{J}_1^*$in a $(NP^{\mathcal{F}})TS\ \mathcal{X}_{\mathcal{P}}$ is called $NP^{\mathcal{F}}$ b-open $NP^{\mathcal{F}b\mathcal{OS}}$ Iff $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcup$
$NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$ $NP^{\mathcal{F}\ b\mathcal{CS}}$ iff $NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcap NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqsubseteq \mathcal{J}_1^*$

Remark 4.1:
For a $NP^{\mathcal{F}}$ set $\mathcal{J}_1^*$ in a $(NP^{\mathcal{F}})TS\mathcal{X}_{\mathcal{P}}$,
(i) $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}b\mathcal{OS}}$ set iff $\mathcal{J}_1^{*c}$ is $NP^{\mathcal{F}\ b\mathcal{CS}}$
(ii) $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}\ b\mathcal{CS}}$set iff $\mathcal{J}_1^{*c}$is $NP^{\mathcal{F}b\mathcal{OS}}$.

Theorem 4.1:
Let $\mathcal{J}_1^*$ and $\mathcal{J}_2^*$are $NP^{\mathcal{F}b\mathcal{OS}s}$. Then
(i)$\mathcal{J}_1^* \sqcup \mathcal{J}_2^*$ is $NP^{\mathcal{F}b\mathcal{OS}s}$
(ii)$\mathcal{J}_1^* \sqcap \mathcal{J}_2^*$ is $NP^{\mathcal{F}\ b\mathcal{CS}s}$

Proof. (i) Since $\mathcal{J}_1^*$ and $\mathcal{J}_2^*$are $NP^{\mathcal{F}b\mathcal{OS}s}$ by the definition of an area $NP^{\mathcal{F}b\mathcal{OS}s}$- open set, we have $\mathcal{J}_1^* \sqsubseteq Cl(Int(\mathcal{J}_1^*)) \sqcup Int(Cl(\mathcal{J}_1^*))$ $\mathcal{J}_2^* \sqsubseteq Cl(Int(\mathcal{J}_2^*)) \sqcup Int(Cl(\mathcal{J}_2^*))$then $\mathcal{J}_1^* \sqcup \mathcal{J}_2^* \sqsubseteq [Cl(Int(\mathcal{J}_1^*)) \sqcup Int(Cl(\mathcal{J}_1^*))] \sqcup [Cl(Int(\mathcal{J}_2^*)) \sqcup Int(Cl(\mathcal{J}_2^*))]$.
Hence, $\mathcal{J}_1^* \sqcup \mathcal{J}_2^* \sqsubseteq Cl(Int(\mathcal{J}_1^* \sqcup \mathcal{J}_2^*)) \sqcup Int(Cl(\mathcal{J}_1^* \sqcup \mathcal{J}_2^*))$.Therefore, by the definition of a $NP^{\mathcal{F}}$ b-open set, $\mathcal{J}_1^* \sqcup \mathcal{J}_2^*$ is an $NP^{\mathcal{F}b\mathcal{OS}s}$.

(ii) By the definition of an$NP^{\mathcal{F}\ b\mathcal{CS}s}$, a set is closed and only if its complement is $NP^{\mathcal{F}}$ b-open set. By De Morgan's law, $(\mathcal{J}_1^*\sqcap\mathcal{J}_2^*)^C=(\mathcal{J}_1^*)^C\sqcup(\mathcal{J}_2^*)^C$.Since $\mathcal{J}_1^*$ and $\mathcal{J}_2^*$are $NP^{\mathcal{F}}$ b-open set, their complements are $NP^{\mathcal{F}}$ b-closed set. Using the corresponding closure property for $NP^{\mathcal{F}}$ b-closed sets, $(\mathcal{J}_1^*)^C\sqcup(\mathcal{J}_2^*)^C$is $NP^{\mathcal{F}}$ b-open set. Consequently,$(\mathcal{J}_1^*\sqcap\mathcal{J}_2^*)^C$ is $NP^{\mathcal{F}}$ b-open. Hence, by the definition of $NP^{\mathcal{F}}$ b-closed sets, $\mathcal{J}_1^*\sqcap\mathcal{J}_2^*$ is an $NP^{\mathcal{F}}$ b-closed sets.

Definition 4.3:
Let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}}$ in a $(NP^{\mathcal{F}})$TS $\mathcal{X}_{\mathcal{P}}$. Then $NP^{\mathcal{F}}$ b-Closure is $NP^{\mathcal{F}bCl}(\mathcal{J}_1^*)=\sqcap\{\mathcal{J}_2^*:\mathcal{J}_2^*$ is a $NP^{\mathcal{F}\ b\mathcal{CS}}$ of $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_1^*\sqsubseteq\mathcal{J}_2^*\}$.

Remark 4.2:
Let $\mathcal{X}_{\mathcal{P}}$ is $(NP^{\mathcal{F}})$TS then following results valid for $NP^{\mathcal{F}\ bCl}$of a set.
(i) $NP^{\mathcal{F}\ bCl}(0_{\mathcal{P}})=(0_{\mathcal{P}})$
(ii) $NP^{\mathcal{F}\ bCl}(1_{\mathcal{P}})=(1_{\mathcal{P}})$
(iii) $NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)$ is a $NP^{\mathcal{F}\ b\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$
(iv) $NP^{\mathcal{F}\ bCl}(NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*))=NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)$

Theorem 4.2:
In a $(NP^{\mathcal{F}})$TS$\mathcal{X}_{\mathcal{P}}$ The next relations are true.
(i)$(NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*\sqcup NP^{\mathcal{F}\ bCl}(\mathcal{J}_2^*)\sqsubseteq NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*\sqcup\mathcal{J}_2^*)$.
(ii) $NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*\sqcap\mathcal{J}_2^*)\sqsubseteq NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)\sqcap NP^{\mathcal{F}\ bCl}(\mathcal{J}_2^*)$
Proof.
(i) $\mathcal{J}_1^*\sqsubseteq\mathcal{J}_1^*\sqcup\mathcal{J}_2^*$or $\mathcal{J}_2^*\sqsubseteq\mathcal{J}_1^*\sqcup\mathcal{J}_2^*\Rightarrow NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*\sqcup\mathcal{J}_2^*)$ or $NP^{\mathcal{F}\ bCl}(\mathcal{J}_2^*)\sqsubseteq NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*\sqcup\mathcal{J}_2^*)$.
Then $NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)\sqcup NP^{\mathcal{F}\ bCl}(\mathcal{J}_2^*)\sqsubseteq NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*\sqcup\mathcal{J}_2^*)$.
(ii) Similar to (i).

Theorem 4.3:
In a $(NP^{\mathcal{F}})$TS $\mathcal{X}_{\mathcal{P}}$, $\mathcal{J}_1^*$ is $NP^{\mathcal{F}b\mathcal{CS}}$iff $\mathcal{J}_1^*=NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)$.
Proof.
Suppose $\mathcal{J}_1^*NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)$.Then $\mathcal{J}_1^*=\sqcap\{\mathcal{J}_3^*:\mathcal{J}_3^*$ is $NP^{\mathcal{F}\ b\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_1^*\sqsubseteq\mathcal{J}_3^*\}$,
$\Rightarrow\mathcal{J}_1^*\sqcap\{\mathcal{J}_3^*:\mathcal{J}_3^*$ is $NP^{\mathcal{F}\ b\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_1^*\sqsubseteq\mathcal{J}_3^*\}$
Hence $\mathcal{J}_1^*$is $NP^{\mathcal{F}\ b\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$. Conversely, suppose $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}\ b\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$. We take $\mathcal{J}_1^*\sqsubseteq\mathcal{J}_1^*$ and $\mathcal{J}_1^*$is $NP^{\mathcal{F}\ b\mathcal{CS}}$. Therefore $\mathcal{J}_1^*\in\{\mathcal{J}_3^*:\mathcal{J}_3^*$ is $NP^{\mathcal{F}\ b\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$ and$\mathcal{J}_1^*\sqsubseteq\mathcal{J}_3^*\}$, which implies, $\mathcal{J}_1^*=\sqcap\{\mathcal{J}_3^*:\mathcal{J}_3^*$ is $NP^{\mathcal{F}\ b\mathcal{CS}}$in $NP^{\mathcal{F}\mathcal{X}}$ and $\mathcal{J}_1^*\sqsubseteq\mathcal{J}_3^*\}=NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)$. Thus $\mathcal{J}_1^*=NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)$.

Remark 4.3:
If $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}}$ subset in $(NP^{\mathcal{F}})$TS $\mathcal{X}_{\mathcal{P}}$, then $NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)$ is smallest $NP^{\mathcal{F}\ b\mathcal{CS}}$ set containing $\mathcal{J}_1^*$.
Thus $NP^{\mathcal{F}\ bCl}(\mathcal{J}_1^*)\sqsupseteq\mathcal{J}_1^*\sqcup(NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)\sqcap NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$.

Definition 4.4:
Let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}}$ set in $\mathcal{X}_{\mathcal{P}}$. Then its $NP^{\mathcal{F}}$b interior is $NP^{\mathcal{F}bInt}(\mathcal{J}_1^*)=\sqcup\{\mathcal{J}_3^*:\mathcal{J}_3^*$ is a $NP^{\mathcal{F}b\mathcal{OS}}$ of $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_3^*\sqsubseteq\mathcal{J}_1^*\}$.

