

Original Article

Prime-Power Transport and a Three-Site Reduction

Payam Danesh¹, Raoul Bianchetti²

¹Department of Biosystems Engineering, University of Tehran, Iran.

²Information Physics Institute, Gosport, Hampshire, United Kingdom.

²Corresponding Author : raoul.bianchetti@informationphysicsinstitute.net

Received: 11 July 2026

Revised: 17 August 2026

Accepted: 10 September 2026

Published: 28 September 2026

Abstract - The least common multiple of a block of consecutive integers records only the largest prime-power valuation at each prime, whereas the associated product records every valuation. Erdős asked whether two disjoint blocks of the same length can nevertheless have equal least common multiples. This paper develops a prime-power transport method for a hypothetical collision. After removing the universal product-to-LCM defect, the collision has coprime coefficients supported on primes not exceeding the block length. Maximal prime powers of the common LCM then form pairwise coprime transport packets whose residual factors have factorial budgets. A quadratic Taylor lift on diagonal packets produces a primitive positive arithmetic progression in which the reduced gap is forced to square-pack. Maximal-valuation extraction limits all losses to one factorial factor, and a finite irrational square-root spectrum shows that every sufficiently large collision requires at least three active progression sites. Exact certificates eliminate the complete zero-, one-, and two-site regimes for lengths five and six. The method also excludes adjacent blocks and lengths three and four, and yields gap and transport-occupation bounds. The remaining obstruction is a configuration with at least three active sites and a moved transport packet; consequently, the argument gives a rigorous reduction, not a resolution of the conjecture. The bounded length-five search does not determine all rational points and does not exclude solutions beyond its search height.

Keywords - Consecutive Integers, Diophantine Equations, Genus-Two Curves, Least Common Multiple, Prime-Power Transport.

1. Introduction

For integers $k \geq 1$ and $n \geq 0$, let $M_k(n)$ denote the least common multiple of the k consecutive positive integers beginning at $n + 1$. Thus

$$M_k(n) = \text{lcm}(n + 1, n + 2, \dots, n + k). \quad (1)$$

Erdős Problem 677 asks whether

$$M_k(n) \neq M_k(m) \quad \text{whenever} \quad m \geq n + k. \quad (2)$$

The separation condition in (2) makes the two blocks disjoint. The question is elementary to state but resists monotonicity: an LCM can decrease when a newly introduced integer contributes no new maximal prime power.

The historical route to the problem begins with the arithmetic of prime divisors in short products. Erdős strengthened the Sylvester–Schur phenomenon for products of consecutive integers in 1934 [1] and returned to the structure of such products in 1939 [2]. Nagura’s explicit prime-in-interval theorem later supplied a particularly efficient obstruction for short separated blocks [3], while Erdős developed a broader large-prime-divisor principle for consecutive intervals [4]. Rosser and Schoenfeld’s explicit estimates for Chebyshev functions [5] provide the uniform LCM bound used below.

Exact Diophantine analysis entered through Ponnudurai’s solution of a quartic product equation [6]. Erdős and Straus then studied equations between products of consecutive integers and observed that Thue–Siegel theory yields finiteness when the block length is fixed [7]. Erdős formulated the equal-LCM conjecture in its present form in 1979 [8]. Fixed-length finiteness, however, does not provide the uniform nonexistence in k required by (2).



Subsequent work clarified several neighboring product problems. Saradha and Shorey analyzed ratios of two consecutive blocks by linear forms and Diophantine approximation [9]. Jia and Liu obtained strong large-prime-factor estimates for products in short intervals [10], complementing the prime-gap theorem of Baker, Harman, and Pintz [11]. Almost-square factorizations were treated by Saradha and Shorey [12]; squarefree parts of consecutive products were studied by Mukhopadhyay and Shorey [13]; and Bennett introduced effective high-degree binomial-Thue methods for related products [14]. More recently, Tengely and Ulas investigated products of disjoint arithmetic-progression blocks [15], Hajdu and Tijdeman placed equal products from bounded blocks in a general polynomial framework [16], and Nguyen gave an elementary treatment of a particular equal-product theorem [17]. These results illuminate product identities, large prime factors, or bounded-length families, but they do not control the variable product-to-LCM defect uniformly.

