

Original Article

Study of the N-Soliton Solutions for the (2+1)-Dimensional Generalized Caudrey–Dodd–Gibbon–Kotera–Sawada Equation

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Received: 27 June 2026

Revised: 28 July 2026

Accepted: 18 August 2026

Published: 30 August 2026

Abstract - This paper investigates the N-soliton solutions of the (2+1)-dimensional generalized Caudrey-Dodd-Gibbon-Kotera-Sawada equation. By virtue of the Hirota bilinear method, the bilinear form of the equation is derived, and the analytical expression for its N-soliton solutions is constructed. When $N \geq 2$, with reasonable parameter constraints and appropriate parameter selections, the multi-soliton solutions can degenerate into other types of solitons and exhibit the nonlinear interaction characteristics between different solitons. For instance, two-line solitons can transform into a periodic soliton, and a two-kink soliton can simplify to a kink breather soliton. Three line solitons can present the interaction between a line soliton and a periodic soliton. In contrast, a three-kink soliton can exhibit the interaction between a single-kink soliton and a kink breather soliton. Meanwhile, the four-soliton solution can effectively describe the collision and evolution processes between two line solitons and one periodic soliton, as well as the interaction between two periodic solitons. In addition, applying the long-wave limit method yields lump soliton solutions, and the interaction mechanism between lump solitons and line solitons is examined. Numerical simulations are conducted for the derived soliton structures, and their spatial evolution diagrams are presented. Furthermore, the dynamic characteristics and propagation rules of these solitons are systematically investigated. This finding enriches the solutions to this equation, broaden research on soliton categories and interaction mechanisms, and offer theoretical references for analyzing relevant nonlinear physical models.

Keywords - (2+1)-dimensional generalized Caudrey-Dodd-Gibbon-Kotera-Sawada equation, Line Soliton, Kink Soliton, Periodic Soliton, Lump Soliton.

1. Introduction

Solitons [1], a type of analytical solution with special physical significance in nonlinear partial differential equations (NPDEs), possess characteristics such as stable form and elastic interaction. These features reflect the fundamental laws governing nonlinear evolution processes in nature. They have extensive application prospects in various fields such as fluid mechanics [2,3], quantum mechanics [4], plasma physics [5,6] and optical fiber communication [7]. Based on the above application value, soliton solutions have become one of the indispensable core research objects in these disciplinary fields. The construction, characteristic analysis and interaction studies of soliton solutions not only enrich the solution system of evolutionary equations but also provide important support for the explanation and solution of various practical nonlinear problems. Since the discovery of the soliton phenomenon, scholars have proposed various methods for constructing soliton solutions. Among them, Hirota's bilinear method [8-11], inverse scattering transformation method [12], Darboux transformation method [13-17], Bäcklund transformation method [18-21] and the long-wave limit method [22-25] have become the most representative research approaches. The application of these methods has promoted the research progress of soliton solutions for a wide range of NPDEs, from the classic KdV equation and Schrödinger equation to various new nonlinear evolution equations. Breakthroughs have been continuously achieved in the construction and characteristic analysis of soliton solutions. However, with the continuous emergence of various complex high-order dynamic models, the construction theories and solving methods for soliton solutions still need further improvement and optimization, and the inherent physical mechanisms behind soliton interactions also require



more in-depth and systematic research. The traditional types of bright solitons, dark solitons, and kink solitons cannot fully cover the rich dynamic phenomena in complex nonlinear systems.

Currently, soliton research is gradually expanding to more specialized types such as high-dimensional localized solitons, composite interacting solitons, and irregular evolving solitons. Numerous scholars at home and abroad have conducted extensive systematic research on various high-order nonlinear evolution equations, and continuously explored diverse local wave solutions contained in these equations as well as their dynamic evolution characteristics. Wang et al.[26] studied the extended (3+1)-dimensional Jimbo-Miwa equation via the Hirota bilinear method and derived its N-soliton solutions. By imposing novel resonance conditions on the obtained N-soliton solutions, they constructed soliton molecules and Y-shaped soliton solutions, and further investigated the complex multi-soliton solutions of this equation. Sachin et al.[27] adopted the Lie symmetry method and unified method to investigate the coupled breaking soliton (cBS) model, and derived polynomial and rational function solutions. They also visualized the dynamical behaviors and characteristics of breather solitons, conical solitons, lump solitons, petal-shaped patterns, periodic solitons, and solitary waves through graphical illustrations. Sachin et al.[6] investigated a nonlinear fifth-order partial differential equation (PDE). By virtue of the simplified technique of the Hirota method, they expressed multiple solitons in the forms of single-soliton, two-soliton, and three-soliton solutions, as well as their interaction behaviors. Yan et al.[28] studied the (2+1)-dimensional Kadomtsev- Pet- viashvili equation with variable coefficients and constructed four different types of solutions based on its bilinear form. In addition, they performed the long-wave limit method on two-soliton, three-soliton, and four-soliton solutions to obtain lump solutions and rational function solutions. Ma et al.[29] investigated the (2+1)-dimensional generalized Benjamin-Ono (gBO) equation, by using the Hirota bilinear method, trial function method and homogeneous balance principle, they derived lump solutions, breather solitons and new soliton solutions of the equation, and analyzed their dynamical behaviors via graphical simulations.

Inspired by the above research achievements, this paper focuses on the construction and characteristics of soliton solutions of nonlinear partial differential equations. Taking the (2+1)-dimensional generalized Caudrey-Dodd-Gibbon- Kotera- Sawada (CDGKS) equation as the specific research object. The equation belongs to the fifth-order KdV integrable hierarchy. Compared with the classical (1+1)-dimensional CDGKS equation, this model introduces a fifth-order dispersion term, which can describe the nonlinear wave phenomena dominated by higher-order dispersion. By introducing a two-dimensional lateral differential coupling term while retaining the core terms such as high-order dispersion and cubic nonlinearity, the physical applicability range of the equation has been expanded. Most of the current research on the (2+1)-dimensional generalized CDGKS equation focuses on the derivation of analytical solutions for a single type of soliton, or simply discusses the propagation characteristics of traveling wave solitons. Most of the literature lacks a systematic analysis of the interaction dynamics between different types of localized waves (striped solitons, lump solitons, breathing solitons).

$$36u_t + (u_{xxxx} + 15uu_{xx} + 15u^3) - av_{yy} - b(u_{xxy} + 3uu_y + 3u_xv_y) = 0 \quad (1)$$

where $u = v_x$, The Eq. (1) is equivalent to the following form

$$36v_{tx} + (v_{xxxx} + 15v_xv_{xx} + 15v_x^3) - av_{yy} - b(v_{xxy} + 3v_x \int v_y dx + 3v_{xx}v_y) = 0 \quad (2)$$

$u(x, y, t)$ and $v(x, y, t)$ are functions of x, y and t , a, b are constants.

Owing to the excellent integrable properties and abundant local wave solutions of the equation, numerous scholars have conducted in-depth discussions on the construction of its various analytical solutions and evolutionary laws in recent years. For example, Ma et al.[30] derived the explicit expressions of N -oscillation wave solutions for the equation. By imposing novel constraint conditions on these solutions, they constructed resonant Y-shaped and X-shaped oscillatory wave solutions. Furthermore, they revealed the interaction behaviors between resonant Y-shaped oscillatory waves and higher-order lump solitons via the long-wave limit method. Based on the Hirota bilinear form of this equation, Peng et al.[31] investigated a class of exact interacting wave solutions by means of the vector symbolic method, and discussed that such solutions can be transformed into pure lump waves or stripe solitons under certain conditions. Deng et al.[32] derived the Nth-order Pfaffian solutions of this equation, from which the one-soliton and two-soliton solutions were obtained. Meanwhile, they constructed one-periodic and two-periodic wave solutions via the Hirota-Riemann method. Although Ref.[33] derived one- to three-soliton solutions for the fixed-coefficient original CDGKS equation via the Hirota bilinear method, it cannot tune soliton behaviours and pays little attention to four-soliton-related localized waves, long-wave-limit lump construction and systematic dynamics of auxiliary field v . Motivated by these gaps, this paper investigates the generalized CDGKS equation with two tunable coefficients (a, b) , whose

bilinear form and dispersion relation depend explicitly on (a, b) . We construct one- to four-soliton solutions, study parameter-degeneration rules, analyse mixed-wave interactions and explore kink-soliton features of field v .

The structure of this article is as follows. In Section 2, the generalized Hirota bilinear method is employed to derive the analytical expression of the N-soliton solution of the (2+1)-dimensional generalized Caudrey-Dodd-Gibbon-Kotera- Sawada equation. By imposing reasonable constraints on the relevant parameters from the two-soliton solution to the four-soliton solution, various localized interaction solutions are constructed, and the morphological characteristics and evolution laws of each soliton are graphically presented. In Section 3, the research results of this article are summarized, and relevant summary comments and prospects are given.

2. The N-Soliton Solutions of the (2+1)-Dimensional gCDGKS Equation

The functions u and v are assumed to take the following form

$$u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_x \quad (3)$$

By virtue of the definition of the bilinear operator, given as

$$D_x^m D_y^n D_t^p (a, b) = (\partial x - \partial x')^m (\partial y - \partial y')^n (\partial t - \partial t')^p a(x, y, t) b(x', y', t')|_{x=x', y=y', t=t'} \quad (4)$$

∂x , ∂y and ∂t denote the first-order partial derivatives with respect to x , y and t , respectively. The bilinear form of Eq. (1) can be obtain , written as

$$(36D_x D_t + D_x^6 - aD_y^2 - bD_x^3 D_y) f \cdot f = 0 \quad (5)$$

Eq. (5) can also be written as

$$2[36f_{tx}f - f_t f_x + f_{xxxxxx} + 15f_{xxxx}f_{xx} - 10f_{xxx}^3 - 6f_{xxxx}f_x - b(f_{xxxy}f - f_{xxx}f_y - 3f_{xxy}f_x + 3f_{xx}f_{xy}) - a(f_{yy}f - f_y^2)] = 0 \quad (6)$$

According to the Hirota bilinear method, the function f can be written as

$$f = f_N = \sum_{\mu_i=0,1} \exp(\sum_{i=1}^N \mu_i \xi_i + \sum_{1 \leq i < j \leq N} \mu_i \mu_j \mathfrak{R}_{ij}) \quad (7)$$

$f(x, y, t)$ is the auxiliary function introduced through the bilinear transformation.

Where

$$\xi_i = k_i(x + s_i y + w_i t) + \xi_{0i}$$

$$36w_i - as_i^2 - bk_i^2 s_i + k_i^4 = 0$$

$$e^{\mathfrak{R}_{ij}} = \frac{a(s_i - s_j)^2 + 5(k_i - k_j)^2(k_i^2 - k_i k_j + k_j^2) - b(k_i - k_j)[s_i(2k_i - k_j) + s_j(k_i - 2k_j)]}{a(s_i - s_j)^2 + 5(k_i + k_j)^2(k_i^2 + k_i k_j + k_j^2) - b(k_i + k_j)[s_i(2k_i + k_j) + s_j(k_i + 2k_j)]}$$

Where $i, j = 1 \cdots N$. ξ_i stands for the phase of the i -th soliton. k_i , s_i are spatial wave numbers; w_i is the angular frequency; ξ_{0i} denotes the initial phase constant. They are all real parameters. $e^{\mathfrak{R}_{ij}}$ represents the phase coupling coefficient between the i -th and j -th solitons. N is the number of solitons. $\sum_{\mu=0,1}$ is the summation of all possible combinations of μ_i , $\mu_j = 0, 1$ which runs over all 2^N binary combinations, i.e. $(\mu_1 \cdots, \mu_N) \in \{0, 1\}^N$.

2.1. One-Soliton Solution

When $N=1$, from Eq. (7), let $i=1$, the following soliton solutions can be obtain

$$f_1 = 1 + e^{\xi_1} \quad (8)$$

By substituting Eq. (8) into Eq. (3), the expressions of u and v are as follows.

$$u = \frac{k_1^2}{2} \sec h^2 \left(\frac{1}{2} \xi_1 \right) \quad (9)$$

$$v = \frac{2k_1 \exp(\xi_1)}{1 + \exp(\xi_1)} \quad (10)$$

Where $\xi_1 = k_1(x + s_1y + w_1t) + \xi_{01}$. The above explicit analytical solutions for $N=1$ are substituted into the Eq.(1), for exact symbolic verification using Maple. The equation holds identically from Eqs. (9) and (10), it can be observe that the value of u is always positive, while the sign of v is determined by k_1 . Their shapes are both independent of the system parameters a and b . Therefore, in order to obtain the image of the one-soliton solution, assume the system parameters to be $a = 2$, $b = 2$. Furthermore, set the other parameters as $k_1 = 3$, $s_1 = \frac{2}{3}$, $\xi_{01} = 0$ and use the mathematical software Maple, the image of Eq. (1) at the time $t = 0$ is presented in Figure 1. It can be observed that u is in the form of a single-line soliton, whereas v appears as a kink soliton. From Eqs. (9) and (10), it can calculate that the amplitudes of u and v are $\frac{k_1^2}{2}$ and $2k_1$ respectively.