Remark 4.4:
In a $(NP^{\mathcal{F}})$TS$\mathcal{X}_{\mathcal{P}}$ The following is valid for $NP^{\mathcal{F}bInt}$
(i) $NP^{\mathcal{F}bInt}(0_{\mathcal{P}})=(0_{\mathcal{P}})$.
(ii)$NP^{\mathcal{F}bInt}(1_{\mathcal{P}})=(1_{\mathcal{P}})$.
(iii) $NP^{\mathcal{F}bInt}(\mathcal{J}_1^*)$ is a $NP^{\mathcal{F}b\mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$.
(iv) $NP^{\mathcal{F}bInt}(NP^{\mathcal{F}bInt}(\mathcal{J}_1^*))=NP^{\mathcal{F}bInt}(\mathcal{J}_1^*)$

Theorem 4.4:
In a $(NP^{\mathcal{F}})$TS$\mathcal{X}_{\mathcal{P}}$, $\mathcal{J}_1^*$is $NP^{\mathcal{F}b\mathcal{OS}}$ iff $\mathcal{J}_1^*=NP^{\mathcal{F}bInt}(\mathcal{J}_1^*)$.
Proof.
Suppose$\mathcal{J}_1^*=NP^{\mathcal{F}bInt}(\mathcal{J}_1^*)$.Then $\mathcal{J}_1^*=\sqcup\{\mathcal{J}_2^*:\mathcal{J}_2^*$is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_2^*\sqsubseteq\mathcal{J}_1^*\}$, $\Rightarrow\mathcal{J}_1^*\in\{\mathcal{J}_2^*:\mathcal{J}_2^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_2^*\sqsubseteq\mathcal{J}_1^*\}$.
Hence $\mathcal{J}_1^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.
Conversely, suppose $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}b\mathcal{OS}}$ in$\mathcal{X}_{\mathcal{P}}$. Consider $\mathcal{J}_1^*\sqsubseteq\mathcal{J}_1^*$ and $\mathcal{J}_1^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$.
Therefore $\mathcal{J}_1^*\in\{\mathcal{J}_2^*:\mathcal{J}_2^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_2^*\sqsubseteq\mathcal{J}_1^*\}$, $\Rightarrow\mathcal{J}_1^*=\sqcup\{\mathcal{J}_2^*:\mathcal{J}_2^*$is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_1^*\sqsubseteq\mathcal{J}_2^*\}=NP^{\mathcal{F}bInt}(\mathcal{J}_1^*)$.

Thus $\mathcal{J}_1^* = \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}\mathcal{J}_1^*)$.

Remark 4.5:
If $\mathcal{J}_1^*$ is a $\mathrm{NP}^{\mathcal{F}}$ subset in $(\mathrm{NP}^{\mathcal{F}})\mathrm{TS}\mathcal{X}_{\mathcal{P}}$, then $\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^*)$ is the largest $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}} \sqsubseteq \mathcal{J}_1^*$. Thus $\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^*) \sqsubseteq$ $\mathcal{J}_1^* \sqcap \mathrm{NP}^{\mathcal{F}\mathrm{int}}\ (\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_1^*) \sqcup \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*))$.

Theorem 4.5:
In a $(\mathrm{NP}^{\mathcal{F}})\mathrm{TS}\mathcal{X}_{\mathcal{P}}$ The following relations hold
(i). $\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^* \sqcup \mathcal{J}_2^*) \sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^*) \sqcup \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_2^*)$.
(ii).$\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^* \sqcap \mathcal{J}_2^*) \sqsupseteq \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^*) \sqcap \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_2^*)$.
Proof.
(i). Consider $\mathcal{J}_1^*$ and $\mathcal{J}_2^*$ belongs to $\mathrm{NP}^{\mathcal{F}}$sets in $\mathcal{X}_{\mathcal{P}}$, then $\mathcal{J}_1^* \sqsubseteq \mathcal{J}_1^* \sqcup \mathcal{J}_2^*$ or $\mathcal{J}_*^* \sqsubseteq \mathcal{J}_1^* \sqcup \mathcal{J}_2^* \Rightarrow \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^*) \sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^* \sqcup \mathcal{J}_*^*)$ or $\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_2^*)\mathcal{P} \sqsubseteq \mathcal{G}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^* \sqcup \mathcal{J}_2^*)$. Therefore $\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^* \sqcup \mathcal{J}_2^*) \sqsupseteq \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^*) \sqcap \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_2^*)$.
(ii). Similar to (i).

Theorem 4.6:
Let $\mathcal{J}_1^*$ is any $\mathrm{NP}^{\mathcal{F}}$ in a $(\mathrm{NP}^{\mathcal{F}})\mathrm{TS}\mathcal{X}_{\mathcal{P}}$. Then
(i) $\mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathcal{J}_1^{*\mathrm{C}}) = \left(\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^*)\right)^{\mathrm{C}}$
(ii) $\mathrm{NP}^{\mathcal{F}\ \mathrm{bInt}}(\mathcal{J}_1^{*\mathrm{C}})\ ) = \left(\mathrm{NP}^{\mathcal{F}\mathrm{bcl}}(\mathcal{J}_1^*)\right)^{\mathrm{C}}$
Proof:
(i) Let $\mathcal{J}_2^*$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$ such that $\mathcal{J}_2^*$. Then for a $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}\left(\mathcal{J}_1^{*\mathrm{C}}\right) = \mathcal{J}_3^*$, $\left(\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^*)\right) = \{\mathcal{J}_2^* : \mathcal{J}_2^*$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_2^* \sqsubseteq \mathcal{J}_1^*\}^{\mathrm{C}} =$ $\{\mathcal{J}_2^{*\mathrm{C}} : \mathcal{J}_2^*$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_2^* \sqsubseteq \mathcal{J}_1^*\} = \{\mathcal{J}_3^* : \mathcal{J}_3^*$ is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_1^{*\mathrm{C}} \sqsubseteq \mathcal{J}_3^*\} = \mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathcal{J}_1^{*\mathrm{C}})$. Therefore $\mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathcal{J}_1^{*\mathrm{C}}) =$ $\left(\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^*)\right)^{\mathrm{C}}$.
(ii) Let $\mathcal{J}_2^*$ is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$ such that $\mathcal{J}_1^* \sqsubseteq \mathcal{J}_2^*$. Then for a $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}\mathcal{J}_4^* \sqsubseteq \mathcal{J}_1^{*\mathrm{C}}$, $(\mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathcal{J}_1^*))^{\mathrm{C}} = \{\mathcal{J}_2^* : \mathcal{J}_2^*$ is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_1^* \sqsubseteq \mathcal{J}_2^*\}^{\mathrm{C}}$ $= \{\mathcal{J}_2^{*\mathrm{C}} : \mathcal{J}_2^*$ is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{J}_1^* \sqsubseteq \mathcal{J}_2^*\} = \{\mathcal{J}_2^* : \mathcal{J}_4^*$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$and $\mathcal{J}_4^* \sqsubseteq \mathcal{J}_1^{*\mathrm{C}}\} = \mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^{*\mathrm{C}})$.
Therefore $\mathrm{NP}^{\mathcal{F}\mathrm{bInt}}(\mathcal{J}_1^{*\mathrm{C}}) = \left(\mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathcal{J}_1^*)\right)^{\mathrm{C}}$.

Theorem 4.7:
In a $(\mathrm{NP}^{\mathcal{F}})\mathrm{TS}\mathcal{X}_{\mathcal{P}}$, every $\mathrm{NP}^{\mathcal{F}\mathrm{Pre}\mathcal{OS}}$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$.
Proof. Let $\mathcal{J}_1^*$ is $\mathrm{NP}^{\mathcal{F}\mathrm{pre}\mathcal{OS}}$ in a $(\mathrm{NP}^{\mathcal{F}})\mathrm{TS}\ \mathcal{X}_{\mathcal{P}}$.
Then $\mathcal{J}_1^* \sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_1^*))$ which implies $\mathcal{J}_1^* \sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_1^*)) \sqcup \mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*) \sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{int}}\left(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_1^*)\right) \sqcup$ $\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}\ (\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*))$. Thus $\mathcal{J}_1^*$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$.