The conjecture is currently recorded as research-open in the Formal Conjectures repository [18] and in the Erdős Problems database [19]. The principal gap in the existing approaches is therefore structural: equality of LCMs must be converted into a product identity without discarding the locations of the maximal prime powers. The method developed here performs that conversion and retains the missing location data as a transport matrix.

The contribution is the retention of prime-power owner positions after normalization. The product-ratio results of Saradha and Shorey [9] impose Diophantine restrictions on block products, while Hajdu and Tijdeman [16] give finiteness criteria for polynomial equations and applications to bounded blocks. The present transport matrix additionally records where each maximal prime power occurs. Propositions 11–12 and Theorem 14 use those constraints to bound the regime with at most two active sites. The length-four exclusion is an application of Ponnudurai’s theorem [6], and the asymptotic bounds combine the transport estimates with the cited prime-distribution theorems [10,11]; these inputs are not claimed as new results.

The main contributions of this work are as follows:

1. An exact carry formula normalizes the product-to-LCM defect to a divisor of $\text{lcm}(1, \dots, k)/k$ and couples its coefficient support to the gap.
2. Every maximal prime power of the common LCM is assigned to a unique edge between its owner positions in the two blocks. The resulting transport packets are pairwise coprime, divide short central differences, and satisfy factorial residual budgets at every vertex.
3. A quadratic edge lift and a primitive rescaling force the reduced gap into a positive arithmetic progression. Maximal-valuation extraction and a finite irrational square-root spectrum prove an effective three-site reduction.
4. The framework excludes adjacent disjoint blocks and gives complete exclusions for $k = 3$ and $k = 4$. For $k = 5$, it reduces the fixed-length problem to seven explicit genus-two curves. Exact rational certificates eliminate all configurations with at most two active sites for $k = 5$ and $k = 6$.
5. Independent consequences include a uniform quadratic lower bound for the gap, an asymptotic $k^{3/2-o(1)}$ bound, linear occupation of transport diagonals, and parity-centred polynomial scale locks.
6. The scope of the result is precise. The arguments below do not eliminate configurations with at least three active primitive sites coupled to a genuinely moved transport packet. They therefore establish a new reduction theorem and several complete subcases, but not a proof of (2) for all k .

Definition 1 (Collision). A disjoint equal-LCM collision is a triple $(n, m, k) \in \mathbb{N}^3$ with $k \geq 1$, $m \geq n + k$, and $M_k(n) = M_k(m)$. Throughout the paper, a hypothetical counterexample means such a collision.

For a prime p and a nonzero integer x , $v_p(x)$ denotes the exponent of p in x ; the notation $p^r \parallel x$ means $v_p(x) = r$. All divisibility and computational arguments below use exact integer or rational arithmetic.

Main Theorem (Structural reduction; theoretical result). Fix $k \geq 3$. For a disjoint equal-LCM collision, let $a < b$ be the coprime coefficients in (13), put $d = m - n$, $h = \gcd(d, b - a)$, $d_0 = d/h$, and $D_0 = (b - a)/h$. Then $\mu_i = D_0(n + i) - ad_0$ is a positive integral progression, $abd_0^2 \mid \prod_{i=1}^k \mu_i$, and the packets of (135) satisfy (136) with a charge dividing $(k - 1)!$. There is an effectively computable constant C_k such that $d_0 > C_k$ forces at least three active packets. Every collision also has a nontrivial off-diagonal transport packet, $m > 5k$, $n \geq k$, and $d > \sqrt{k(m + 1)}/6$. For each fixed $\eta > 0$, it satisfies $d \geq c_\eta k^{3/2-\eta}$ for all sufficiently large k , with $c_\eta > 0$. Adjacent blocks and lengths three and four admit no collision.

The progression and effective threshold follow from Propositions 11–12 and Theorem 14; the transport, location, gap, and fixed-length assertions follow from Proposition 3, Propositions 18 and 22, and Theorems 17, 19–21, and 23. The length-three

assertion uses the unconditional Thue calculation in Table 3. Finite Exact Certificate: Table 1 removes the entire at-most-two-site regime for lengths five and six. Neither statement excludes the remaining regime with three or more active sites.