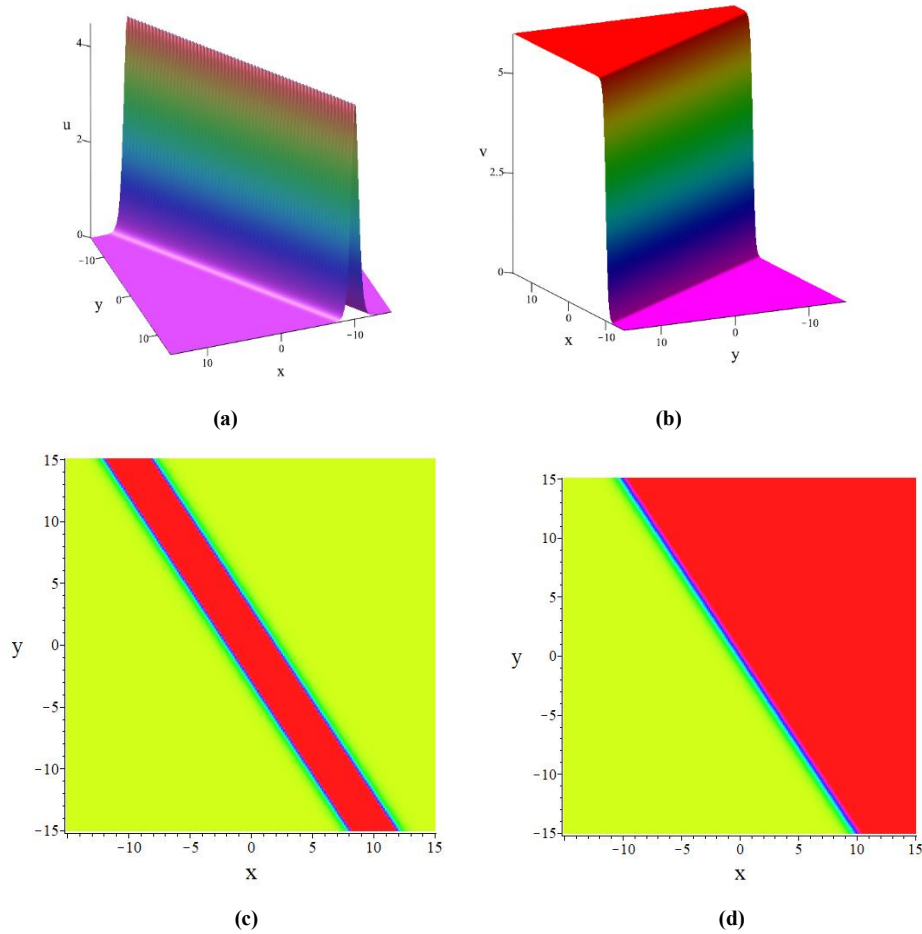


Fig. 1 The one-soliton solution of Eq. (1) at $t = 0$ when the parameters are chosen as follows: $k_1 = 3, s_1 = \frac{3}{2}, \xi_{01} = 0, a = 2, b = 2$ a, 3D structure of u ; b, 3D structure of v ; c, 2D density plot of u ; d, 2D density plot of v

2.2. Two-Soliton Solution

In order to obtain the two-soliton solution of Eq. (1), set $N=2$ and then use Eq. (7), f_2 can be as follows.

$$f_2 = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + \mathfrak{R}_{12}} \quad (11)$$

Substituting Eq. (11) into Eq. (3), the expressions of u and v can be obtained, given as follows:

$$u = 2 \left[\frac{k_1 e^{\xi_1 + k_2 e^{\xi_2} + (k_1 + k_2) e^{\xi_1 + \xi_2 + \Re_{12}}}}{1 + e^{\xi_1 + e^{\xi_2} + e^{\xi_1 + \xi_2 + \Re_{12}}}} - \left(\frac{k_1 e^{\xi_1 + k_2 e^{\xi_2} + (k_1 + k_2) e^{\xi_1 + \xi_2 + \Re_{12}}}}{1 + e^{\xi_1 + e^{\xi_2} + e^{\xi_1 + \xi_2 + \Re_{12}}}} \right)^2 \right] \quad (12)$$

$$v = 2 \frac{k_1 e^{\xi_1 + k_2 e^{\xi_2} + (k_1 + k_2) e^{\xi_1 + \xi_2 + \Re_{12}}}}{1 + e^{\xi_1 + e^{\xi_2} + e^{\xi_1 + \xi_2 + \Re_{12}}}} \quad (13)$$

Substituting Eqs. (11) and (12) into Eq.(1), for exact symbolic verification using Maple. The equation holds identically, which proves that the derived low-order soliton solutions strictly satisfy the governing equation and demonstrates the validity of the proposed construction approach. By constructing different types of parameter relations, different forms of the two-soliton solution can be obtained. The following four cases were obtained in this study.

2.2.1. The Line and Kink Solitons

When the values of k_i and s_i are real parameters, u exhibits the form of a two-line soliton, while v corresponds to a two-kink soliton structure. To visually demonstrate the dynamic behavior of the solitons, specific parameter values $k_1 = 3$, $k_2 = 2$, $s_1 = \frac{2}{3}$, $s_2 = 3$, $\xi_{01} = \xi_{02} = 0$, $a = 1$, $b = 1$ were selected to draw the three-dimensional space evolution plots and density distribution diagrams of u and v . Figure 2 displays these results: (a) and (c) show soliton u , (b) and (d) show soliton v .

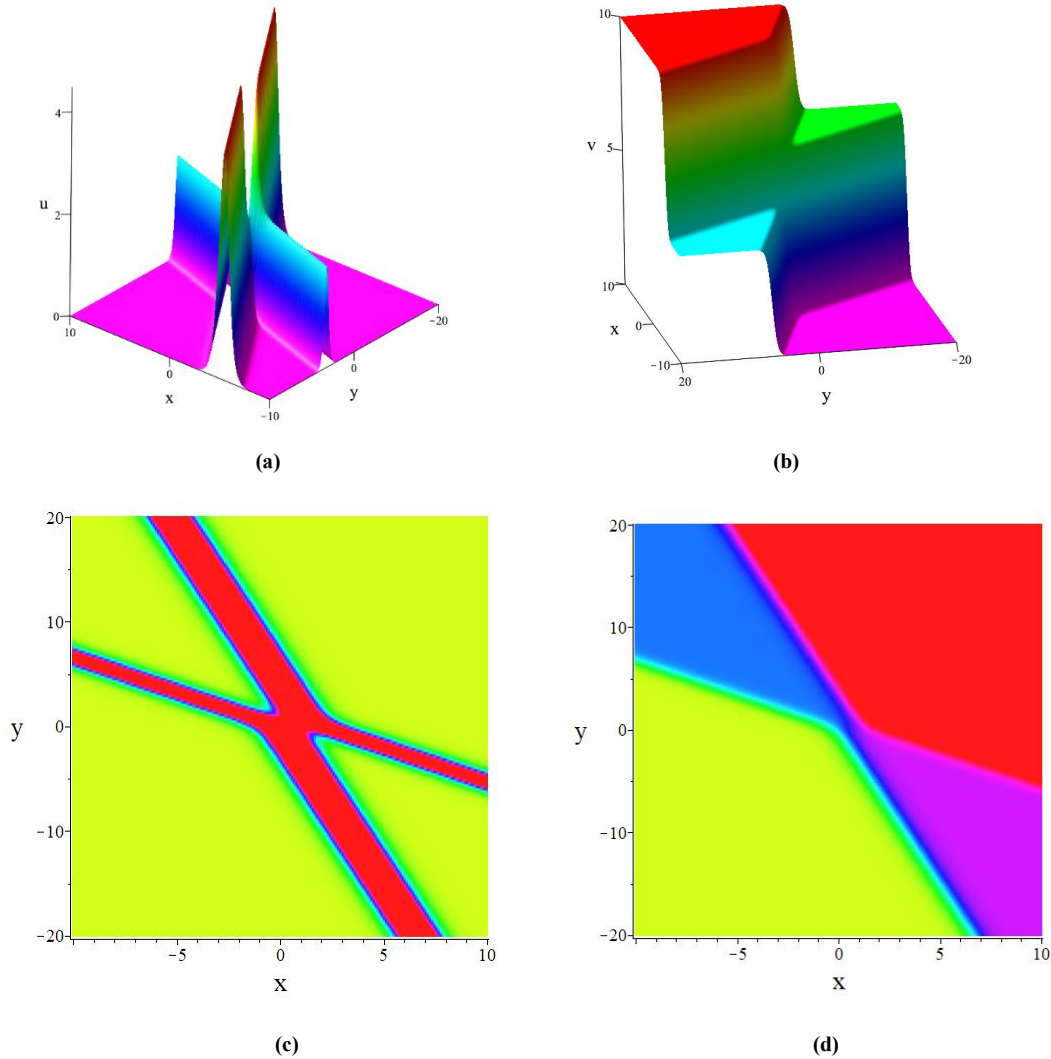


Fig. 2 Two-soliton solution of Eq. (1) at $t=0$ when the parameters are chosen as follows: $k_1 = 3$, $k_2 = 2$, $s_1 = \frac{2}{3}$, $s_2 = 3$, $\xi_{01} = \xi_{02} = 0$, $a = 1$, $b = 1$. 1. 3D structure of u ; b, 3D structure of v ; c, 2D density plot of u ; d, 2D density plot of v

2.2.2. The Periodic and Kink Breather Solitons

To obtain periodic solitons, the complex conjugate method can be employed in Eq. (11). By setting the relevant phase parameters $k_1 = k_2 = d_1$, $s_1 = s_2^* = m_1 + i\delta_1$, $\xi_{01} = \xi_{02} = 0$ ($i^2 = -1$), then the f can be re-expressed as

$$f_2 = 1 + e^{2\alpha_1 + \Re_{12}} + 2e^{\alpha_1} \cos(\delta_1 d_1 y + \psi_1 t), \quad (14)$$

where

$$e^{\Re_{12}} = \frac{4a\delta_1^2}{60d_1^3 - 4a\delta_1^2 - 12bm_1d_1^2},$$

$$\alpha_1 = d_1 x + d_1 m_1 y - \left(d_1 \frac{am_1^2 - a\delta^2 + bd_1^2 m_1 - d_1^4}{36} \right) t$$

$$\psi_1 = -d \left(\frac{2am_1\delta_1 + bd_1^2\delta_1}{36} \right)$$

Set parameters s_1 and s_2 as pure imaginary numbers, namely $m = 0$, which enables the construction of a periodic soliton perpendicular to the x -axis. Further, by taking specific parameter values $d_1 = \frac{2}{3}$, $\delta_1 = \frac{9}{4}$, $a = 2$, $b = -2$, the periodic soliton u and the kink breather soliton v can be obtained. Their images are presented in the Figure. 3.

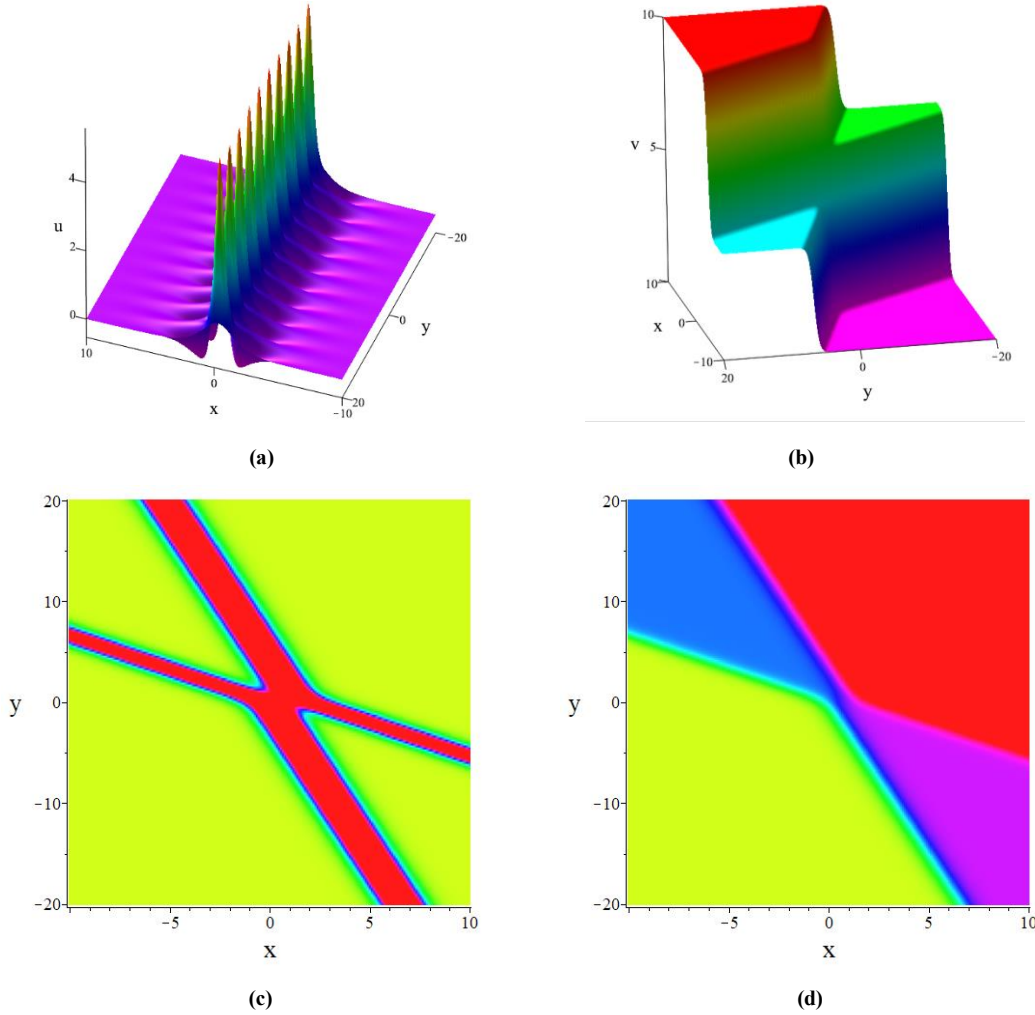


Fig. 3 The y -periodic soliton and kink breather soliton at $t = 0$ when the parameters are chosen as follows: $d_1 = \frac{2}{3}$, $\delta_1 = \frac{9}{4}$, $\xi_{01} = \xi_{02} = 0$, $a = 2$, $b = -2$, a, 3D structure of u ; b, 3D structure of v ; c, 2D density plot of u ; d, 2D density plot of v

2.2.3. The (x, y) -Period and Kink Breather Solitons

Unlike the results obtained in subsection 2.2.2, when the parameter s_i takes a complex value, the equation has a (x, y) -periodic soliton. In the specific construction, by reasonably selecting $d_1 = \frac{3}{5}$, $\delta_1 = \frac{5}{2}$, $m_1 = \frac{5}{3}$, $\xi_{01} = \xi_{02} = 0$, $a = 2$, $b = -2$ and employing Maple software, the three-dimensional and density graphs of u and v were obtained, shown in Figure. 4. During propagation, both the waveform and amplitude remain invariant.