Example 4.1:
Let $\mathcal{X}_{\mathcal{P}} = \{\mathrm{j}_1, \mathrm{j}_2\}$ and $\forall \mathrm{x} \in \mathcal{X}_{\mathcal{P}}$ the $\mathrm{NP}^{\mathcal{F}}$sets are follows
$$\mathcal{J}_1^* = \langle \mathrm{x}, \frac{0.2}{\mathrm{j}_1} + \frac{0.25}{\mathrm{j}_1} + \frac{0.35}{\mathrm{j}_1} + \frac{0.2}{\mathrm{j}_1}, \frac{0.25}{\mathrm{j}_2} + \frac{0.25}{\mathrm{j}_2} + \frac{0.35}{\mathrm{j}_2} + \frac{0.15}{\mathrm{j}_2} \rangle$$
$$\mathcal{J}_2^* = \langle \mathrm{x}, \frac{0.35}{\mathrm{j}_1} + \frac{0.25}{\mathrm{j}_1} + \frac{0.25}{\mathrm{j}_1} + \frac{0.15}{\mathrm{j}_1}, \frac{0.25}{\mathrm{j}_2} + \frac{0.25}{\mathrm{j}_2} + \frac{0.25}{\mathrm{j}_2} + \frac{0.25}{\mathrm{j}_2} \rangle$$
$\tau_{\mathrm{NPF}} = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \mathcal{J}_1^*\}$ is be $(\mathrm{NP}^{\mathcal{F}})\mathrm{T}$ on $\mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_2^*$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}.}$ but not $\mathrm{NP}^{\mathcal{F}\mathrm{pre}\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Theorem 4.8:
In a $(\mathrm{NP}^{\mathcal{F}})\mathrm{TS}\ \mathcal{X}_{\mathcal{P}}$ every $\mathrm{NP}^{\mathcal{F}\mathrm{Semi}\mathcal{OS}}$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$ set.
Proof. Let $\mathcal{J}_1^*$is $\mathrm{NP}^{\mathcal{F}\mathrm{Semi}\mathcal{OS}}$ in a $(\mathrm{NP}^{\mathcal{F}})\mathrm{TS}\ \mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_1^* \sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*))$
$\Rightarrow \mathcal{J}_1^* \sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_2^*) \sqcup \mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*) \sqcup \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*)) \sqcup \mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_1^*))$.
Thus $\mathcal{J}_1^*$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Example 4.2:
Let $\mathcal{X}_{\mathcal{P}} = \{\mathrm{j}_1, \mathrm{j}_2\}$ the $\mathrm{NP}^{\mathcal{F}}$sets are follows
$$\mathcal{J}_1^* = \langle \mathrm{x}, \frac{0.15}{\mathrm{j}_1} + \frac{0.25}{\mathrm{j}_1} + \frac{0.4}{\mathrm{j}_1} + \frac{0.2}{\mathrm{j}_1}, \frac{0.2}{\mathrm{j}_2} + \frac{0.25}{\mathrm{j}_2} + \frac{0.4}{\mathrm{j}_2} + \frac{0.15}{\mathrm{j}_2} \rangle$$

$$\mathcal{J}_2^* = \langle x, \frac{0.1}{j_1} + \frac{0.25}{j_1} + \frac{0.45}{j_1} + \frac{0.2}{j_1}, \frac{0.1}{j_1} + \frac{0.25}{j_1} + \frac{0.45}{j_1} + \frac{0.2}{j_1} \rangle$$

$\tau_{NPF} = \{0_{NP^{\mathcal{F}}}, 1_{NP^{\mathcal{F}}}, \mathcal{J}_1^*\}$ is be $(NP^{\mathcal{F}})T$ on $\mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_2^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$ but not $NP^{\mathcal{F}\text{Semi }\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Relation other $NP^{\mathcal{F}\ \mathcal{CS}}$ sets based on the above results shown in figure-2.

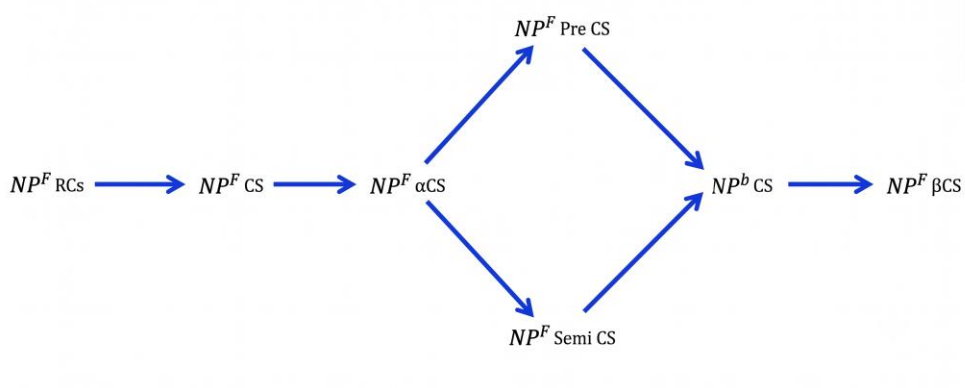

Fig. 2 Comparison diagram

Theorem 4.9:
In a $(NP^{\mathcal{F}})TS\mathcal{X}_{\mathcal{P}}$, every $NP^{\mathcal{F}b\mathcal{OS}}$ is $NP^{\mathcal{F}\beta\mathcal{OS}}$.
Proof. Let $NP^{\mathcal{F}}\mathcal{J}_1^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$
Then $\mathcal{J}_1^* \sqsubset NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqcup NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqsubset NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqcup NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)))$.
Thus $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\beta\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Example 4.3:
Let $\mathcal{X}_{\mathcal{P}} = \{j_1, j_*\}$ the $NP^{\mathcal{F}}$ sets are
$$\mathcal{J}_1^* = \langle x, \frac{0.2}{j_1} + \frac{0.25}{j_1} + \frac{0.4}{j_1} + \frac{0.15}{j_1}, \frac{0.25}{j_2} + \frac{0.25}{j_2} + \frac{0.35}{j_2} + \frac{0.15}{j_2} \rangle,$$
$$\mathcal{J}_2^* = \langle x, \frac{0.3}{j_1} + \frac{0.25}{j_1} + \frac{0.25}{j_1} + \frac{0.2}{j_1}, \frac{0}{j_2} + \frac{0}{j_2} + \frac{1}{j_2} + \frac{0}{j_2} \rangle$$
$\tau_{NPF} = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \mathcal{J}_1^*\}$ is be $(NP^{\mathcal{F}})T$ on $\mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_2^*$ is $NP^{\mathcal{F}\beta\mathcal{OS}}$ but not $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.
Figure 2 represents the comparison of $NP^{\mathcal{F}b\mathcal{OS}}$ with other existing $NP^{\mathcal{F}\mathcal{OS}}$ Moreover, the reverse implication is not true.

Theorem 4.10:
Let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in a $(NP^{\mathcal{F}})TS\mathcal{X}_{\mathcal{P}}$.
(i) If $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}R\mathcal{CS}}$, then $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}\text{ Pre}\mathcal{OS}}$
(ii) If $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}R\ \mathcal{OS}}$, then $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}\text{Semi }\mathcal{OS}}$.
Proof.
Let $\mathcal{J}_1$ be $NP^{\mathcal{F}b\ \mathcal{OS}}$, then $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*) \sqcup NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)$.
(i) Now let $\mathcal{J}_1^*$ be $NP^{\mathcal{F}R\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_1^* = NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$.
Then $\mathcal{J}_1^* \sqsubseteq \mathcal{J}_1^* \sqcup (NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))$, which gives $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))$.
Thus, $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\text{ Pre}\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.
(ii) Let $\mathcal{J}_1$ be $NP^{\mathcal{F}R\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$. Therefore $\mathcal{J}_1^* = NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))$.
Then $\mathcal{J}_1^* = NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$, $\Rightarrow \mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$.
Hence $\mathcal{J}_1^*$s $NP^{\mathcal{F}\text{Semi}\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Theorem 4.11:
Let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\ b\mathcal{CS}}$ in a $(NP^{\mathcal{F}})TS\ \mathcal{X}_{\mathcal{P}}$.
(i) If $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}R\ \mathcal{CS}}$, then $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}\text{Semi }\mathcal{CS}}$
(ii) If $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}R\ \mathcal{OS}}$, then $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}\text{ Pre}\mathcal{CS}}$
Proof. Let $\mathcal{J}_1^*$ be $NP^{\mathcal{F}\ b\mathcal{CS}}$ then $NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcap NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqsubseteq \mathcal{J}_1^*$.
(i) Now let $\mathcal{J}_1^*$ be $NP^{\mathcal{F}R\ \mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_1^* = NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$.

$\therefore$, $NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcap \mathcal{J}_1^* \sqsubseteq \mathcal{J}_1^*$, $\Rightarrow NP^{\mathcal{F}int}(\mathcal{J}_1^*) \sqcap \mathcal{J}_1^*$.
Hence $\mathcal{J}_1^*$ is $NP^{\mathcal{F}Semi\,\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$.