2. Normalization and the Carry Cocycle

Put

$$P_k(n) = \prod_{i=1}^k (n+i), \quad D_k(n) = \frac{P_k(n)}{M_k(n)}, \quad (3)$$

and write

$$\lambda_k = \text{lcm}(1, 2, \dots, k), \quad B_k = \frac{k!}{\lambda_k}, \quad Q_k = \frac{\lambda_k}{k}. \quad (4)$$

Let $\psi(x) = \sum_{p^r \leq x} \log p$, where the sum ranges over prime powers, denotes Chebyshev's second function. Since $\log \lambda_k = \psi(k)$, the estimate $\psi(x) < 1.03883x$ of Rosser and Schoenfeld [5], together with $1.03883 < \log 3$, gives

$$\lambda_k < 3^k \quad (5)$$

The next valuation estimate supplies the normalization used throughout.

Lemma 1 (Defect bounds). For every $k \geq 1$ and $n \geq 0$,

$$B_k \mid D_k(n) \mid (k-1)!. \quad (6)$$

Proof. Fix a prime p and choose an index $i_p \in \{1, \dots, k\}$ at which $v_p(n+i)$ is maximal. Removing this maximal valuation from the product gives

$$v_p(D_k(n)) = \sum_{i \neq i_p} v_p(n+i). \quad (7)$$

For $i \neq i_p$, the ultrametric inequality implies $v_p(n+i) \leq v_p(i - i_p)$. Hence

$$v_p(D_k(n)) \leq v_p((i_p - 1)!(k - i_p)!) \leq v_p((k-1)!), \quad (8)$$

because $(i_p - 1)!(k - i_p)!$ divides $(k-1)!$.

For the lower bound, put $A_p = \lfloor \log_p k \rfloor$. An interval of k consecutive integers contains at least $\lfloor k/p^r \rfloor$ multiples of p^r for every $1 \leq r \leq A_p$, and it contains at least one such multiple. Counting valuations level by level, therefore, yields

$$v_p(D_k(n)) \geq \sum_{r=1}^{A_p} \left(\left\lfloor \frac{k}{p^r} \right\rfloor - 1 \right) = v_p(k!) - A_p = v_p(B_k). \quad (9)$$

The upper and lower inequalities hold for every prime p , proving (6). $\square$

Consequently, there is an integer $E_k(n)$ such that

$$D_k(n) = B_k E_k(n), \quad E_k(n) \mid Q_k. \quad (10)$$

This removes the universal factor B_k from the collision equation. If $n < m$ and $M_k(n) = M_k(m) = L$, set

$$e = E_k(n), \quad f = E_k(m), \quad d = m - n. \quad (11)$$

Since P_k is strictly increasing, $f > e$. From $P_k(n) = B_k e L$ and $P_k(m) = B_k f L$ one obtains the exact normalized equation

$$e P_k(m) = f P_k(n), \quad e < f, \quad e, f \mid Q_k. \quad (12)$$

The coefficients in (12) are divisors of $Q_k = \lambda_k/k$, rather than divisors of $(k-1)!$. This is an exponential-height reduction of the coefficient space.

There is an equivalent near-square formulation that retains the common-factor structure. Put

$$g = \gcd(e, f), \quad a = \frac{e}{g}, \quad b = \frac{f}{g}. \quad (13)$$

Then $\gcd(a, b) = 1$, $a < b$, and $ab \mid Q_k$. Equation (12) becomes

$$aP_k(m) = bP_k(n). \quad (14)$$

Consequently, there is an integer R for which

$$\begin{aligned} P_k(n) &= aR, \\ P_k(m) &= bR, \\ R &= \gcd(P_k(n), P_k(m)) = B_k g L. \end{aligned} \quad (15)$$

In particular,

$$P_k(n)P_k(m) = abR^2, \quad ab \mid Q_k. \quad (16)$$

Thus, the product of the two disjoint blocks is a perfect square times a coefficient supported entirely on primes at most k . Equivalently, its squarefree kernel divides Q_k . This paired valuation constraint is stronger than merely saying that the two products have the same large prime divisors.