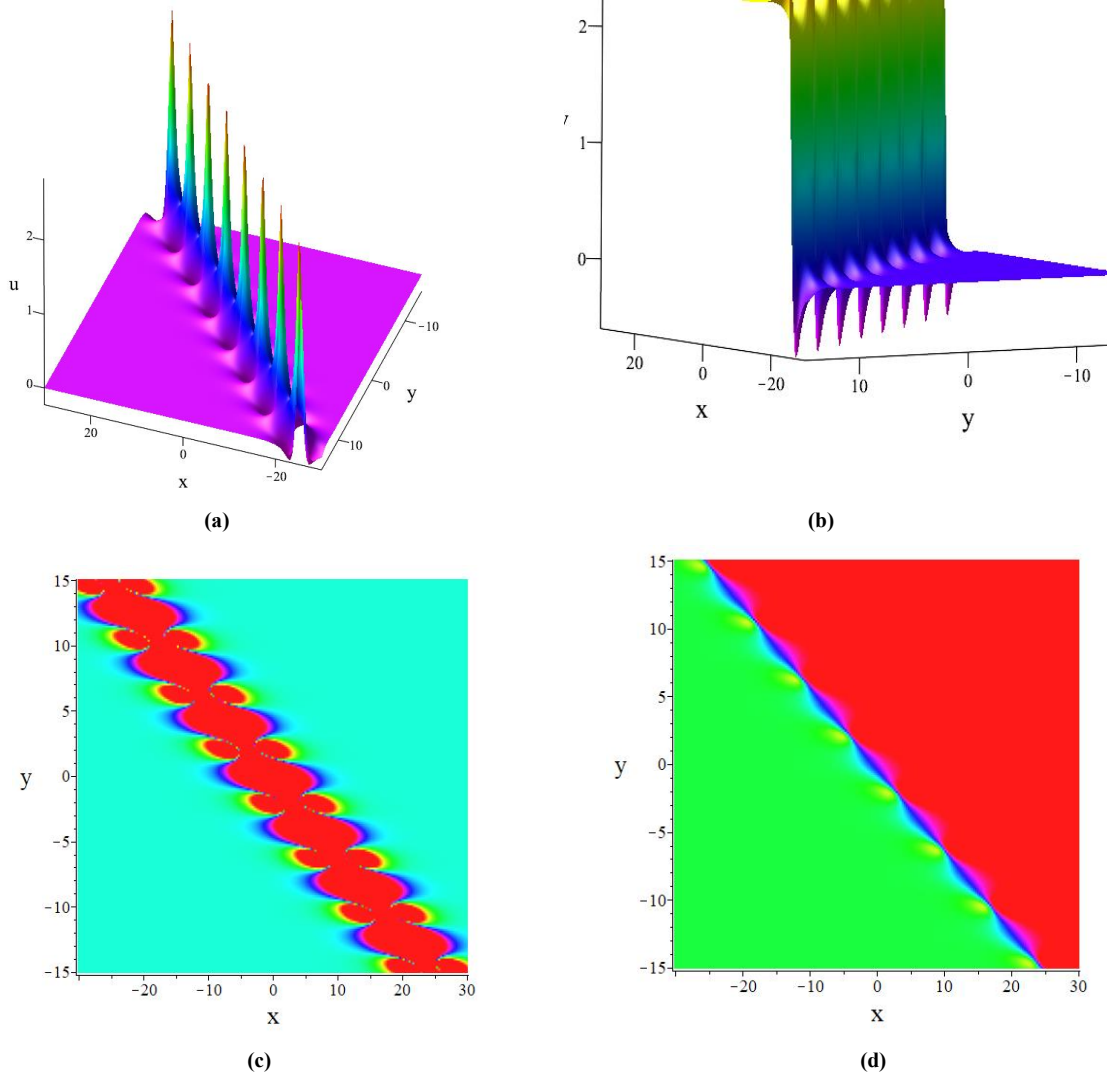


Fig. 4 The (x, y) -periodic and kink breather soliton solutions at $t = 0$ when the parameters are chosen as follows: $d_1 = \frac{3}{5}$, $\delta_1 = \frac{5}{2}$, $m_1 = \frac{5}{3}$, $\xi_{01} = \xi_{02} = 0$, $a = 2$, $b = -2$. a, 3D structure of u ; b, 3D structure of v ; c, 2D density plot of u ; d, 2D density plot of v

2.2.4. The Lump Soliton

In this subsection, applied the long-wave limit method to obtain the lump soliton. First, set $\xi_{01} = \xi_{02} = i\pi$ when $k_i \rightarrow 0$

$$e^{\Re_{12}} = 1 + \frac{6bk_1k_2(s_1 + s_2)}{a(s_1 + s_2)^2} + O(k)$$

Eq. (11) can be rewritten as

$$f_2 = k_1k_2 \left[\theta_1\theta_2 + \frac{6b(s_1+s_2)}{a(s_1-s_2)^2} \right]. \quad (15)$$

Ignoring the common factor k_1 and, k_1 Eq. (16) was derived,

$$f_2 = \theta_1 \theta_2 + \Re_{12}. \quad (16)$$

Where

$$\theta_i = x + s_i y + w_i t, \quad w_i = \frac{as_i^2}{36}, \quad i=1, 2$$

$$\Re_{12} = \frac{6b(s_1 + s_2)}{a(s_1 - s_2)^2}$$

By substituting Eq. (16) into Eq. (3), u and v can be written as

$$u = \frac{4}{\theta_1 \theta_2 + \Re_{12}} - 2 \left(\frac{\theta_1 + \theta_2}{\theta_1 \theta_2 + \Re_{12}} \right)^2 \quad (17)$$

$$v = 2 \frac{\theta_1 + \theta_2}{\theta_1 \theta_2 + \Re_{12}} \quad (18)$$

The parameters satisfy the following relationship $s_1 = s_2^* = c + di$, and then can construct a lump soliton degenerated from the two-soliton solution. Specifically, by taking $k_1 = k_2 = 0.01$, $s_1 = s_2^* = 1 + i$, $a = 2$, $b = -2$, the image of the lump soliton can be obtained and shown in the Figure. 5. From Figure 5, it can observe that u is localized in the two-dimensional plane, with the energy mainly concentrated in the central area and rapidly decaying towards all directions. The image of the v component (Figure b) is significantly different from that of u . It fails to exhibit a symmetric double-peak structure and instead features one dominant peak accompanied by a secondary peak.

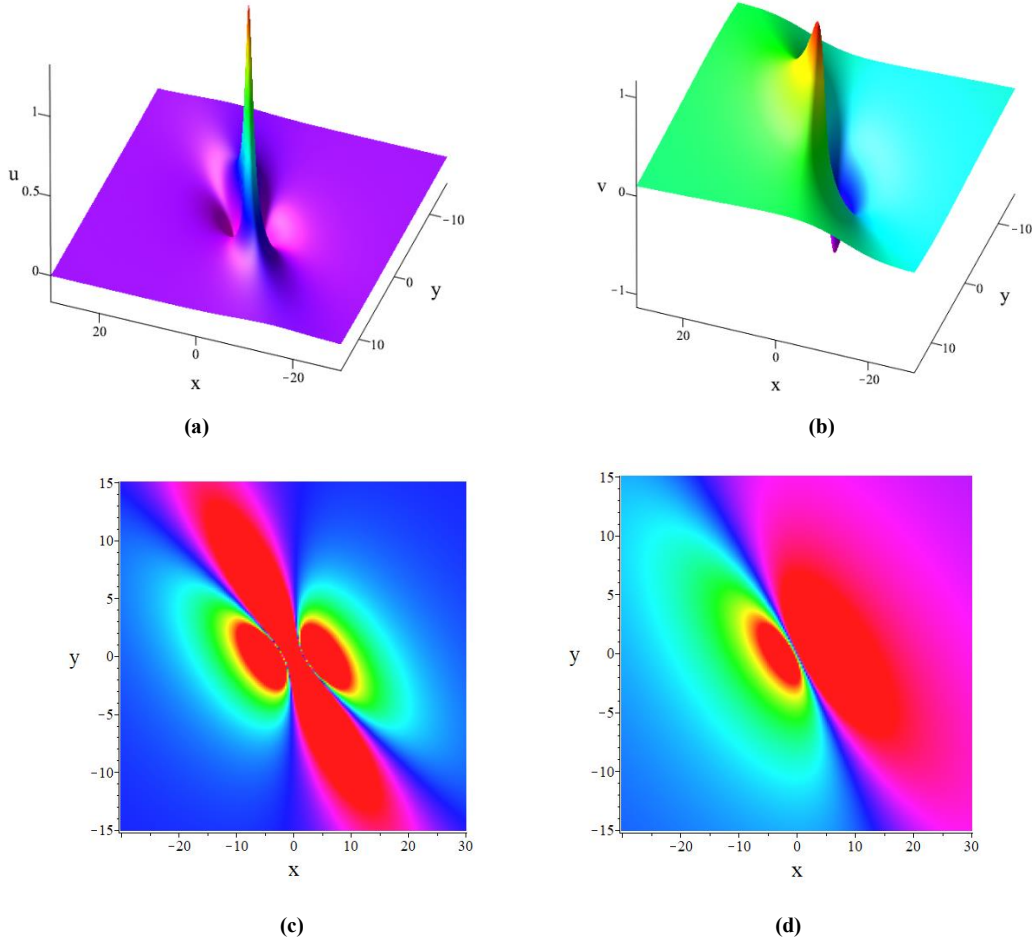


Fig. 5 The lump soliton at $t=0$ when the parameters are chosen as follows: $k_1 = k_2 = 0.01$, $s_1 = s_2^* = 1 + i$, $\xi_{01} = \xi_{02} = i\pi$, $a = 2$, $b = -2$. a, 3D structure of u ; b, 3D structure of v ; c, 2D density plot of u ; d, 2D density plot of v

To visually reveal the regulatory effects of parameter relationships and their values on the dynamic morphology of two-soliton solution, the aforementioned analytical results are systematically summarized in Table 1. In this table, the first column corresponds to the order of N -solitons, the second column presents the structural characteristics of the localized waves, and the last column shows the parameter values corresponding to each structure.