(ii) Let $\mathcal{J}_1^*$ be $NP^{\mathcal{F}R\,\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_1^* = NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))$.
Then $\mathcal{J}_1^* NP^{\mathcal{F}\,bCl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*) \sqcap \mathcal{J}_1^*$, which implies $NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqsubseteq \mathcal{J}_1^*$.
Hence $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\,Pre\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Theorem 4.12:
For any $NP^{\mathcal{F}b\mathcal{OS}}$ set $\mathcal{J}_1^*$ in $(NP^{\mathcal{F}})TS\ \mathcal{X}_{\mathcal{P}}$, $NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}R\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$.
Proof.
Let $\mathcal{J}_1^*$ be $NP^{\mathcal{F}b\mathcal{OS}}$ in $NP^{\mathcal{F}\mathcal{X}}$. Then $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcup NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$.
Therefore $NP^{\mathcal{F}Cl}(\mathcal{J}_1^*) \sqsubseteq NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcup NP^{\mathcal{F}Cl} NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqsubseteq NP^{\mathcal{F}Cl}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))$.
Also $NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))) \sqsubseteq NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)$. Hence $(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) = NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)$. Therefore $NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}R\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Theorem 4.13:
For any $NP^{\mathcal{F}\,b\mathcal{CS}}$ set $\mathcal{J}_1^*$ in a $(NP^{\mathcal{F}})TS\mathcal{X}_{\mathcal{P}}$. $NP^{\mathcal{F}int}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}R\,\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.
Proof. Let $\mathcal{J}_1^*$ be $NP^{\mathcal{F}\,b\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$. Then $NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcap NP^{\mathcal{F}Cl} NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqsubseteq \mathcal{J}_1^*$, that is
$NP^{\mathcal{F}int}(NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))) \sqcap NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))) \sqsubseteq NP^{\mathcal{F}int}(\mathcal{J}_1^*)$. Thus $NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))) \sqsubseteq NP^{\mathcal{F}int}(\mathcal{J}_1^*)$.
But $NP^{\mathcal{F}int}(\mathcal{J}_1^*) \sqsubseteq NP^{\mathcal{F}int}(NP^{\mathcal{F}int} NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)))$. Therefore $NP^{\mathcal{F}int}(\mathcal{J}_1^*) = NP^{\mathcal{F}int}(NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)))$. Thus $NP^{\mathcal{F}int}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}R\,\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Theorem 4.14:
Let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in a $(NP^{\mathcal{F}})TS\ \mathcal{X}_{\mathcal{P}}$ such that $NP^{\mathcal{F}int}(\mathcal{J}_1^*) = 0_{\mathcal{P}}$ then $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\,Pre\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.
Proof. Let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in a $(NP^{\mathcal{F}})TS\mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcup NP^{\mathcal{F}Cl} NP^{\mathcal{F}int}(\mathcal{J}_1^*)$
If $NP^{\mathcal{F}int}(\mathcal{J}_1^*) = 0_{\mathcal{P}}$ then $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*))$.
Hence $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}Pre\,\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Theorem 4.15:
Let $\mathcal{J}_1^*$ is $\mathcal{NP}F^{\mathcal{F}b\mathcal{OS}.}$ in a $(NP^{\mathcal{F}})TS\ \mathcal{X}_{\mathcal{P}}$, such that $NP^{\mathcal{F}Cl}(\mathcal{J}_1^*) = 0_{\mathcal{P}}$ then $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}Semi\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.
Proof. Let $\mathcal{J}_1^*$s $NP^{\mathcal{F}b\mathcal{OS}.}$ in a $(NP^{\mathcal{F}})TS\mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcup NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$.
If $NP^{\mathcal{F}Cl}(\mathcal{J}_1^*) = 0_{\mathcal{P}}$, then $\mathcal{J}_1^* \sqsubseteq NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*))$. Hence $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}Semi\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Theorem 4.16:
Let $\mathcal{J}_1^*$ is a subset of a $(NP^{\mathcal{F}})TS\ \mathcal{X}_{\mathcal{P}}$. Then,
(i) $NP^{\mathcal{F}\,bCl}(\mathcal{J}_1^*) \sqsubseteq (NP^{\mathcal{F}SemiCl}(\mathcal{J}_1^*) \sqcap (NP^{\mathcal{F}Pre\,Cl}(\mathcal{J}_1^*))$
(ii) $NP^{\mathcal{F}bInt}(\mathcal{J}_1^*) \sqsupseteq (NP^{\mathcal{F}Semi\,Int}(\mathcal{J}_1^*)) \sqcup (NP^{\mathcal{F}Pre\,Int}(\mathcal{J}_1^*))$.
Proof.
(i) $NP^{\mathcal{F}SemiCl}(\mathcal{J}_1^*) \sqcap NP^{\mathcal{F}\,PreCl}(\mathcal{J}_1^*)\ (\mathcal{J}_1^* \sqcup NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^* \sqcap (\mathcal{J}_1^* \sqcup NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)))$
$= \mathcal{J}_1^* \sqcup NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*) \sqcap NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*) = NP^{\mathcal{F}bCl}(\mathcal{J}_1^*)$.
Therefore $NP^{\mathcal{F}\,bCl}(\mathcal{J}_1^*) \sqsubseteq (NP^{\mathcal{F}SemiCl}(\mathcal{J}_1^*)) \sqcap (NP^{\mathcal{F}\,PreCl}(\mathcal{J}_1^*))$
(ii) $NP^{\mathcal{F}SemiInt}(\mathcal{J}_1^*) \sqcup NP^{\mathcal{F}bInt}(\mathcal{J}_1^*) \sqsubseteq \mathcal{J}_1^* \sqcap NP^{\mathcal{F}Cl} NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqcap (\mathcal{J}_1^* \sqcap NP^{\mathcal{F}int}$
$NP^{\mathcal{F}Cl}(\mathcal{J}_1^*) = \mathcal{J}_1^* \sqcap NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqcup NP^{\mathcal{F}int} NP^{\mathcal{F}Cl}(\mathcal{J}_1^*) = NP^{\mathcal{F}bInt}(\mathcal{J}_1^*)$
Therefore $NP^{\mathcal{F}bInt}(\mathcal{J}_1^*) \sqsupseteq (NP^{\mathcal{F}SemiInt}(\mathcal{J}_1^*) \sqcup (NP^{\mathcal{F}\,Preint}(\mathcal{J}_1^*)$.

Theorem 4.17:
Let $\mathcal{X}_{\mathcal{P}}$ is $(NP^{\mathcal{F}})TS$. If $\mathcal{J}_1^*$ is a $NP^{\mathcal{F}\,\mathcal{OS}}$ and $\mathcal{J}_2^*$ is a $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$. Then $\mathcal{J}_1^* \sqcap \mathcal{J}_2^*$ is a $NP^{\mathcal{F}b\mathcal{OS}}$ set in $\mathcal{X}_{\mathcal{P}}$.
Proof. Let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\,\mathcal{OS}}\mathcal{X}_{\mathcal{P}}$. Then, $\mathcal{J}_1^*$ is $NP^{\mathcal{F}Semi\mathcal{OS}}$ and $NP^{\mathcal{F}\,Pre\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.
Hence for $NP^{\mathcal{F}b\mathcal{OS}}$ set $\mathcal{J}_2^*$ in $\mathcal{X}_{\mathcal{P}}$, $\mathcal{J}_1^* \sqcap \mathcal{J}_2^* \sqsubseteq \mathcal{J}_1^* \sqcap NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_2^*)) \sqcup NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_2^*))$
$= (\mathcal{J}_1^*) \sqcap NP^{\mathcal{F}Cl}(NP^{\mathcal{F}int}(\mathcal{J}_2^*)) \sqcup (\mathcal{J}_1^*) \sqcap NP^{\mathcal{F}int} NP^{\mathcal{F}cl}(\mathcal{J}_2^*)$
$= NP^{\mathcal{F}Cl} NP^{\mathcal{F}int}(\mathcal{J}_1^*) \sqcap NP^{\mathcal{F}Cl} NP^{\mathcal{F}int}(\mathcal{J}_2^*) \sqcup NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^*)) \sqcap NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_2^*))$
$= NP^{\mathcal{F}Cl\,(}NP^{\mathcal{F}int}(\mathcal{J}_1^* \sqcap \mathcal{J}_2^*)) \sqcup NP^{\mathcal{F}int}(NP^{\mathcal{F}Cl}(\mathcal{J}_1^* \sqcap \mathcal{J}_2^*))$.
Therefore $\mathcal{J}_1^* \sqcap \mathcal{J}_2^*$ is a $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.

Theorem 4.18:
Let $\mathcal{X}_{\mathcal{P}}$ be $(\mathrm{NP}^{\mathcal{F}})$TS. If $\mathcal{J}_1^*$ is a $\mathrm{NP}^{\mathcal{F}\alpha\mathcal{OS}}$ and $\mathcal{J}_2^*$is a in $\mathcal{X}_{\mathcal{P}}$, then $\mathcal{J}_1^*\sqcap\mathcal{J}_2^*$is a $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$.
Proof. Let $\mathcal{J}_1^*$ is $\mathrm{NP}^{\mathcal{F}\alpha\mathcal{OS}}$ and $\mathcal{J}_2^*$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}s}$in $\mathcal{J}_1^*$.
Then $\mathcal{J}_1^*\sqcap\mathcal{J}_2^* \sqsubseteq (\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*)))\sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_2^*)) \sqcup \mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_2^*))$
$= \mathrm{NP}^{\mathcal{F}\mathrm{int}}\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*)))\sqcap \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_2^*))\sqcup \mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*)))\sqcap\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_2^*))$
$\sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*)) \sqcap\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_2^*) \sqcup \mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_1^*))\sqcap\mathrm{NP}^{\mathcal{F}\mathrm{int}}\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_2^*)$
$= \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathcal{J}_1^*\sqcap\mathcal{J}_2^*)) \sqcup\mathrm{NP}^{\mathcal{F}\mathrm{int}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_1^*\sqcap\mathcal{J}_2^*)$.
Therefore $\mathcal{J}_1^*\sqcap\mathcal{J}_2^*$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}\cdot}$ in $\mathcal{J}_1^*$.

5. Neutrosophic picture fuzzy b-Continuous

Definition 5.1:
A maps $\mathrm{f}_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{\mathrm{NPF}}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{\mathrm{NPF}})$ is named as $\mathrm{NP}^{\mathcal{F}\mathrm{b}}$continuous ($\mathrm{NP}^{\mathcal{F}\mathrm{b}}$- continuity), iff$_{\mathcal{P}}{}^{-1}$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$, for each $\mathrm{NP}^{\mathcal{F}\mathcal{OS}}$ $\mathcal{J}_1^*$ in $\mathcal{Y}_{\mathcal{P}}$.

Definition 5.2:
A maps $\mathrm{f}_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{\mathrm{NPF}}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{\mathrm{NPF}})$ is named as $\mathrm{NP}^{\mathcal{F}\mathrm{b}^*}$Continuous($\mathrm{NP}^{\mathcal{F}\mathrm{b}^*}$-continuity), if $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}\cdot}$ in $\mathcal{X}_{\mathcal{P}}$, for each $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}\cdot}$set $\mathcal{J}_1^*$ in $\mathcal{Y}_{\mathcal{P}}$.