2.1. The Prime-Power Carry Formula

The divisor condition in (10) can be sharpened to an exact formula. Fix a prime $p \leq k$, and put

$$A_p = \lfloor \log_p k \rfloor, \quad u_p = v_p(k). \quad (17)$$

For $1 \leq r \leq A_p$, let

$$N_{p,r}(n) = \#\{1 \leq i \leq k: p^r \mid n + i\}. \quad (18)$$

Counting valuations in the product and then removing the largest valuation in the block gives

$$v_p(D_k(n)) = \sum_{r=1}^{A_p} (N_{p,r}(n) - 1). \quad (19)$$

Indeed, for $p^r > k$ there is at most one multiple of p^r in the block, so its contribution to “sum minus maximum” is zero. Since

$$v_p(B_k) = \sum_{r=1}^{A_p} \lfloor \frac{k}{p^r} \rfloor - A_p, \quad (20)$$

Equation (19) yields

$$v_p(E_k(n)) = \sum_{r=1}^{A_p} \left(N_{p,r}(n) - \lfloor \frac{k}{p^r} \rfloor \right). \quad (21)$$

Each summand is either 0 or 1. It vanishes identically when $p^r \mid k$; otherwise, writing $[n]_{p^r}$ for the least nonnegative residue of n modulo p^r , it equals

$$\epsilon_{p,r}(n) = \mathbf{1}_{[n]_{p^r} \geq p^r - (k \bmod p^r)}. \quad (22)$$

Consequently

$$E_k(n) = \prod_{p \leq k} p^{\sum_{r=u_p+1}^{A_p} \epsilon_{p,r}(n)}. \quad (23)$$

Formula (23) both proves $E_k(n) \mid Q_k$ and exhibits the omitted information: the exponent is a sum of nested prime-power carry bits, not a free choice between 0 and $v_p(Q_k)$.

In particular, $E_k(n)$ is periodic modulo λ_k . More locally, the bit at level p^r is unchanged under $n \mapsto n + d$ whenever $p^r \mid d$. Put

$$s_p = \min\{v_p(d), A_p\}. \quad (24)$$

Only the levels $r > \max\{u_p, s_p\}$ can contribute to $v_p(f) - v_p(e)$. Since a and b are coprime, this proves the gap-sensitive coefficient bound.

$$ab \mid \hat{Q}_k(d) := \prod_{p \leq k} p^{A_p - \max\{u_p, s_p\}} = \frac{\lambda_k}{\text{lcm}(k, \gcd(d, \lambda_k))}. \quad (25)$$

Define the effective defect period

$$\tau_k = \prod_{\substack{p \leq k \\ A_p > u_p}} p^{A_p}. \quad (26)$$

Formula (23) shows that E_k is periodic modulo τ_k , which can be strictly smaller than λ_k . Since $a < b$, the right side of (25) is greater than 1. Equivalently,

$$\tau_k \nmid d. \quad (27)$$

Replacing Q_k by the gap-sensitive quantity $\hat{Q}_k(d)$ The proof of (36) also gives

$$n < k \left(\sqrt{\hat{Q}_k(d)} + 1 \right) d - 1. \quad (28)$$

3. Coefficient Geometry and Gap Constraints

Define

$$\alpha = \left(\frac{f}{e} \right)^{1/k} > 1. \quad (29)$$

Then $\alpha \notin \mathbb{Q}$. Indeed, for every prime p ,

$$v_p(Q_k) = \lfloor \log_p k \rfloor - v_p(k) < k. \quad (30)$$

After cancelling $\gcd(e, f)$, every nonzero prime exponent in f/e therefore has absolute value strictly less than k . A rational k -th power has every prime exponent divisible by k . Since $f/e > 1$, these conditions are incompatible.

Equation (12) also gives the elementary corridor

$$1 + \frac{d}{n+k} < \alpha < 1 + \frac{d}{n+1}, \quad (31)$$

and hence

$$\frac{d}{\alpha-1} - k < n < \frac{d}{\alpha-1} - 1. \quad (32)$$

Using

$$\frac{f-e}{kf} \leq \alpha - 1 \leq \frac{f-e}{ke}, \quad (33)$$

one obtains the rational form

$$\frac{ke}{f-e} d - k < n < \frac{kf}{f-e} d - 1. \quad (34)$$

In particular, because $f \leq Q_k$ Every hypothetical collision satisfies.