Table 1. The localized wave patterns for two-soliton solution

N-soliton	Localized waves patterns	Variable values
N = 2	The line and kink solitons	$k_1 = 3, k_2 = 2, s_1 = \frac{2}{3}, s_2 = 3, \xi_{01} = \xi_{02} = 0, a = 1, b = 1$ $k_1 = k_2 = \frac{2}{3}, s_1 = s_2^* = \frac{9}{4}i, \xi_{01} = \xi_{02} = 0, a = 2, b = -2$
	The y -periodic and kink breather solitons	$k_1 = k_2 = \frac{3}{5}, s_1 = s_2^* = \frac{5}{3} + \frac{5}{2}i$ $\xi_{01} = \xi_{02} = 0, a = 2, b = -2$
	The (x, y) -periodic and kink breather solitons	$k_1 = k_2 = 0.01, s_1 = s_2^* = 1 + i$ $\xi_{01} = \xi_{02} = i\pi, a = 2, b = -2$
	The lump soliton	

2.3. The Three-Soliton Solution

Let $N=3$, based on Eq. (7), the three-soliton solution expression is derived as follows.

$$f_3 = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_1+\xi_2+\Re_{12}} + e^{\xi_1+\xi_3+\Re_{13}} + e^{\xi_2+\xi_3+\Re_{23}} + e^{\xi_1+\xi_2+\xi_3+\Re_{12}+\Re_{13}+\Re_{23}} \quad (19)$$

substituting Eq. (19) into Eq. (3), Eq. (20) can be obtain

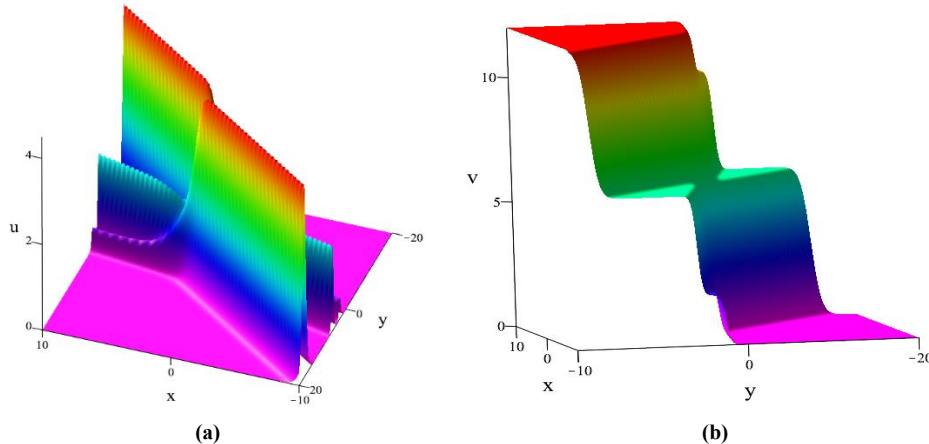
$$u = 2 \cdot \left[\frac{f_{xx}}{f} - \left(\frac{f_x}{f} \right)^2 \right], v = 2 \cdot \frac{f_x}{f} \quad (20)$$

where

$$\begin{aligned} f_{xx} &= k_1^2 e^{\xi_1} + k_2^2 e^{\xi_2} + k_3^2 e^{\xi_3} + (k_1 + k_2) e^{\xi_1+\xi_2+\Re_{12}} + (k_1 + k_3)^2 e^{\xi_1+\xi_3+\Re_{13}} \\ &\quad + (k_2 + k_3) e^{\xi_2+\xi_3+\Re_{23}} + (k_1 + k_2 + k_3)^2 e^{\xi_1+\xi_2+\xi_3+\Re_{12}+\Re_{13}+\Re_{23}} \\ f_x &= k_1 e^{\xi_1} + k_2 e^{\xi_2} + k_3 e^{\xi_3} + (k_1 + k_2) e^{\xi_1+\xi_2+\Re_{12}} + (k_1 + k_3) e^{\xi_1+\xi_3+\Re_{13}} \\ &\quad + (k_2 + k_3) e^{\xi_2+\xi_3+\Re_{23}} + (k_1 + k_2 + k_3) e^{\xi_1+\xi_2+\xi_3+\Re_{12}+\Re_{13}+\Re_{23}}, \end{aligned}$$

2.3.1. The Line and Kink Solitons

Without loss of generality, when take $k_1 = 3, k_2 = 2, k_3 = 1, s_1 = \frac{2}{3}, s_2 = 2, s_3 = 4, \xi_{01} = \xi_{02} = \xi_{03} = 0, a = 2, b = 2$, it can obtain u as a three-line soliton and v as a three-kink soliton. The corresponding waveform images are shown in Figure 6. Figure. 6(a) and 6(b) are the three-dimensional spatial structure diagrams of the field quantities u and v , respectively, while Figure. 6(c) and 6(d) are the corresponding two-dimensional density diagrams.



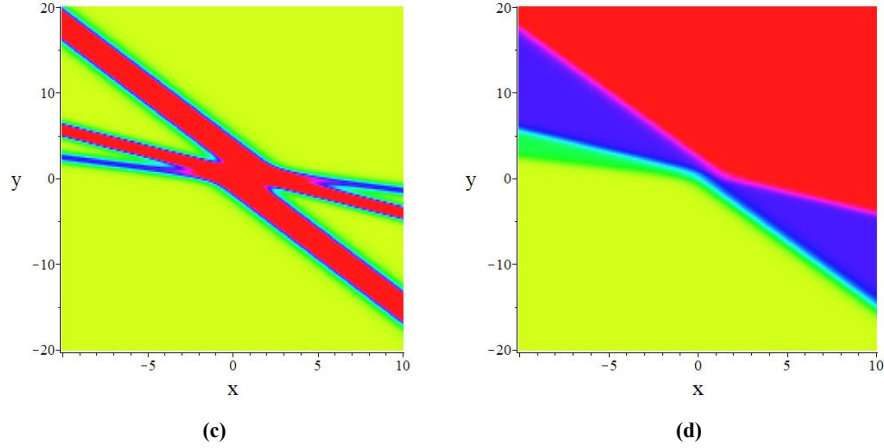


Fig. 6 Three-soliton solution of Eq. (1) at $t = 0$ when the parameters are chosen as follows: $k_1 = 3$, $k_2 = 2$, $k_3 = 1$, $s_1 = \frac{2}{3}$, $s_2 = 2$, $s_3 = 4$, $\xi_{01} = \xi_{02} = \xi_{03} = 0$, $a = 2$, $b = 2$, a, 3D structure of u ; b, 3D structure of v ; c, 2D density plot of u ; d, 2D density plot of v

2.3.2. The Line, Periodic, Kink and Kink Breather Solitons (I)

To investigate the nonlinear interaction characteristics between periodic soliton and line soliton, as well as kink soliton and kink breather soliton, set the parameter relationship as $k_2 = k_3 = m$, $s_2 = s_3^* = p + qi$. Under this condition, the soliton u degenerates into a line soliton and a periodic soliton, while the soliton v degenerates into a one-kink soliton and a kink breather soliton. Further, by taking $m = \frac{33}{40}p=0$, $q = \frac{20}{11}$ other parameters $k_1 = \frac{4}{3}$, $s_1 = 0$, $\xi_{01} = \xi_{02} = \xi_{03} = 0$, $a = 18$, $b = -18$ it can obtained the three-dimensional structure images and two-dimensional density plots of u and v at different times, which are presented in Figure. 7 and 8, respectively. Analyzing Figure. 7, it can be seen that both the line soliton and the periodic soliton are parallel to the x -axis and propagate along the x -axis direction. At $t = 0$, they collided. During the collision process, the waveforms of both show significant deformation and distortion, after the collision, they can maintain their original shapes and separate, demonstrating good elastic collision characteristics. For the v component in Figure 8, the kink soliton and the kink breather soliton also exhibit the characteristic of propagation along the x -axis. The dynamic evolution process of their interaction is similar to that of u and this will not be elaborated further here.

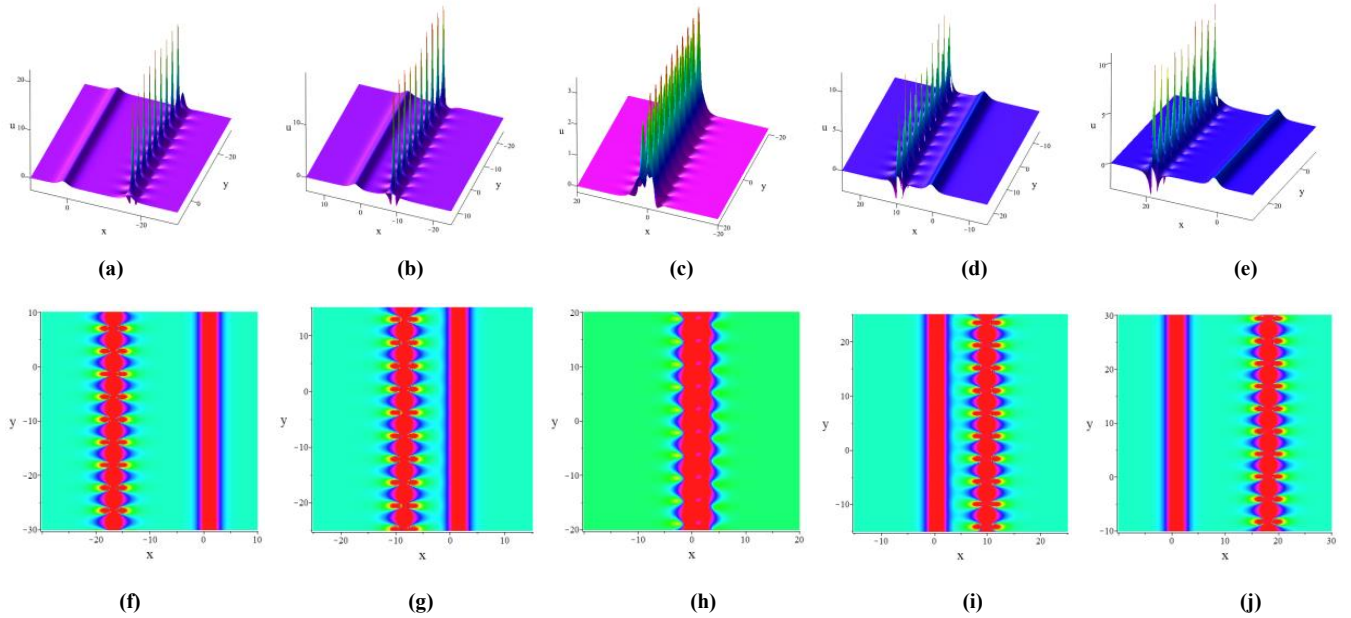


Fig. 7 One line and periodic solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = \frac{4}{3}$, $k_2 = k_3 = \frac{33}{40}$, $s_1 = 0$, $s_2 = s_3^* = \frac{20}{11}i$, $\xi_{01} = \xi_{02} = \xi_{03} = 0$, $a = 18$, $b = -18$. a-e, 3D structures of u ; f-j, 2D structures of u ; a, f $t = -10$; b, g $t = -5$; c, h $t = 0$; d, i $t = 5$; e, j $t = 10$

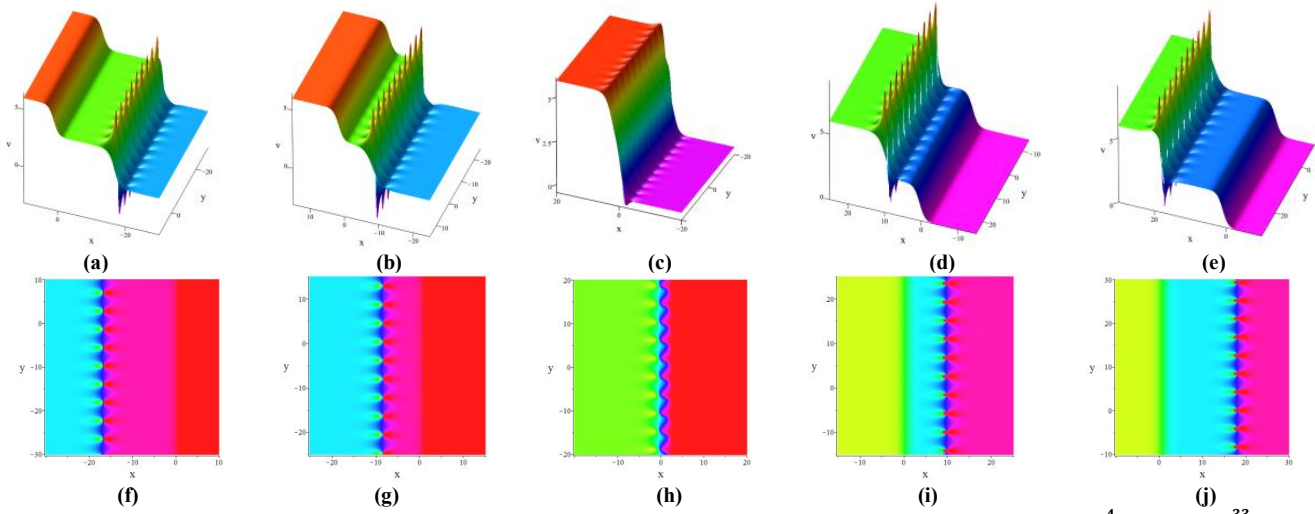


Fig. 8 One kink and kink breather solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = \frac{4}{3}$, $k_2 = k_3 = \frac{33}{40}$, $s_1 = 0$, $s_2 = s_3^* = \frac{20}{11}i$, $\xi_{01} = \xi_{02} = \xi_{03} = 0$, $a = 18, b = -18$. a-e, 3D structures of v ; f-j, 2D structures of v ; a, $t = -10$; b, $t = -5$; c, $t = 0$; d, $t = 5$; e, $t = 10$

2.3.3. The Line, Periodic, Kink and Kink Breather Solitons (II)

In this subsection, set the parameter $s_1 = \frac{3}{4}$, and keep the other parameters consistent with those in subsection 2.3.2. In this case, u degenerates into an intersecting line soliton and periodic soliton, while v degenerates into an intersecting kink soliton and kink breather soliton. Figure. 9 (a)-(e) show the three-dimensional plots of soliton u at different times and Figure. 9 (f)-(j) show their corresponding two-dimensional density distribution plots. The time parameters from left to right are $t = -8$, $t = -4$, $t = 0$, $t = 4$, $t = 8$. As illustrated in the figures, a periodic soliton travels along the positive x -axis, while the line soliton intersects both the x -axis and y -axis in the (x, y) -plane. As time increases, this line soliton propagates from the upper right toward the lower left. In Figure 10, (a)-(e) are the three-dimensional plots of soliton v , and (f)-(j) are their corresponding density plots. It can be observed that the kink breather soliton is also parallel to the y -axis, while the kink line of the kink soliton intersects both the x and y -axes in the (x, y) -plane. The overall movement trajectory of soliton v is similar to that of soliton u .