Theorem 5.1:
A maps $\mathrm{f}_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{\mathrm{NPF}}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{\mathrm{NPF}})$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}^*}$continuity, iff $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)$ is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$, for each $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}\mathcal{J}_1^*$in $\mathcal{Y}_{\mathcal{P}}$.
Proof. Suppose the maps $\mathrm{f}_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{\mathrm{NPF}}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{\mathrm{NPF}})$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}^*}$- continuity. Let $\mathcal{J}_1^*$is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$in $\mathcal{Y}_{\mathcal{P}}$. Then ${\mathcal{J}_1^*}^{\mathrm{C}}$is $\mathrm{NP}^{\mathcal{F}\ \mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$. Since $\mathrm{f}_{\mathcal{P}}$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}^*}$- continuity, $\mathrm{f}_{\mathcal{P}}{}^{-1}({\mathcal{J}_1^*}^{\mathrm{C}})$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$. Now $\mathrm{f}_{\mathcal{P}}{}^{-1}({\mathcal{J}_1^*}^{\mathrm{C}})=\left(\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)\right)^{\mathrm{C}}$. Thus $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)$is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}in\mathcal{X}_{\mathcal{P}}$. Therefore $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)$is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$, for each $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$set $\mathcal{J}_1^*$ in $\mathcal{Y}_{\mathcal{P}}$.
Conversely, now assume $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}\cdot}$in $\mathcal{X}_{\mathcal{P}}$, for each $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}\mathcal{J}_1^*$ in $\mathcal{Y}_{\mathcal{P}}$. By hypothesis $\mathrm{f}_{\mathcal{P}}{}^{-1}({\mathcal{J}_1^*}^{\mathrm{C}})= ((\mathrm{f}_{\mathcal{P}})^{-1})^{\mathrm{C}}$is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$, which denotes $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$.
Thus the maps $\mathrm{f}_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{\mathrm{NPF}}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{\mathrm{NPF}})$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}^*}$-continuity.

Theorem 5.2:
If the maps $\mathrm{f}_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{\mathrm{NPF}}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{\mathrm{NPF}})$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}^*}$-continuity then $\mathrm{f}_{\mathcal{P}}$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}}$- continuity.
Proof. Let $\mathcal{J}_1$is $\mathrm{NP}^{\mathcal{F}\ \mathcal{OS}}$set in $\mathcal{Y}_{\mathcal{P}}$, then $\mathcal{J}_1^*$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Since $\mathrm{f}_{\mathcal{P}}$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}^*}$continuity, $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}\cdot}$in $\mathcal{X}_{\mathcal{P}}$. Thus $\mathrm{f}_{\mathcal{P}}$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}}$- continuity.

Theorem 5.3:
If the maps $\mathrm{f}_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{\mathrm{NPF}}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{\mathrm{NPF}})$ is $(\mathrm{NP}^{\mathcal{F}\ \mathrm{Pre}})$-continuity then $\mathrm{f}_{\mathcal{P}}$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}}$-continuity.
Proof. Let $\mathcal{J}_1^*$ be $\mathrm{NP}^{\mathcal{F}\ \mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Since $\mathrm{f}_{\mathcal{P}}$ is $(\mathrm{NP}^{\mathcal{F}\ \mathrm{Pre}})$ - continuity $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)$is a $\mathrm{NP}^{\mathcal{F}\mathrm{Pre}\mathcal{OS}}$ inf$_{\mathcal{P}}\mathcal{X}_{\mathcal{P}}$ and hence $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*)$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}\mathcal{OS}}$. (by Theorem 3.16). Thus $\mathrm{f}_{\mathcal{P}}$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}}$- continuity.

Theorem 5.4:
Let $\mathrm{f}_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{\mathrm{NPF}}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{\mathrm{NPF}})$. At that point, the accompanying explanations are identical.
(i) $\mathrm{f}_{\mathcal{P}}$is $\mathrm{NP}^{\mathcal{F}\mathrm{b}}$-continuity
(ii) $\mathrm{f}_{\mathcal{P}}(\mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathcal{J}_1^*)) \sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{f}_{\mathcal{P}}(\mathcal{J}_1^*))$, for every $\mathrm{NP}^{\mathcal{F}}$set $\mathcal{J}_1^*$ in $\mathcal{X}_{\mathcal{P}}$.
Proof.
(i) ⇒(ii) Let $\mathcal{J}_1^*$is $(\mathrm{NP}^{\mathcal{F}})$in $\mathcal{X}_{\mathcal{P}}$. Then $\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{f}_{\mathcal{P}}(\mathcal{J}_1^*)$ is $\mathrm{NP}^{\mathcal{F}\mathcal{CS}}$. Since $\mathrm{f}_{\mathcal{P}}$ is $\mathrm{NP}^{\mathcal{F}\mathrm{b}}$-TS, $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{f}_{\mathcal{P}}(\mathcal{J}_1^*))$ is $\mathrm{NP}^{\mathcal{F}\ \mathrm{b}\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$ and so $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}\mathrm{f}_{\mathcal{P}}(\mathcal{J}_1^*)) = \mathrm{NP}^{\mathcal{F}\mathrm{bCl}}\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{f}_{\mathcal{P}}(\mathcal{J}_1^*))$.
Since $\mathcal{J}_1^*\sqsubseteq\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathrm{f}_{\mathcal{P}}(\mathcal{J}_1^*))$we have $\mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathcal{J}_1^*) \sqsubseteq \mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathrm{f}_{\mathcal{P}}(\mathcal{J}_1^*)) \sqsubseteq \mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathrm{f}_{\mathcal{P}}{}^{-1}\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{f}_{\mathcal{P}}(\mathcal{J}_1^*)))$
$= \mathrm{f}_{\mathcal{P}}{}^{-1}\ (\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{f}_{\mathcal{P}}(\mathcal{J}_1^*))$.
Hence $\mathrm{f}_{\mathcal{P}}\ ((\mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathcal{J}_1^*)))\sqsubseteq \mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{f}_{\mathcal{P}}\ (\mathcal{J}_1^*))$.
(ii)⇒(i) Let $\mathcal{J}_2^*$is $\mathrm{NP}^{\mathcal{F}\ \mathcal{CS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Let $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_2^*)$,then by (ii),
$\mathrm{f}_{\mathcal{P}}(\mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_2^*)))\sqsubseteq\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{f}_{\mathcal{P}}(\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_2^*))),\Rightarrow \mathrm{NP}^{\mathcal{F}\mathrm{bCl}}(\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathrm{f}_{\mathcal{P}}))$
$\sqsubseteq \mathrm{f}_{\mathcal{P}}{}^{-1}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathrm{f}_{\mathcal{P}}(\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_1^*))) \sqsubseteq\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathrm{NP}^{\mathcal{F}\mathrm{Cl}}(\mathcal{J}_2^*)\sqsubseteq\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_2^*)$.
But $\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_2^*)\sqsubseteq\mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}\left(\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_2^*)\right)$,so$\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_2^*) = \mathrm{NP}^{\mathcal{F}\ \mathrm{bCl}}(\mathrm{f}_{\mathcal{P}}{}^{-1}(\mathcal{J}_2^*)$.

Hence $f_{\mathcal{P}}^{-1}(\mathcal{J}_2^*)$ is $NP^{\mathcal{F}\ b\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$. Therefore $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$- continuity.

Results:
Let $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ The following are equivalent.
(i) $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b^*}$- continuity
(ii) $f_{\mathcal{P}}(NP^{\mathcal{F}\ bCl}(\mathcal{J}_1) \sqsubseteq NP^{\mathcal{F}Cl}(f_{\mathcal{P}}(\mathcal{J}_1^*))$, for every $(NP^{\mathcal{F}})$set $\mathcal{J}_1^*$in $\mathcal{X}_{\mathcal{P}}$.

Theorem 5.5:
Let $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ The following are equivalent.
(i) $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$- continuity.
(ii) for every $NP^{\mathcal{F}\ \mathcal{CS}}$ set $\mathcal{J}_1^*$in $\mathcal{Y}_{\mathcal{P}}$,$f_{\mathcal{P}}$ is $NP^{\mathcal{F}\ b\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$.
Proof. (i) ⇒ (ii) Suppose $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$is $NP^{\mathcal{F}b}$-continuity.
Let $\mathcal{J}_1^*$be $NP^{\mathcal{F}\ \mathcal{CS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Then ${\mathcal{J}_1^*}^C$ is $NP^{\mathcal{F}\ \mathcal{OS}}$in $\mathcal{Y}_{\mathcal{P}}$. Since $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$- continuity, $f_{\mathcal{P}}^{-1}({\mathcal{J}_1^*}^C)$ is $NP^{\mathcal{F}b\mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$. But $f_{\mathcal{P}}^{-1}\left({\mathcal{J}_1^*}^C\right) = \left(f_{\mathcal{P}}^{-1}(\mathcal{J}_1^*)\right)^C$Thus $f_{\mathcal{P}}^{-1}(\mathcal{J}_1^*)$ $NP^{\mathcal{F}\ b\mathcal{CS}}$in $\mathcal{X}_{\mathcal{P}}$.
(ii) ⇒ (i) Let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\ b\mathcal{CS}}$in $\mathcal{Y}_{\mathcal{P}}$, that implies $\overline{\mathcal{J}_1^*}$is $NP^{\mathcal{F}b\mathcal{OS}}$in $\mathcal{Y}_{\mathcal{P}}$.
Thus $f_{\mathcal{P}}^{-1}(\overline{\mathcal{J}_1^*})$ is $NP^{\mathcal{F}b\mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$, which implies $\overline{(f_{\mathcal{P}})^{-1}(\mathcal{J}_1^*)}$ is $NP^{\mathcal{F}b\mathcal{OS}}$in $\mathcal{X}_{\mathcal{P}}$. For every $NP^{\mathcal{F}\ \mathcal{OS}}$set $\overline{\mathcal{J}_1^*}$in $\mathcal{Y}_{\mathcal{P}}$, there $\mathcal{X}_{\mathcal{P}}$is a $NP^{\mathcal{F}b\mathcal{OS}}$ $\overline{(f_{\mathcal{P}})^{-1}(\mathcal{J}_1^*)}$in $\mathcal{X}_{\mathcal{P}}$. Hence $f_{\mathcal{P}}$is $NP^{\mathcal{F}b}$- continuity.