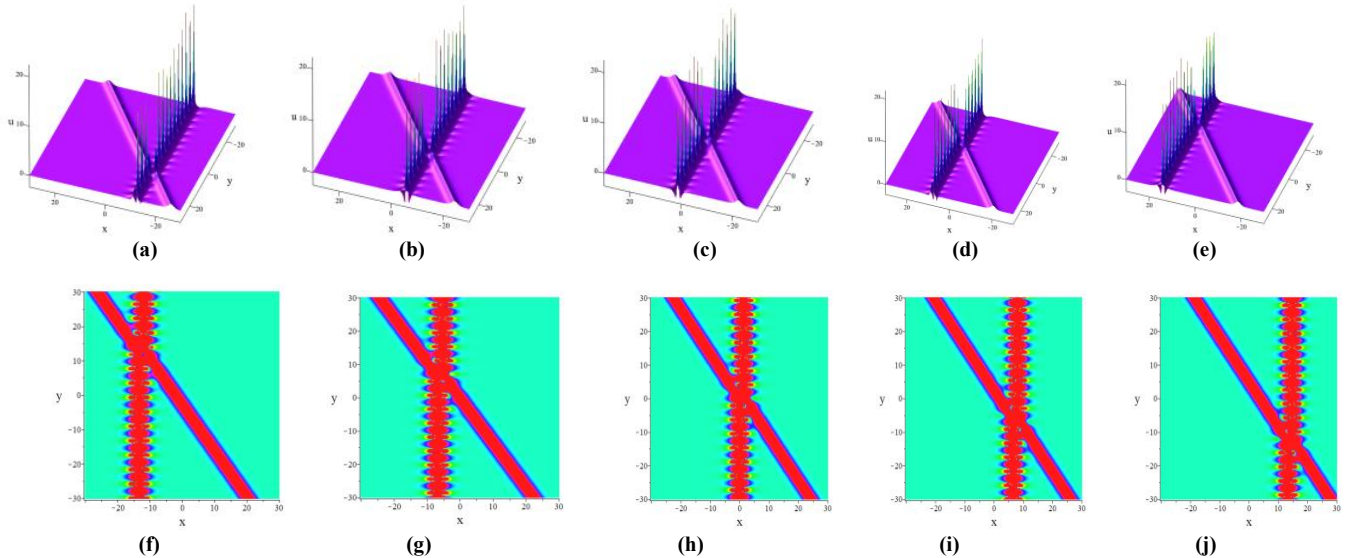


Fig. 9 One line and periodic solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = \frac{4}{3}$, $k_2 = k_3 = \frac{33}{40}$, $s_1 = \frac{3}{4}$, $s_2 = s_3^* = \frac{20}{11}i$, $\xi_{01} = \xi_{02} = \xi_{03} = 0$, $a = 18$, $b = -18$. a-e, 3D structures of u ; f-j, 2D structures of u ; a, $t = -8$; b, $t = -4$; c, $t = 0$; d, $t = 4$; e, $t = 8$

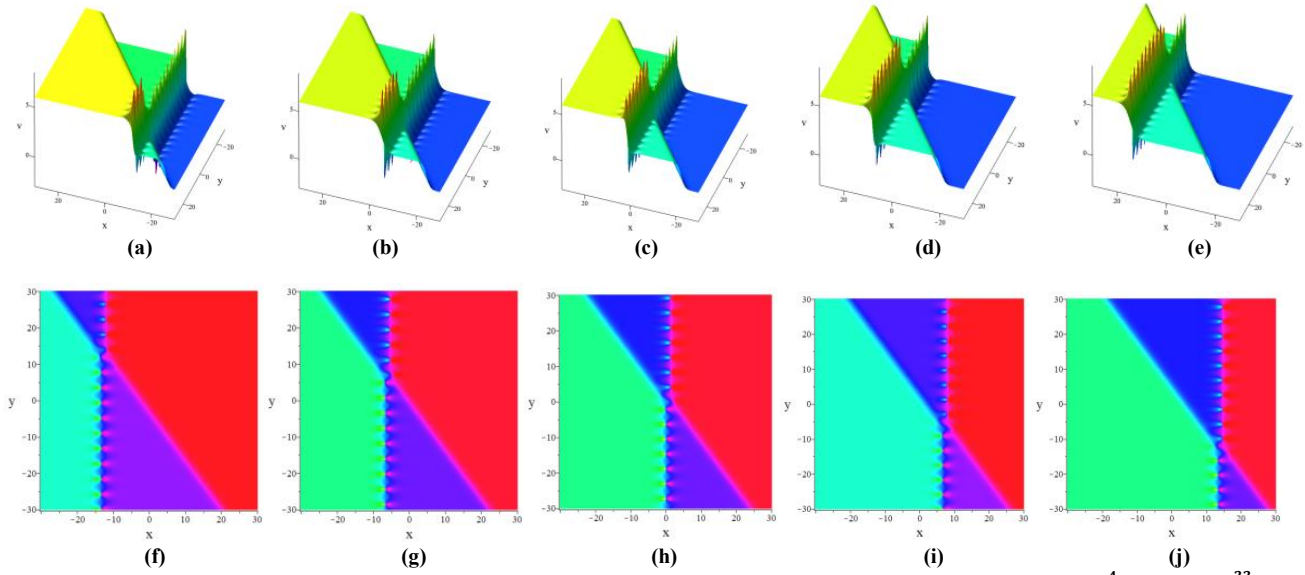


Fig. 10 One kink and kink breather solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = \frac{4}{3}$, $k_2 = k_3 = \frac{33}{40}$, $s_1 = \frac{3}{4}$, $s_2 = s_3^* = \frac{20}{11}i$, $\xi_{01} = \xi_{02} = \xi_{03} = 0$, $a = 18$, $b = -18$. a-e, 3D structures of v ; f-j, 2D structures of v ; a, f $t = -8$; b, g $t = -4$; c, h $t = 0$; d, i $t = 4$; e, j $t = 8$

2.3.4. The Lump, Line and Kink Solitons

In this subsection, adopt the long-wave limit method to obtain mixed solitons composed of line or kink solitons and lump soliton. First, set $\xi_{01} = \xi_{02} = i\pi$ when $k_i \rightarrow 0$, $i = 1, 2$, and the Eq. (19) can be expressed in the following form

$$f_3 = \theta_1 \theta_2 + \Re_{12} + e^{\xi_3} (\theta_1 \theta_2 + \Re_{12} + \Re_{13} \theta_2 + \Re_{23} \theta_1 + \Re_{13} \Re_{23}), \quad (21)$$

where

$$\theta_i = x + s_i y + w_i t, \quad w_i = \frac{as_i^2}{36}, \quad i = 1, 2$$

$$\xi_3 = k_3(x + s_3 y + w_3 t) + \xi_{03},$$

$$w_3 = \frac{as_3^2 + bk_3^2 s_3 - k_3^4}{36},$$

$$\Re_{12} = \frac{6b(s_1 + s_2)}{a(s_1 - s_2)^2}$$

$$\Re_{12} = \frac{-6k_j(5k_i^2 - bs_i - bs_j)}{a(s_i - s_j)^2 + k_j^2(5k_i^2 - bs_i - 2bs_j)}, \quad i = 1, 2, j = 3$$

When take the parameters relationship as $s_1 = s_2^*$, mixed solitons consisting of lump soliton and other types of solitons can be derived. To visually demonstrate the evolution process of the interaction between these hybrid solitons, select a specific set of parameters as $k_1 = k_2 = 0.01$, $k_3 = \frac{6}{5}$, $s_1 = s_2^* = 1 + i$, $s_3 = \frac{5}{6}$, $a = 2$, $b = -2$ and conduct a numerical simulation using the Maple software, the relevant results are shown in Figure. 11 and 12. Figure 11 depicts the interaction behavior between the line soliton and the lump soliton. By combining the three-dimensional evolution plots (a)-(e), it can be seen that the two types of solitons exhibit opposite propagation directions. The line soliton propagates from the upper right to the lower left, while the lump soliton moves from the lower left to the upper right. At $t=0$, the two solitons collide with only a slight variation in amplitude. After separation, they restore their original wave profiles and continue to propagate along their initial trajectories. Figure 12 displays the three-dimensional structures and density distributions of soliton v , illustrating the interaction characteristics between the lump solitons and the kink soliton. The figures show that, as time progresses, the lump soliton and the kink soliton intersect in the upper and lower spatial regions, respectively. Moreover, the amplitudes and wave shapes of both solitons remain stable without obvious deformation before and after collision.

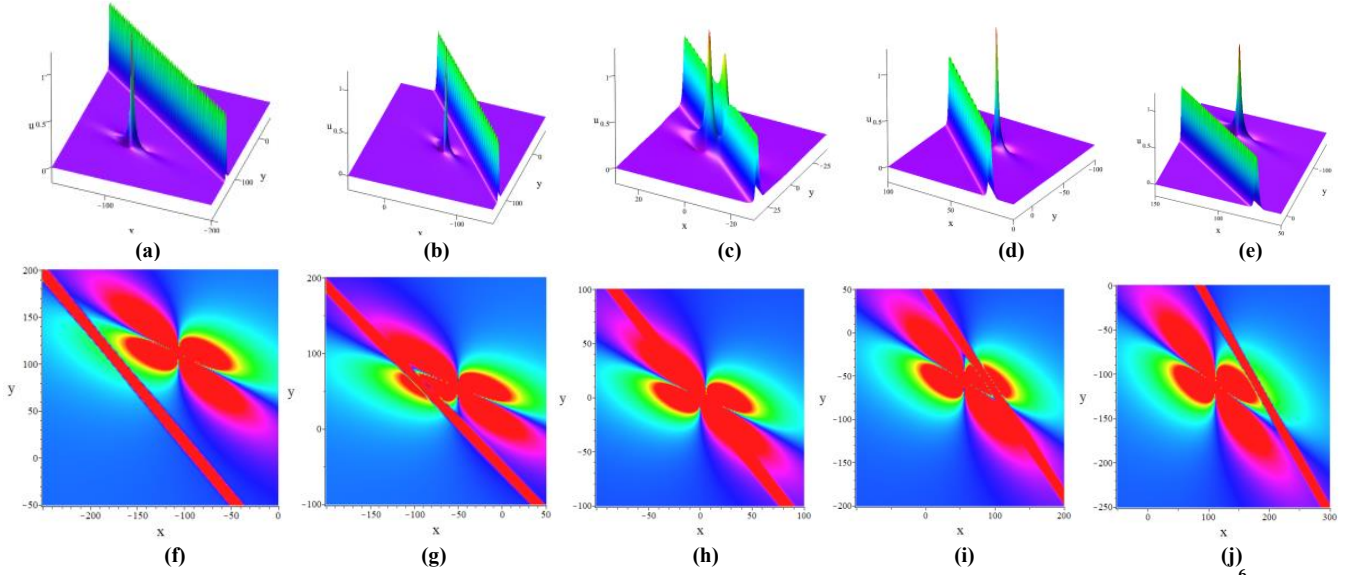


Fig. 11 One line and one lump solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = k_2 = 0.01$, $k_3 = \frac{6}{5}$, $s_1 = s_2^* = 1 + i$, $s_3 = \frac{5}{6}$, $a = 2$, $b = -2$, a-e, 3D structures of u ; f-j, 2D structures of u ; a, f $t = -1000$; b, g $t = -500$; c, h $t = 0$; d, i $t = 500$; e, j $t = 1000$

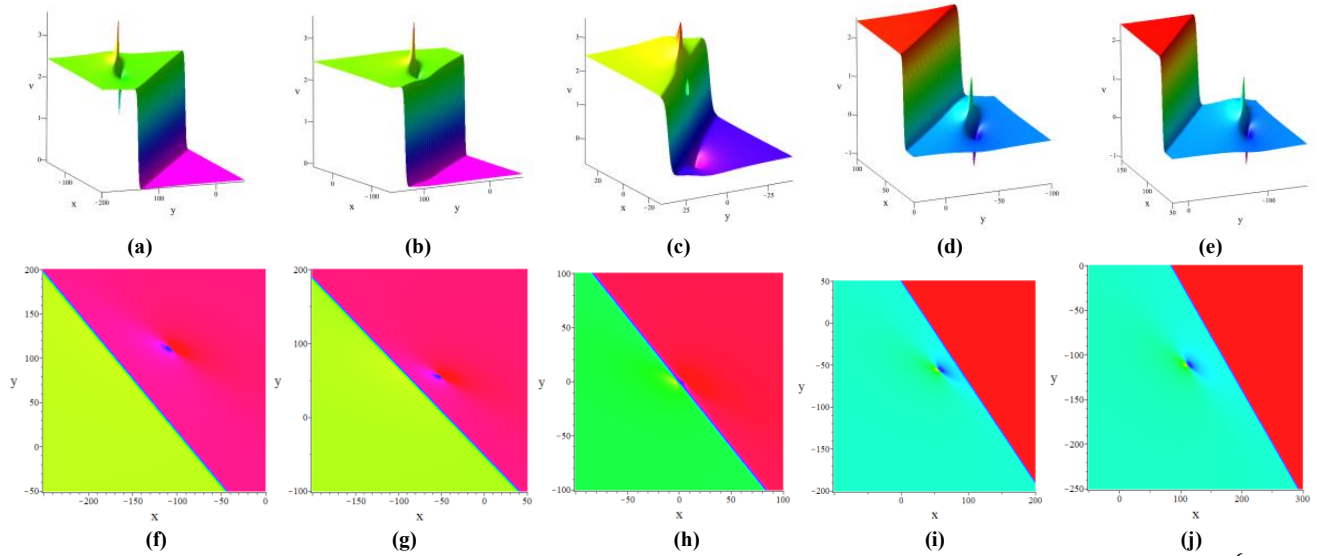


Fig. 12 One kink and one lump solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = k_2 = 0.01$, $k_3 = \frac{6}{5}$, $s_1 = s_2^* = 1 + i$, $s_3 = \frac{5}{6}$, $a = 2$, $b = -2$, a-e, 3D structures of v ; f-j, 2D structures of v ; a, f $t = -1000$; b, g $t = -500$; c, h $t = 0$; d, i $t = 500$; e, j $t = 1000$

Consistent with Table 1, the soliton types corresponding to the three-soliton solution under four distinct parameter sets are summarized in Table 2. The second column presents the wave structure characteristics of the soliton solutions, while the third column lists the specific values of the relevant parameters.

2.4. Four-Soliton Solution

Set $N=4$, and the following formula can be obtained from Eq. (7)

$$\begin{aligned}
 f_4 = & 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_1 + \xi_2 + \Re_{12}} + e^{\xi_1 + \xi_3 + \Re_{13}} + e^{\xi_1 + \xi_4 + \Re_{14}} + e^{\xi_2 + \xi_3 + \Re_{23}} \\
 & + e^{\xi_3 + \xi_4 + \Re_{34}} + e^{\xi_1 + \xi_2 + \xi_3 + \Re_{12} + \Re_{13} + \Re_{23}} + e^{\xi_1 + \xi_2 + \xi_4 + \Re_{12} + \Re_{14} + \Re_{24}} \\
 & + e^{\xi_1 + \xi_3 + \xi_4 + \Re_{13} + \Re_{14} + \Re_{34}} + e^{\xi_2 + \xi_3 + \xi_4 + \Re_{23} + \Re_{24} + \Re_{34}}
 \end{aligned}$$

$$+e^{\xi_1+\xi_2+\xi_3+\xi_4+\Re_{12}+\Re_{13}+\Re_{14}+\Re_{23}+\Re_{24}+\Re_{34}} \quad (22)$$

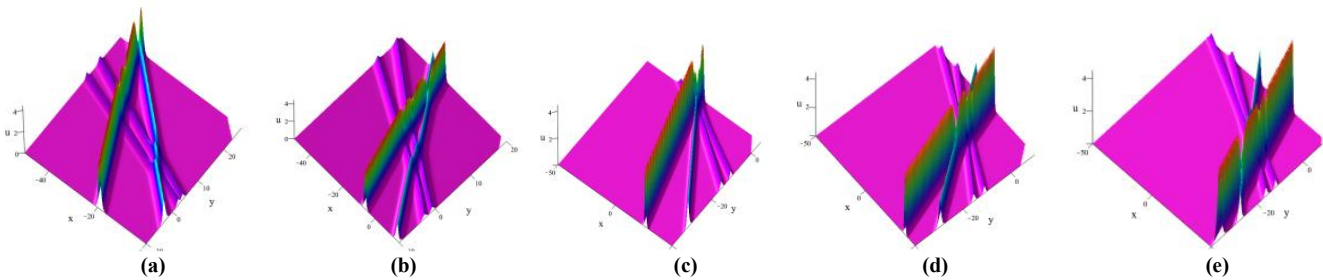
Since the form of f_4 is relatively complex, the explicit expressions of u and v are omitted here.

Table 2. The localized wave patterns for three-soliton solution

N-soliton	Localized waves patterns	Variable values
N = 3	The line and kink solitons	$k_1 = 3, k_2 = 2, k_3 = 1, s_1 = \frac{2}{3},$ $s_2 = 2, s_3 = 4, a = 2, b = 2, \xi_{01} =$ $\xi_{02} = \xi_{03} = 0,$
	The line, periodic, kink and kink breather solitons (I)	$k_1 = \frac{4}{3}, k_2 = k_3 = \frac{33}{44}, s_1 = 0$ $s_2 = s_3^* = \frac{20}{11}i, a = 18, b = -18,$ $\xi_{01} = \xi_{02} = \xi_{03} = 0$
	The line, periodic, kink and kink breather solitons (II)	$k_1 = \frac{4}{3}, k_2 = k_3 = \frac{33}{44}, s_1 = \frac{4}{3},$ $s_2 = s_3^* = \frac{20}{11}i, a = 18, b = -18,$ $\xi_{01} = \xi_{02} = \xi_{03} = 0$
	The lump, line, and kink solitons	$k_1 = k_2 = 0.01, k_3 = \frac{6}{5}, s_3 = \frac{5}{6},$ $s_1 = s_2^* = 1 + i, \xi_{02} = \xi_{03} = i\pi,$ $\xi_{01} = 0, a = 2, b = -2$

2.4.1. The Line and Kink Solitons

In this subsection, the parameters $k_i, s_i, i = 1 \cdots 4$ are taken as real numbers. Thus, u can be expressed as four bell-shaped line solitons, while v represents a kind of four-kink soliton structure. To investigate the time evolution patterns of these two types of solitons, fixed parameter values $k_1 = 1, k_2 = 2, k_3 = 1, k_4 = 3, s_1 = 3, s_2 = \frac{3}{2}, s_3 = 4, s_4 = \frac{2}{3}, \xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0, a = 2, b = -2$, and, performed numerical simulations, the results are presented in the following figures. Figure 13 (a)-(e) present the three-dimensional evolution plots of u , while Figure 13 (f)-(j) show the corresponding two-dimensional density plots. The time parameters are selected from left to right as $t = -10, t = -5, t = 0, t = 5, t = 10$. The results indicate that these four line solitons move independently of each other and do not have any interfering effects. The evolution result of the soliton v is shown in Figure 14, with the parameters remaining consistent with those of u . By analyzing the images, it can be concluded that as time progresses, the kink soliton v only undergoes minor shape adjustments at the kink distribution positions, while the overall waveform structure remains stable.



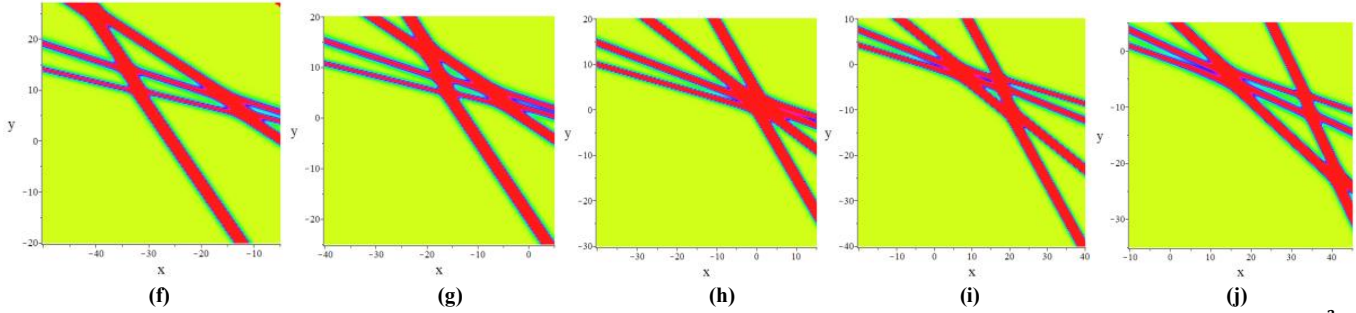


Fig. 13 Four line solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = 1, k_2 = 2, k_3 = 1, k_4 = 3, s_1 = 3, s_2 = \frac{3}{2}, s_3 = 4, s_4 = \frac{2}{3}, \xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0, a = 2, b = -2$. a-e, 3D structures of u ; f-j, 2D structures of u ; a, f $t = -10$; b, g $t = -5$; c, h $t = 0$; d, i $t = 5$; e, j $t = 10$

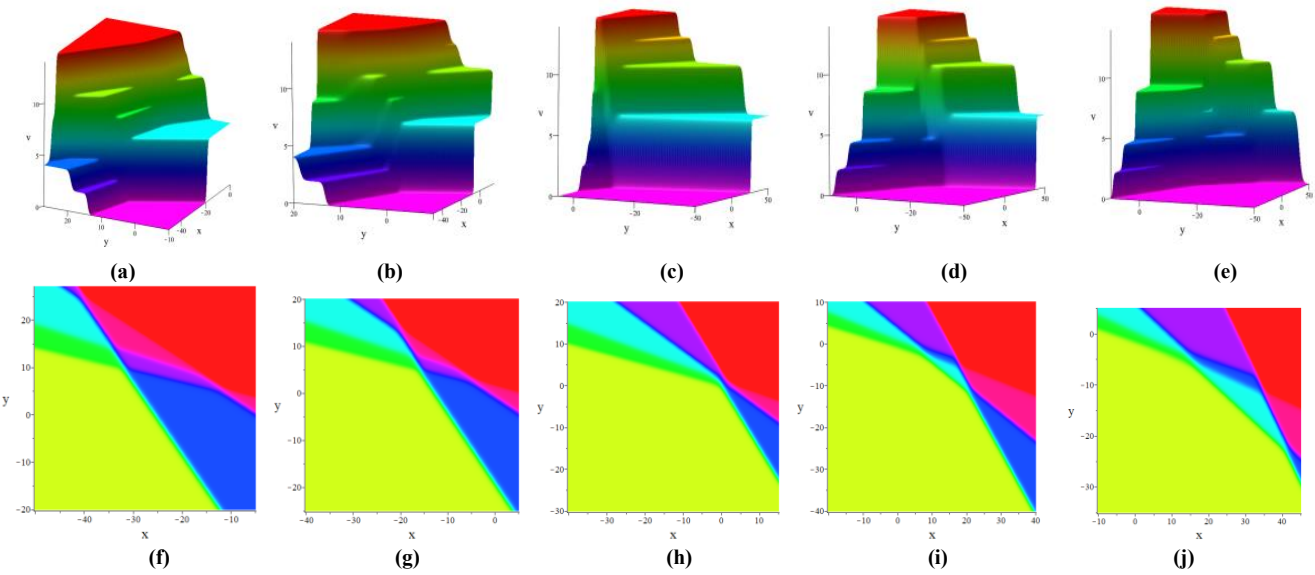


Fig. 14 Four-kink soliton of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = 1, k_2 = 2, k_3 = 1, k_4 = 3, s_1 = 3, s_2 = \frac{3}{2}, s_3 = 4, s_4 = \frac{2}{3}, \xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0, a = 2, b = -2$. a-e, 3D structures of v ; f-j, 2D structures of v ; a, f $t = -10$; b, g $t = -5$; c, h $t = 0$; d, i $t = 5$; e, j $t = 10$

2.4.2. The Line, Periodic, Kink and Kink Breather Solitons

When the parameter values are set as: $k_1 = k_2 = \frac{4}{5}, k_3 = k_4 = \frac{5}{4}, s_1 = s_2^* = \frac{5}{3}i, s_3 = \frac{12}{5}, s_4 = -\frac{12}{5}, \xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0, a = 18, b = -18$, the four-soliton solution u displays a mixed structure consisting of two line solitons and one periodic soliton (as shown in Figure 15), v degenerates into a mixed structures consisting of a two-kink soliton and a kink breather soliton (as shown in Figure 16). To clearly observe the interaction behavior between solitons in the two types of field quantities, evolutionary analysis is performed at five distinct time points. Observations on Figure 15 indicate that the periodic soliton is distributed parallel to the x -axis and intersects the two line solitons separately.

At $t=0$, the three types of solitons meet at the same point. As time t increases from left to right, the periodic soliton propagates along the positive direction of the x -axis, while the intersection point of the two line solitons moves along the negative x -axis and positive y -axis directions. Observations on Figure 16 show that the kink breather soliton is also distributed parallel to the x -axis, with its kink position moving along the positive direction of the x -axis over time. The positions of the two-kink soliton also undergo dynamic adjustments with time, and its overall morphological characteristics can be directly observed through the three-dimensional evolution plots.

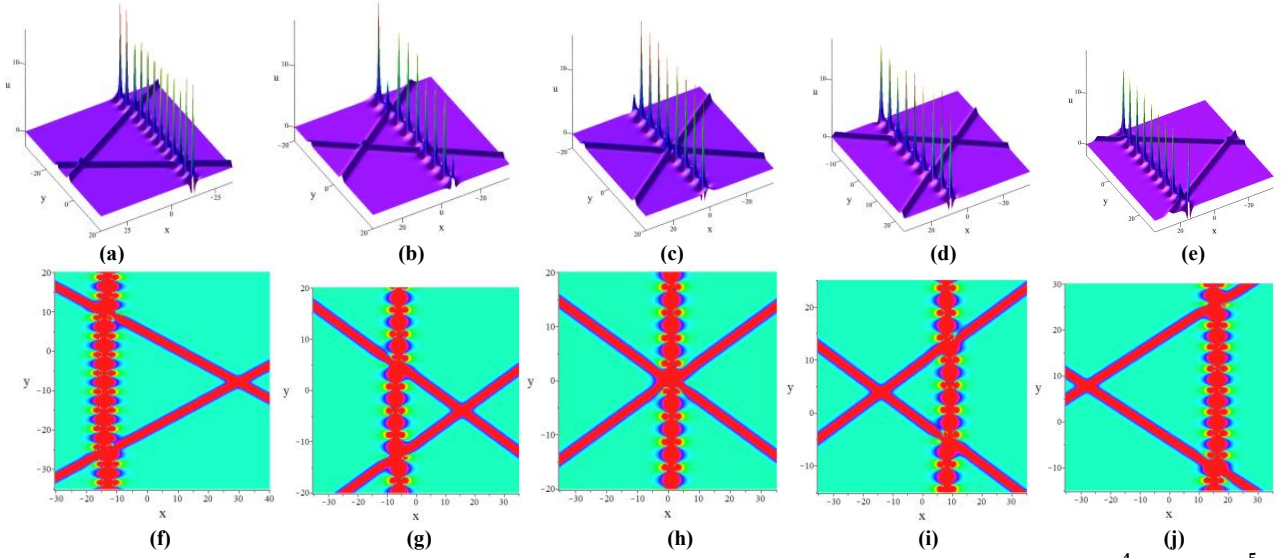


Fig. 15 Two-line and one periodic solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = \frac{5}{4}$, $s_1 = s_2^* = \frac{5}{3}i$, $s_3 = \frac{12}{5}$, $s_4 = -\frac{12}{5}$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$, $a = 18$, $b = -18$, a-e, 3D structures of u ; f-j, 2D structures of u ; a, f $t = -10$; b, g $t = -5$; c, h $t = 0$; d, i $t = 5$; e, j $t = 10$.