Theorem 5.6:
If $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$is $NP^{\mathcal{F}b}$- continuity and $g_{\mathcal{P}}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ is$(NP^{\mathcal{F}})$ -continuity,then $(g \circ f)_{\mathcal{P}}$: $(\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ is $NP^{\mathcal{F}b}$ continuity.
Proof. Let $\mathcal{J}_2^*$be $NP^{\mathcal{F}\ \mathcal{OS}}$ of $\mathcal{Z}_{\mathcal{P}}$. Since $g_{\mathcal{P}}$is $(NP^{\mathcal{F}})$-continuity, $g_{\mathcal{P}}^{-1}(\mathcal{J}_*^*)$is $NP^{\mathcal{F}\ \mathcal{OS}}$in $\mathcal{Y}_{\mathcal{P}}$. Now $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$continuity,$f_{\mathcal{P}}^{-1}(g_{\mathcal{P}}^{-1}(\mathcal{J}_*^*))$is $NP^{\mathcal{F}b\mathcal{OS}}$ in$\mathcal{X}_{\mathcal{P}}$. Hence $(g \circ f)_{\mathcal{P}}^{-1}$ $(\mathcal{J}_2^*)$ is $NP^{\mathcal{F}b\mathcal{OS}.}$in $\mathcal{X}_{\mathcal{P}}$. Therefore$(f \circ g)_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$-continuity.

Theorem 5.7:
If $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ and $g_{\mathcal{P}}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ are both $NP^{\mathcal{F}b^*}$-continuity, then $(g \circ f)_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ is $NP^{\mathcal{F}b^*}$-continuity.
Proof. Let $\mathcal{J}_2^*$be$NP^{\mathcal{F}b\mathcal{OS}}$ in$\mathcal{Z}_{\mathcal{P}}$. Since $g_{\mathcal{P}}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$is $NP^{\mathcal{F}b^*}$-continuity,$g_{\mathcal{P}}^{-1}(\mathcal{J}_2^*)$ is $NP^{\mathcal{F}b\mathcal{OS}.}$in $\mathcal{Y}_{\mathcal{P}}$. Also since$f_{\mathcal{P}}: \mathcal{X}_{\mathcal{P}} \longrightarrow \mathcal{Y}_{\mathcal{P}}$ is $NP^{\mathcal{F}b^*}$-continuity; $f_{\mathcal{P}}^{-1}(g_{\mathcal{P}}^{-1}(\mathcal{J}_2^*))$ is $NP^{\mathcal{F}b\mathcal{OS}.}in\mathcal{X}_{\mathcal{P}}$.
Hence$(g \circ f)_{\mathcal{P}}^{-1}(\mathcal{J}_2^*)$is $NP^{\mathcal{F}b\mathcal{OS}.}$in $\mathcal{X}_{\mathcal{P}}$. Therefore $(g \circ f)_{\mathcal{P}}$is $NP^{\mathcal{F}b^*}$- continuity.

Theorem 5.8:
If $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ is $NP^{\mathcal{F}b^*}$- continuity and $g_{\mathcal{P}}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ is $(NP^{\mathcal{F}})$-continuity, then $(g \circ f)_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ is $NP^{\mathcal{F}b}$ continuity.
Proof. Let $\mathcal{J}_2^*$be $NP^{\mathcal{F}\ \mathcal{OS}}in\mathcal{Z}_{\mathcal{P}}$. Since $NP\mathcal{F}^{(g)}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$is $NP^{\mathcal{F}b}$-continuity, $g_{\mathcal{P}}^{-1}$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Since $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b^*}$- continuity, $f_{\mathcal{P}}^{-1}(g_{\mathcal{P}}^{-1}(\mathcal{J}_2^*))$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in$\mathcal{X}_{\mathcal{P}}$.
Hence $(g \circ f)_{\mathcal{P}}^{-1}(\mathcal{J}_2^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$. Therefore $(g \circ f)_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$-continuity.

6. Neutrosophic Picture Fuzzy b-Open and b-Closed Maps

Definition 6.1:
A maps $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ is named as $NP^{\mathcal{F}b\mathcal{OS}}$map) if the image of every $NP^{\mathcal{F}\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$.

Theorem 6.1:
A maps $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ is named as $NP^{\mathcal{F}\ b\mathcal{CS}}$ if the image of every $NP^{\mathcal{F}\ \mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$ is $NP^{\mathcal{F}b\mathcal{CS}}$ in $\mathcal{Y}_{\mathcal{P}}$.
Proof. Use 5.1.

Definition 6.2:
A maps $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ is named as $NP^{\mathcal{F}b^*\mathcal{OS}}$ if the image of every $NP^{\mathcal{F}b\mathcal{OS}.}$ in $\mathcal{X}_{\mathcal{P}}$ is $NP^{\mathcal{F}b\mathcal{OS}.}$ in $\mathcal{Y}_{\mathcal{P}}$.

Theorem 6.2:
A maps $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ is named as $NP^{\mathcal{F}b^*\mathcal{C}}$map iff $f_{\mathcal{P}}^{-1}(\mathcal{J}_1^*)NP^{\mathcal{F}\ b\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$for $NP^{\mathcal{F}\ b\mathcal{CS}}\mathcal{J}_1^*$ in $\mathcal{Y}_{\mathcal{P}}$.
Proof. by using 5.1.

Remark 6.1:
(i) Every $NP^{\mathcal{F}b^*\mathcal{O}}$ maps is $NP^{\mathcal{F}b^*\mathcal{O}}$ maps.(ii) Each of $NP^{\mathcal{F}Pre\mathcal{O}}$ and $NP^{\mathcal{F}Semi\mathcal{O}}$ maps is $NP^{\mathcal{F}b\mathcal{O}}$ maps.

Example 6.1:
Let $\mathcal{X}_{\mathcal{P}} = \{j_1, j_2\}$ the $NP^{\mathcal{F}}$ sets are

$$\mathcal{J}_1^* = \langle x, \frac{0.35}{j_1} + \frac{0.3}{j_1} + \frac{0.15}{j_1} + \frac{0.2}{j_1}, \frac{0.4}{j_2} + \frac{0.3}{j_2} + \frac{0.1}{j_2} + \frac{0.2}{j_2} \rangle$$

$$\mathcal{J}_2^* = \langle x, \frac{0.25}{j_1} + \frac{0.3}{j_1} + \frac{0.3}{j_1} + \frac{0.15}{j_1}, \frac{0.25}{j_2} + \frac{0.3}{j_2} + \frac{0.35}{j_2} + \frac{0.1}{j_2} \rangle$$

$$\mathcal{J}_3^* = \langle x, \frac{0.25}{j_1} + \frac{0.3}{j_1} + \frac{0.3}{j_1} + \frac{0.15}{j_1}, \frac{0}{j_2} + \frac{0}{j_2} + \frac{1}{j_2} + \frac{0}{j_2} \rangle$$

$$\mathcal{J}_4^* = \langle x, \frac{0.25}{j_1} + \frac{0.3}{j_1} + \frac{0.3}{j_1} + \frac{0.15}{j_1}, \frac{0.25}{j_2} + \frac{0.3}{j_2} + \frac{0.3}{j_2} + \frac{0.15}{j_2} \rangle$$

Let $\tau_{NPF} = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \mathcal{J}_1^*\}$, $\sigma_{NPF} = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \mathcal{J}_2^*\}$ $\gamma_{NPF} = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \mathcal{J}_3^*\}$ and $\mu_{NPF} = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \mathcal{J}_4^*\}$
The map $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{PGF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{PGF})$ in identity map is $NP^{\mathcal{F}b\mathcal{OS}}$. but not $NP^{\mathcal{F}P\mathcal{OS}}$ because $(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ but not $NP^{\mathcal{F}Pre\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$.
The map $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \mu_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \gamma_{NPF})$ in identity map is $NP^{\mathcal{F}\beta\mathcal{OS}}$ but not $NP^{\mathcal{F}b\mathcal{OS}}$ because $f_{\mathcal{P}}(\mathcal{J}_4^*)$ is $NP^{\mathcal{F}\beta\mathcal{OS}}$ but not $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$.
The map $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \mu_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ in identity map is $NP^{\mathcal{F}\beta\mathcal{OS}}$ but not $NP^{\mathcal{F}b\mathcal{OS}}$ because $f_{\mathcal{P}}(\mathcal{J}_4^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ but not $NP^{\mathcal{F}Pre\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$.