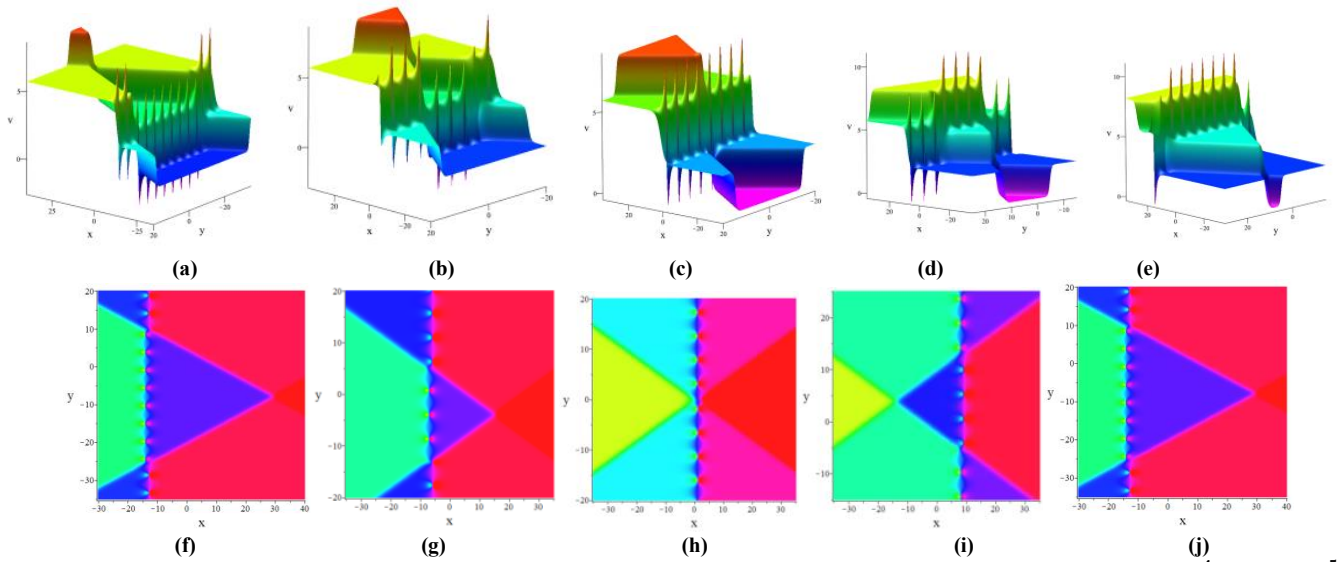


Fig. 16 Two-kink and one kink breather solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = \frac{5}{4}$, $s_1 = s_2^* = \frac{5}{3}i$, $s_3 = \frac{12}{5}$, $s_4 = -\frac{12}{5}$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$, $a = 18$, $b = -18$, a-e, 3D structures of v ; f-j, 2D structures of v ; a, f $t = -10$; b, g $t = -5$; c, h $t = 0$; d, i $t = 5$; e, j $t = 10$.

2.4.3. The Periodic and Kink Breather Solitons (I)

In this subsection, by setting specific parameter conditions and making the parameters satisfy, $k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = 1$, $s_1 = s_2^* = \frac{5}{3}i$, $s_3 = \frac{7}{6}i$, $s_4 = -\frac{7}{6}i$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$, $a = 18$, $b = -18$, it is found that the field quantity degenerates into two periodic soliton structures, and its evolutionary behavior is presented in Figure 17. Meanwhile, the field quantity v degenerates into two kink breather soliton structures, with the corresponding dynamical characteristics illustrated in Figure 18. In Figures 17 and 18, panels (a)-(e) denote the three-dimensional spatial evolution distributions and (f)-(j) are the density plots at the corresponding moments. Five typical time points, including $t = -10$, $t = -5$, $t = 0$, $t = 5$, and $t = 10$, are selected to systematically investigate the interaction and evolution laws of the two types of localized excitation structures. According to the evolutionary characteristics in Figure 17, the two periodic solitons propagate along the positive direction of the x -axis with distinct propagation velocities. The fast-moving soliton catches up and collides with the slow one. During the collision, the amplitudes of the solitons change significantly and instantaneously. After the interaction, the two solitons separate gradually, and their profiles, morphologies, and propagation velocities return to the initial states before collision. As observed from Figure

18, the two kink breather solitons exhibit identical dynamical evolution behaviors. They collide at $t = 0$, with only a transient amplitude disturbance, and their structural morphologies can be completely restored after collision. In conclusion, the interaction processes of periodic solitons and kink breather solitons are merely accompanied by transient amplitude adjustments, while key properties such as structural profiles and propagation velocities undergo no permanent variations. It is verified that the collisions of the two types of localized structures in this evolutionary system belong to elastic collisions.

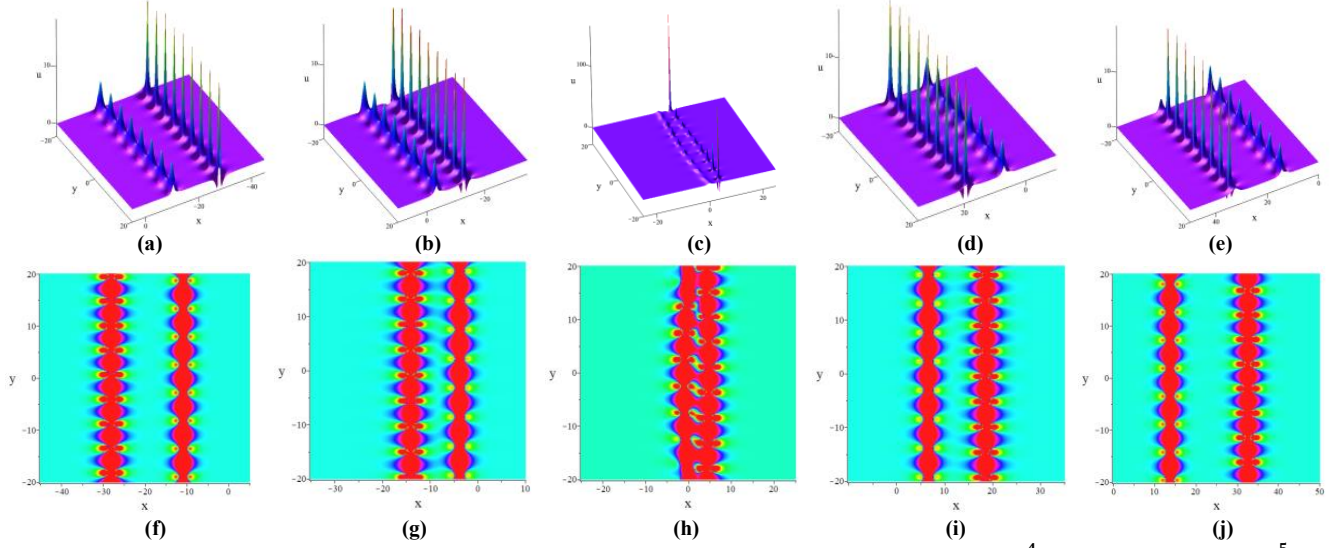


Fig. 17 Two periodic solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = 1$, $s_1 = s_2^* = \frac{5}{3}i$, $s_3 = \frac{7}{6}i$, $s_4 = -\frac{7}{6}i$, $a = 18$, $b = -18$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$. a-e, 3D structures of u ; f-j, 2D structures of u ; a, f $t = -20$; b, g $t = -10$; c, h $t = 0$; d, i $t = 10$; e, j $t = 20$

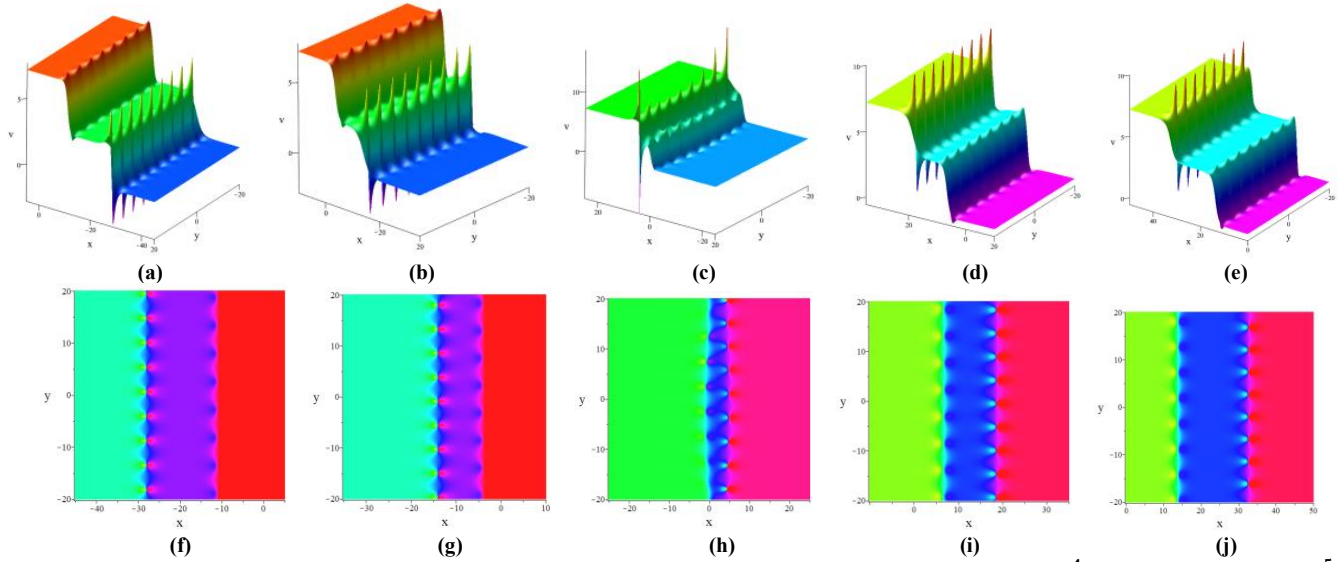


Fig. 18 Two kink breather solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = 1$, $s_1 = s_2^* = \frac{5}{3}i$, $s_3 = \frac{7}{6}i$, $s_4 = -\frac{7}{6}i$, $a = 18$, $b = -18$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$. a-e, 3D structures of v ; f-j, 2D structures of v ; a, f $t = -20$; b, g $t = -10$; c, h $t = 0$; d, i $t = 10$; e, j $t = 20$

2.4.4. The Periodic and Kink Breather Solitons (II)

When the parameter values are taken as $k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = 1$, $s_1 = s_2^* = \frac{5}{4} + \frac{5}{3}i$, $s_3 = 1 + \frac{7}{6}i$, $s_4 = 1 - \frac{7}{6}i$, $a = 18$, $b = -18$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$, degenerates into two intersecting periodic solitons. One of the solitons is parallel to the y -axis, which is called a y -periodic soliton, the other one, which has intersections with both the x -axis and the y -axis, is called the (x, y) -periodic soliton. The corresponding images are shown in Figure 19. From the figures, it can be seen that the y -periodic soliton propagates along the positive x -axis direction, while the (x, y) -periodic soliton propagates in the direction from the lower left to the upper right on the (x, y) -plane. During the entire propagation process, these two solitons do not affect each other.

Similarly, the soliton v splits into two kink breather solitons. One is parallel to the y -axis, and the other has intersections with both the x -axis and the y -axis. Their evolution are shown in Figure 20, from which it can be seen that the propagation of these two kink breather solitons is also independent of each other.

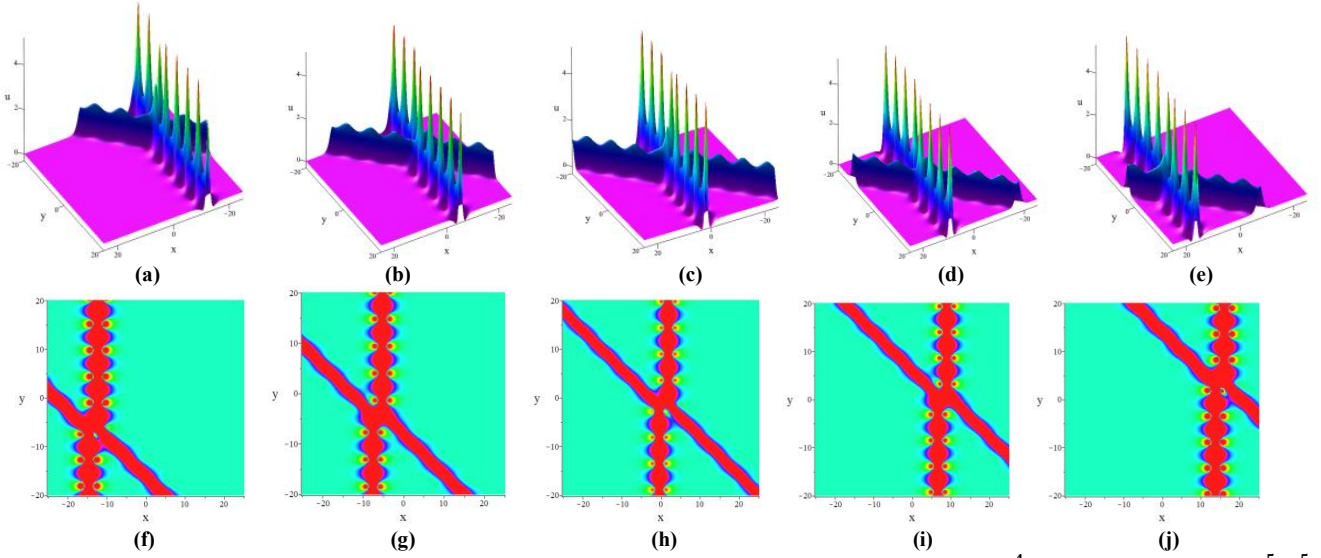


Fig. 19 Two periodic solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = 1$, $s_1 = s_2^* = \frac{5}{4} + \frac{5}{3}i$, $s_3 = s_4^* = 1 + \frac{7}{6}i$, $a = 18$, $b = -18$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$. a-e, 3D structures of u ; f-j, 2D structures of u ; a, f $t = -20$; b, g $t = -10$; c, h $t = 0$; d, i $t = 10$; e, j $t = 20$

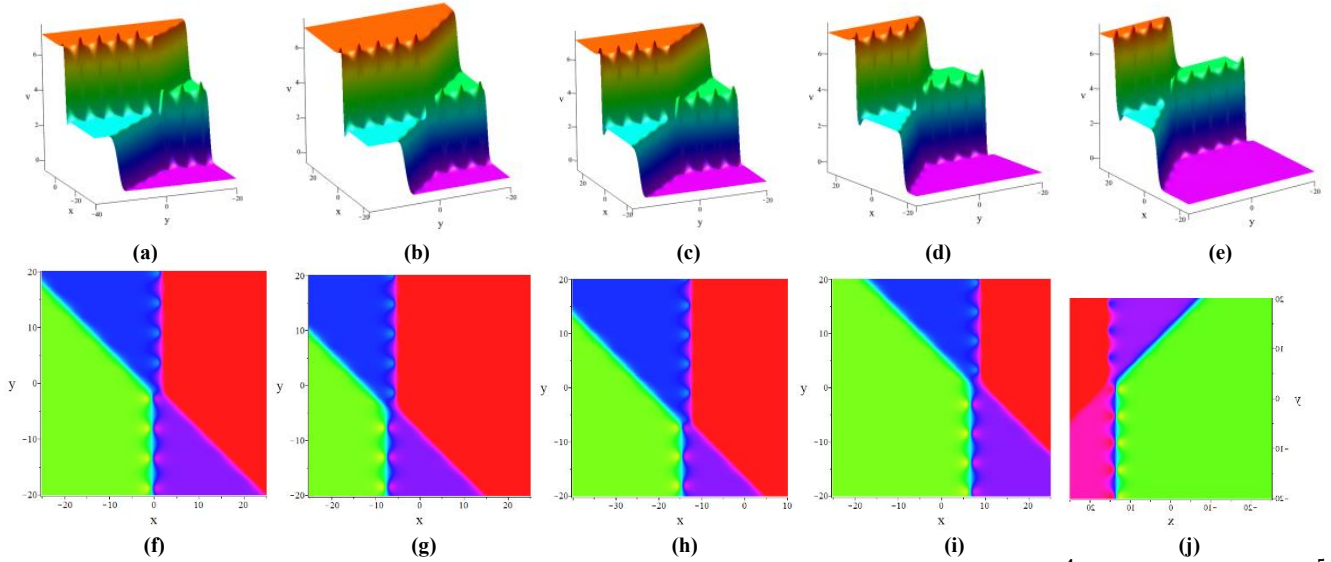


Fig. 20 Two kink breather solitons of Eq. (1) at different times when the parameters are chosen as follows: $k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = 1$, $s_1 = s_2^* = \frac{5}{4} + \frac{5}{3}i$, $s_3 = s_4^* = 1 + \frac{7}{6}i$, $a = 18$, $b = -18$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$. a-e, 3D structures of v ; f-j, 2D structures of v ; a, f $t = -20$; b, g $t = -10$; c, h $t = 0$; d, i $t = 10$; e, j $t = 20$

2.4.5. Two-Lump Solitons

In Eq. (22), set $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = i\pi$ f_4 can be expressed in the form of Eq. (23). Then, substituting it into Eq. (3), when the condition $s_1 = s_2^*$ $s_3 = s_4^*$ are satisfied, two lump solitons can be constructed.

$$f_4 = \theta_1 \theta_2 \theta_3 \theta_4 + \Re_{12} \theta_3 \theta_4 + \Re_{13} \theta_2 \theta_4 + \Re_{14} \theta_2 \theta_3 + \Re_{23} \theta_1 \theta_4 + \Re_{24} \theta_1 \theta_3 + \Re_{34} \theta_1 \theta_2 + \Re_{12} \Re_{34} + \Re_{13} \Re_{24} + \Re_{14} \Re_{23} \quad (23)$$

Where

$$\theta_i = x + s_i y + w_i t, \quad w_i = \frac{as_i^2}{36}, \quad i=1, 2, 3, 4$$

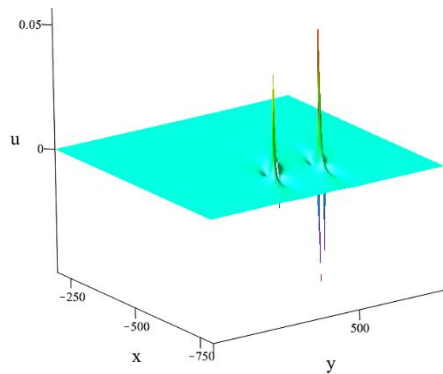
$$\mathfrak{R}_{12} = \frac{6b(s_1+s_2)}{a(s_1-s_2)^2}, \quad 1 \leq i < j \leq 4.$$

In order to obtain the images of the lump soliton, we selected specific parameter values as $k_1 = k_2 = 0.01$, $k_3 = k_4 = 0.01$, $s_1 = s_2^* = 0.01 + 0.01i$, $s_3 = s_4^* = \frac{4}{3} + \frac{4}{3}i$, $a = 2$, $b = -2$. Their images are presented in Figure 21, where a and c correspond to soliton u and b and d correspond to soliton v .

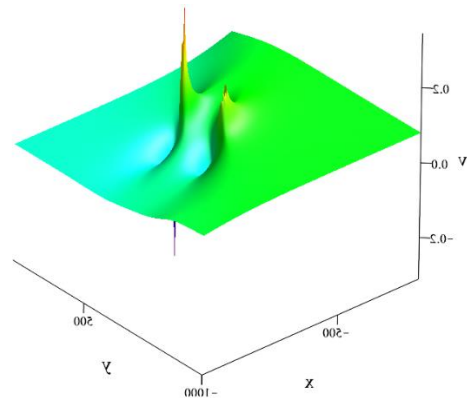
Table 3 summarizes the relationship between the parameters and the soliton shapes when $N=4$.

Table 3. The localized wave patterns for four-soliton solution

N-soliton	Localized waves patterns	Variable values
N = 4	The line and kink solitons	$k_1 = k_3 = 1$, $k_2 = 2$, $k_4 = 3$, $s_1 = 3$, $s_3 = 4$, $s_2 = \frac{3}{2}$, $s_4 = \frac{2}{3}$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$ $a = 2$, $b = -2$,
	The line, periodic, kink and kink breather solitons	$k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = \frac{5}{4}$, $s_1 = s_2 = \frac{5}{3}i$, $s_3 = s_4 = \frac{12}{5}$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$, $a = 18$, $b = -18$
	The periodic and kink breather solitons (I)	$k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = 1$, $s_1 = s_2^* = \frac{5}{3}i$, $s_3 = s_4^* = \frac{7}{6}i$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$, $a = 18$, $b = -18$
	The periodic and kink breather solitons (II)	$k_1 = k_2 = \frac{4}{5}$, $k_3 = k_4 = 1$, $s_1 = s_2^* = \frac{5}{4} + \frac{5}{3}i$, $s_3 = s_4^* = \frac{7}{6}i$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = 0$, $a = 18$, $b = -18$
	Two-lump solitons	$k_1 = k_2 = k_3 = k_4 = 0.01$, $s_1 = s_2^* = 0.01 + 0.01i$, $s_3 = s_4^* = \frac{4}{3} + \frac{4}{3}i$, $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = i\pi$, $a = 2$, $b = -2$



(a)



(b)

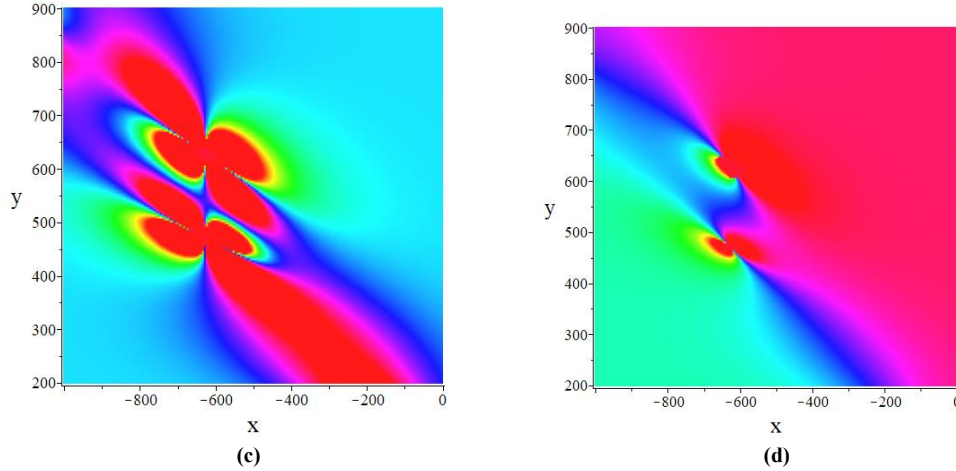


Fig. 21 Two lump solitons of Eq. (1) at $t = 0$ when the parameters are chosen as follows: $k_1 = k_2 = k_3 = k_4 = 0.01$, $s_1 = s_2^* = 0.01 + 0.01i$, $s_3 = s_4^* = \frac{4}{3} + \frac{4}{3}i$, $a = 2$, $b = -2$ $\xi_{01} = \xi_{02} = \xi_{03} = \xi_{04} = i\pi$. a, 3D structure of u ; b, 3D structure of v ; c, 2D density plot of u ; d, 2D density plot of v

3. Conclusion

This paper systematically investigates the N -soliton solutions of (2+1)-dimensional generalized Caudrey-Dodd-Gibbon-Kotera-Sawada equation, with focused analysis on four typical cases where $N = 1 \cdots 4$. The results reveal that reasonable constraint and regulation of the key parameters in soliton solutions can induce solution degeneration, thereby generating various degenerate soliton structures and diverse evolutionary patterns of nonlinear wave interactions. For $N=2$, the original two-soliton solution can degenerate into one periodic soliton and kink breather soliton structures. When $N=3$, coupled interactions and dynamic evolution can be realized between a one-line soliton and a periodic soliton, as well as between a kink soliton and kink breather. In the high-order soliton system with $N=4$, compound interaction processes of multi-component nonlinear waves are further observed, including two line solitons with one periodic soliton, two-kink soliton with kink breather, two periodic solitons, and two kink breathers solitons. To construct lump solitons and their multitype coupling models, the long-wave limit method is adopted for derivation in this work. Single lump solitons are derived from two-soliton solution; coupled structures of straight-line soliton-lump soliton and kink soliton-lump soliton are obtained based on three-soliton solution; double lump solitons are constructed via four-soliton solution, and the interaction laws between lump soliton and straight-line solitons are systematically explored. Through numerical simulation, three-dimensional diagrams and density distribution plots of various solitons are plotted. Combined with graphical analysis, the propagation characteristics, motion laws, and evolutionary properties of different nonlinear waves are quantitatively analyzed. Meanwhile, to clarify the regulatory effect of parameters on soliton morphology, this paper summarizes three sets of tables to intuitively present the morphological differences among solitons corresponding to different parameter combinations. Based on previous studies, this paper constructs multi-soliton analytical solutions using the Hirota bilinear method, expanding the types of known soliton solutions for this equation. Different from the fixed-coefficient CDGKS model in Ref.[34], our generalized CDGKS equation contains tunable coefficients (a, b) to modulate soliton amplitudes and velocities under valid parameter constraints. Beyond the results in Ref.[34], We obtain four-soliton-degenerated localized waves, construct lump solutions via the long-wave-limit technique, and complete the systematic analysis for auxiliary field v and its kink-soliton dynamics, which enriches the research of CDGKS-type integrable equations.

Acknowledgements

This work is supported by the Doctoral Fund of Henan Polytechnic University (No. B2020-35) and the Fundamental Research Funds for the Universities of Henan Province (No. NSFRF230430).

Data Availability Statement

This manuscript has no associated data. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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