Theorem 6.3:
A maps $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ is $NP^{\mathcal{F}b\mathcal{OS}}$ iff $f_{\mathcal{P}}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)) \sqsubseteq NP^{\mathcal{F}bInt}(f_{\mathcal{P}}(\mathcal{J}_1^*))$, for every $NP^{\mathcal{F}}$ set $\mathcal{J}_1^*$ of $NP^{\mathcal{F}\mathcal{X}}$.
Proof.
Let $\mathcal{J}_1^* NP^{\mathcal{F}\,\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$, then $NP^{\mathcal{F}int}(\mathcal{J}_1^*) \sqsubseteq \mathcal{J}_1^*$ Since $f_{\mathcal{P}}$ is a $NP^{\mathcal{F}b\mathcal{O}}$ map $f_{\mathcal{P}}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$.
Hence $f_{\mathcal{P}}(NP^{\mathcal{F}int}(\mathcal{J}_1^*)) = NP^{\mathcal{F}int}(f_{\mathcal{P}} NP^{\mathcal{F}int}(J_1^*) \sqsubseteq NP^{\mathcal{F}bInt}(f_{\mathcal{P}}(\mathcal{J}_1^*)$.
Conversely, let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$. by hypothesis $f_{\mathcal{P}}(\mathcal{J}_1^*) = f_{\mathcal{P}}(NP^{\mathcal{F}int})(\mathcal{J}_1^*) \sqsubseteq NP^{\mathcal{F}bInt}(f_{\mathcal{P}}(\mathcal{J}_1^*)$)which implies $f_{\mathcal{P}}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Hence $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b\mathcal{OS}}$.

Theorem 6.4:
If $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ is $NP^{\mathcal{F}\mathcal{OS}}$ and $g_{\mathcal{P}}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ is $NP^{\mathcal{F}b\mathcal{OS}}$, then $(g \circ f)_{\mathcal{P}}$ is $NP^{\mathcal{F}b\mathcal{OS}}$.
Proof. Suppose $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\,\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$. Since $f_{\mathcal{P}}: \mathcal{X}_{\mathcal{P}} \longrightarrow \mathcal{Y}_{\mathcal{P}}$ is $NP^{\mathcal{F}\mathcal{O}}$ maps, $f_{\mathcal{P}}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}\,\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Now $g_{\mathcal{P}}: \mathcal{Y}_{\mathcal{P}} \longrightarrow \mathcal{Z}_{\mathcal{P}}$ is $NP^{\mathcal{F}b\mathcal{OS}}$, Hence $g_{\mathcal{P}}(f_{\mathcal{P}}(\mathcal{J}_1^*)) = (g \circ f)_{\mathcal{P}}(\mathcal{J}_1^*)$ $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Z}_{\mathcal{P}}$. Therefore $(g \circ f)_{\mathcal{P}}$ is $NP^{\mathcal{F}b\mathcal{OS}}$.

Theorem 6.5:
If $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ is $NP^{\mathcal{F}\,\mathcal{OS}}$ and $g_{\mathcal{P}}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ is $NP^{\mathcal{F}b^*\mathcal{OS}}$, then $(g \circ f)_{\mathcal{P}}$ is $NP^{\mathcal{F}b^*\mathcal{OS}}$.
Proof: By definition its proved.

Theorem 6.6:
Let $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ and $g_{\mathcal{P}}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ be two maps. If $f_{\mathcal{P}}$ is $f_{\mathcal{P}}$-continuity, onto and $(g \circ f)_{\mathcal{P}} NP^{\mathcal{F}\,b\mathcal{CS}}$, then $g_{\mathcal{P}}$ is $NP^{\mathcal{F}\,b\mathcal{CS}}$.
Proof.
Let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}\mathcal{CS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Since $f_{\mathcal{P}}$ is $f_{\mathcal{P}}$-continuity, then ${f_{\mathcal{P}}}^{-1}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}\,\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$. Now $(g \circ f)_{\mathcal{P}}$ is $NP^{\mathcal{F}\,b\mathcal{CS}}$ and $f_{\mathcal{P}}$ is onto so,$(g \circ f)_{\mathcal{P}}({f_{\mathcal{P}}}^{-1}(\mathcal{J}_1^*)) = g_{\mathcal{P}}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}\,b\mathcal{CS}}$ in $\mathcal{Z}_{\mathcal{P}}$. Hence $g_{\mathcal{P}}$ is $NP^{\mathcal{F}\,b\mathcal{CS}}$ maps.

Theorem 6.7:
Let $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ and $g_{\mathcal{P}}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ be two maps. If $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$- continuity, onto and $(g \circ f)_{\mathcal{P}}$ is $NP^{\mathcal{F}b^*\mathcal{CS}}$, then $g_{\mathcal{P}}$ is $NP^{\mathcal{F}\,b\mathcal{CS}}$.
Proof: Similar 5.6.

Theorem 6.8:
Let $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ and $g_{\mathcal{P}}: (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF}) \longrightarrow (\mathcal{Z}_{\mathcal{P}}, \rho_{NPF})$ be two maps. Iff is $NP^{\mathcal{F}b^*}$- continuity, onto and $(g \circ f)_{\mathcal{P}}$ is $NP^{\mathcal{F}\,\mathcal{CS}}$, then $g_{\mathcal{P}}$ is $NP^{\mathcal{F}b^*\mathcal{CS}}$.

Proof: Similar 6.6.

Definition 6.3:
A bijective map$f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$is called $NP^{\mathcal{F}b}$-homeomorphism $NP^{\mathcal{F}b}$- homeomorphism) if $f_{\mathcal{P}}$ and $f_{\mathcal{P}}{}^{-1}$are are $NP^{\mathcal{F}b}$ continuity.

Definition 6.4:
A bijective map $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ is called $NP^{\mathcal{F}b^*}$homeomorphism ($NP^{\mathcal{F}b^*}$- homeomorphism) if $f_{\mathcal{P}}$ and $f_{\mathcal{P}}{}^{-1}$are $NP^{\mathcal{F}b^*}$-continuity.

Remark 6.2:
(i) Every ($NP^{\mathcal{F}}$) - homeomorphism is $NP^{\mathcal{F}b}$- homeomorphism.
(ii) Every $NP^{\mathcal{F}b^*}$- homeomorphism is $NP^{\mathcal{F}b}$- homeomorphism.

Example 6.16.
Let $\mathcal{X}_{\mathcal{P}}$={$j_1$, j_2} the $NP^{\mathcal{F}}$sets are

$$\mathcal{J}_1^* = \langle x, \frac{0}{j_1} + \frac{0}{j_1} + \frac{1}{j_1} + \frac{0}{j_1}, \frac{0}{j_2} + \frac{0}{j_2} + \frac{1}{j_2} + \frac{0}{j_2} \rangle$$
$$\mathcal{J}_2^* = \langle x, \frac{0}{j_1} + \frac{0}{j_1} + \frac{1}{j_1} + \frac{0}{j_1}, \frac{0}{j_2} + \frac{0}{j_2} + \frac{1}{j_2} + \frac{0}{j_2} \rangle$$

is $NP^{\mathcal{F}b}$- homeomorphism but not ($NP^{\mathcal{F}}$) - homeomorphism, since $f_{\mathcal{P}}{}^{-1}: \mathcal{Y}_{\mathcal{P}} \longrightarrow \mathcal{X}_{\mathcal{P}}$is not ($NP^{\mathcal{F}}$) - continuity.

Example 6.17.
Let $\mathcal{X}_{\mathcal{P}}$={$j_1$, j_2} the $NP^{\mathcal{F}}$sets are

$$\mathcal{J}_1^* = \langle x, \frac{1}{j_1} + \frac{0}{j_1} + \frac{0}{j_1} + \frac{0}{j_1}, \frac{0}{j_2} + \frac{0}{j_2} + \frac{1}{j_2} + \frac{0}{j_2} \rangle$$
$$\mathcal{J}_2^* = \langle x, \frac{1}{j_1} + \frac{0}{j_1} + \frac{0}{j_1} + \frac{0}{j_1}, \frac{0}{j_2} + \frac{0}{j_2} + \frac{1}{j_2} + \frac{0}{j_2} \rangle$$

Let $\tau_{NPF} = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \mathcal{J}_1^*\}$,$\sigma_{NPF} = \{0_{\mathcal{P}}, 1_{\mathcal{P}}, \mathcal{J}_2^*\}$ Then the maps $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ defined by$f_{\mathcal{P}}(j_1) = j_2$and $f_{\mathcal{P}}(j_2) = j_1$is $NP^{\mathcal{F}b}$- homeomorphism but not $NP^{\mathcal{F}b^*}$- homeomorphism, since $f_{\mathcal{P}}^{-1} : \mathcal{Y}_{\mathcal{P}} \longrightarrow \mathcal{X}_{\mathcal{P}}$ is not $NP^{\mathcal{F}b^*}$- continuity.

Theorem 6.9:
Every $NP^{\mathcal{F}b^*}$- homeomorphism is $NP^{\mathcal{F}b}$- homeomorphism.
Proof.
Let $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$be $NP^{\mathcal{F}b^*}$- homeomorphism. Then $f_{\mathcal{P}}$and $f_{\mathcal{P}}$ are $NP^{\mathcal{F}b^*}$- continuity maps. By Theorem 4.4, $f_{\mathcal{P}}$ and $f_{\mathcal{P}}^{-1}$are $NP^{\mathcal{F}b}$-continuity. Hence $f_{\mathcal{P}}: \mathcal{X}_{\mathcal{P}} \longrightarrow \mathcal{Y}_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$ homeomorphism.

Theorem 6.10:
Let $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$is a bijective map. At that point, the accompanying explanations are identical
(i) $f_{\mathcal{P}}$is $NP^{\mathcal{F}b}$- homeomorphism
(ii) $f_{\mathcal{P}}$is $NP^{\mathcal{F}b}$-continuity and $NP^{\mathcal{F}b\mathcal{OS}}$
(iii) $f_{\mathcal{P}}$is $NP^{\mathcal{F}b}$-continuity and $NP^{\mathcal{F}\,b\mathcal{CS}}$.
Proof. (i) ⇒ (ii) Let $f_{\mathcal{P}}$ be $NP^{\mathcal{F}b}$-homeomorphism. Then $f_{\mathcal{P}}$and $f_{\mathcal{P}}^{-1}$are $NP^{\mathcal{F}b}$-continuity. To prove that $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b\mathcal{OS}}$. Let $\mathcal{J}_1^*$is $NP^{\mathcal{FOS}}$ in $\mathcal{X}_{\mathcal{P}}$. Then since $f_{\mathcal{P}}{}^{-1}: \mathcal{Y}_{\mathcal{P}} \longrightarrow \mathcal{X}_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$- continuity $\left(f_{\mathcal{P}}{}^{-1}\right)^{-1}(\mathcal{J}_1^*) = f_{\mathcal{P}}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Hence $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b\mathcal{OS}}$.
(ii) ⇒ (i) Let $f_{\mathcal{P}}$be $NP^{\mathcal{F}b}$continuity and $NP^{\mathcal{F}b\mathcal{OS}}$. To prove that $f_{\mathcal{P}}{}^{-1}$:$\mathcal{Y}_{\mathcal{P}} \longrightarrow \mathcal{X}_{\mathcal{P}}$is $NP^{\mathcal{F}b}$-continuity. Let $\mathcal{J}_1^*$is $NP^{\mathcal{FOS}}$ in $\mathcal{X}_{\mathcal{P}}$. Then $f_{\mathcal{P}}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$since $f_{\mathcal{P}}{}^{-1}NP^{\mathcal{F}b\mathcal{OS}}$, which implies$f_{\mathcal{P}}(\mathcal{J}_1^*)$= $\left(f_{\mathcal{P}}{}^{-1}\right)^{-1}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Therefore $f_{\mathcal{P}}{}^{-1}: \mathcal{Y}_{\mathcal{P}} \longrightarrow \mathcal{X}_{\mathcal{P}}$is $NP^{\mathcal{F}b}$- continuity. Hence $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b}$-homeomorphism.
(ii)⇒ (iii) Let $f_{\mathcal{P}}$be $NP^{\mathcal{F}b}$-continuity and $NP^{\mathcal{F}b\mathcal{OS}}$. To prove that $f_{\mathcal{P}}$ is $NP^{\mathcal{F}\,b\mathcal{CS}}$. Let $\mathcal{J}_2^*$is $NP^{\mathcal{F}\,\mathcal{CS}}$ in $\mathcal{X}_{\mathcal{P}}$. Then $1_{\mathcal{P}}$-$\mathcal{J}_2^*$is a $NP^{\mathcal{FOS}}$ in $\mathcal{X}_{\mathcal{P}}$. Since $f_{\mathcal{P}}{}^{-1} : \mathcal{X}_{\mathcal{P}} \longrightarrow \mathcal{Y}_{\mathcal{P}}NP^{\mathcal{F}b\mathcal{O}}$ map, $f_{\mathcal{P}}(1_{\mathcal{P}}$-$\mathcal{J}_2^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$in $\mathcal{Y}_{\mathcal{P}}$.Now $f_{\mathcal{P}}(1_{\mathcal{P}}$-$\mathcal{J}_2^*)$ =$1_{\mathcal{P}}$-$f_{\mathcal{P}}(\mathcal{J}_2^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$, which implies $f_{\mathcal{P}}(\mathcal{J}_2^*)$ is $NP^{\mathcal{F}\,b\mathcal{CS}}$in $\mathcal{Y}_{\mathcal{P}}$. Hence $f_{\mathcal{P}}$is $NP^{\mathcal{F}\,b\mathcal{CS}}$.

Theorem 6.11:
Let $f_{\mathcal{P}}: (\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$is a bijective function. At that point, the accompanying explanations are identical

(i) $f_{\mathcal{P}}$is $NP^{\mathcal{F}b^*}$- homeomorphism
(ii) $f_{\mathcal{P}}$ is $NP^{\mathcal{F}b^*}$- continuity and $NP^{\mathcal{F}b^*\mathcal{OS}}$
(iii) $f_{\mathcal{P}}$is $NP^{\mathcal{F}b^*}$-continuity and $NP^{\mathcal{F}b^*\mathcal{C}}$maps

Theorem 6.12:
Iff$_{\mathcal{P}}$: $(\mathcal{X}_{\mathcal{P}}, \tau_{NPF}) \longrightarrow (\mathcal{Y}_{\mathcal{P}}, \sigma_{NPF})$ be $NP^{\mathcal{F}b^*}$-homeomorphism, then their composition $(g \circ f)_{\mathcal{P}} \longrightarrow \mathcal{Z}_{\mathcal{P}}$ is $NP^{\mathcal{F}b^*}$homeomorphism.
Proof.
Let $\mathcal{J}_1$is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Z}_{\mathcal{P}}$. Then since ${f_{\mathcal{P}}}^{-1}$: $\mathcal{Y}_{\mathcal{P}} \longrightarrow \mathcal{Z}_{\mathcal{P}}$and ${f_{\mathcal{P}}}^{-1}$: $\mathcal{X}_{\mathcal{P}} \longrightarrow \mathcal{Y}_{\mathcal{P}}$ are $NP^{\mathcal{F}b^*}$- homeomorphism, $f_{\mathcal{P}}$is $NP^{\mathcal{F}b^*}$- continuity $NP\mathcal{F}^{(g)^{-1}}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Also since $f_{\mathcal{P}}$: $\mathcal{X}_{\mathcal{P}} \longrightarrow \mathcal{Y}_{\mathcal{P}}$is $NP^{\mathcal{F}b^*}$- continuity, $({f_{\mathcal{P}}}^{-1}{g_{\mathcal{P}}}^{-1}(\mathcal{J}_1^*)) = {(g \circ f)_{\mathcal{P}}}^{-1}(\mathcal{J}_1^*)$is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$. Therefore $(g \circ f)_{\mathcal{P}}$: $\mathcal{X}_{\mathcal{P}} \longrightarrow \mathcal{Z}_{\mathcal{P}}$is $NP^{\mathcal{F}b}$-continuity. Again, let $\mathcal{J}_1^*$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{X}_{\mathcal{P}}$. Then since${f_{\mathcal{P}}}^{-1}$: $\mathcal{Y}_{\mathcal{P}} \longrightarrow \mathcal{X}_{\mathcal{P}}$isNP$^{\mathcal{F}b}$ continuity,$\left({f_{\mathcal{P}}}^{-1}\right)^{-1}(\mathcal{J}_1^*)$=$f_{\mathcal{P}}(\mathcal{J}_1^*)$is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Y}_{\mathcal{P}}$. Also ${g_{\mathcal{P}}}^{-1}$: $\mathcal{Z}_{\mathcal{P}} \longrightarrow \mathcal{Y}_{\mathcal{P}}$is $NP^{\mathcal{F}b}$-continuity, $({g_{\mathcal{P}}}^{-1})^{-1}f_{\mathcal{P}}(\mathcal{J}_1^*) = g_{\mathcal{P}}(f_{\mathcal{P}}(\mathcal{J}_1^*)) = (g \circ f)_{\mathcal{P}}(\mathcal{J}_1^*)$ is $NP^{\mathcal{F}b\mathcal{OS}}$ in $\mathcal{Z}_{\mathcal{P}}$. Therefore$(g \circ f)_{\mathcal{P}}^{-1}$: $\mathcal{Z}_{\mathcal{P}}$ • $\mathcal{X}_{\mathcal{P}}$is $NP^{\mathcal{F}b}$-continuity. Hence$(g \circ f)_{\mathcal{P}}$: $\mathcal{X}_{\mathcal{P}} \longrightarrow \mathcal{Z}_{\mathcal{P}}$is $NP^{\mathcal{F}b^*}$-homeomorphism.

6. Conclusion

In this paper, we introduced and studied the concept of Neutrosophic Picture Fuzzy Sets (NPFS), which generalized the fuzzy set concept by considering four functions: YES (membership), ABSTAIN(indeterminacy), NO(non-membership), and REFUSAL(refusal membership). We investigated their basic properties and operations. We also introduced the concept of Neutrosophic Picture Fuzzy Topological spaces (NPFTS) as a mathematical model to represent and study NPFS structures. We believe that NPFTS could be used to represent and analyze various problems in many fields because of the generality of NPFS and fuzzy sets. The future work may focus on proposing computational algorithms and methods for NPFTS, and applications of NPFS in computer fields such as image processing, artificial intelligence, decision support systems, and optimization. We hope that our work will contribute to the study of neutrosophic and fuzzy sets and their applications.

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