$$n < \lambda_k d - 1. \quad (35)$$

The coprimality exposed in (15) improves this exponential corridor. Since $ab \mid Q_k$, one has $ab \leq Q_k$. If $t = b - a$, then

$$\frac{b}{b-a} \leq \sqrt{Q_k} + 1. \quad (36)$$

Indeed, this is immediate if $b \leq \sqrt{Q_k} + 1$. Otherwise $a \leq Q_k/b < \sqrt{Q_k}$, and the function $x/(x - \sqrt{Q_k})$ is decreasing for $x > \sqrt{Q_k}$. Applying (36) to the upper bound in (34) gives

$$n < k(\sqrt{Q_k} + 1)d - 1. \quad (37)$$

Thus, the balanced-factor form $ab \mid Q_k$ saves a square root in the coefficient height; this is stronger than (35), although still far from a contradiction.

There is also an exact congruence coupling the gap to the coefficient difference. Since $P_k(n + d) \equiv P_k(n) \pmod{d}$, equation $aP_k(m) = bP_k(n)$ implies

$$\frac{d}{\gcd(d, P_k(n))} \mid b - a. \quad (38)$$

Here $\gcd(d, P_k(n)) = \gcd(d, P_k(m))$. Thus, every part of the gap not already represented in the block product must fit into the small difference of the two coprime carry coefficients. Equations (25) and (38) are complementary: the former limits the product ab using the prime powers already present in d , while the latter constrains the part of d absent from $P_k(n)$.

4. The Prime-Power Transport Method

Let

$$K_k(d) = \text{lcm}(d - k + 1, d - k + 2, \dots, d + k - 1). \quad (39)$$

All entries are positive when $d \geq k$.

Proposition 2 (Common-LCM bridge). For arbitrary $n < m$ with $d = m - n \geq k$,

$$\gcd(M_k(n), M_k(m)) \mid K_k(d). \quad (40)$$

Proof. Fix a prime p , put

$$p^a \parallel \gcd(M_k(n), M_k(m)), \quad (41)$$

and choose $1 \leq i, j \leq k$ for which $p^a \mid n + i$ and $p^a \mid m + j$. Then

$$p^a \mid (m + j) - (n + i) = d + j - i. \quad (42)$$

The last integer lies in $[d - k + 1, d + k - 1]$. Hence $K_k(d)$ contains p to the exponent at least a . This holds for every p , proving (40). $\square$

In particular, a collision gives

$$L \mid K_k(d). \quad (43)$$

There is a useful binomial version. Write

$$C_k(x) = \binom{x+k}{k}. \quad (44)$$

Equation (10) gives

$$M_k(x) = \frac{\lambda_k}{E_k(x)} C_k(x), \quad (45)$$

so $C_k(x) \mid M_k(x)$. Proposition 2, therefore, implies, without assuming a collision,

$$\gcd(C_k(n), C_k(m)) \mid K_k(d). \quad (46)$$

Under a collision, both binomial coefficients divide L , whence

$$\text{lcm}(C_k(n), C_k(m)) \mid L \mid K_k(d). \quad (47)$$

The factorization in (15) makes (47) exact. Since $k! \mid aR$ and $k! \mid bR$, while $(a, b) = 1$, a prime-by-prime argument gives $k! \mid R$. With $S = R/k!$,

$$\begin{aligned} C_k(n) &= aS, \\ C_k(m) &= bS, \\ \text{lcm}(C_k(n), C_k(m)) &= abS. \end{aligned} \quad (48)$$

Since $S = gL/\lambda_k$ and $abg = \text{lcm}(e, f)$, one has the exact quotient

$$\frac{L}{\text{lcm}(C_k(n), C_k(m))} = \frac{\lambda_k}{\text{lcm}(e, f)} = k \frac{Q_k}{\text{lcm}(e, f)}. \quad (49)$$

In particular,

$$k \text{lcm}(C_k(n), C_k(m)) \mid L \mid K_k(d). \quad (50)$$

Thus, a counterexample must make the LCM of two normalized binomial coefficients divide the LCM of only $2k - 1$ consecutive integers centred at the gap.

4.1. Maximal Prime-Power Packets

The bridge admits a factorization that retains the direction of every prime-power transport. For each prime $p \mid L$, put

$$q_p = p^{v_p(L)}. \quad (51)$$

Choose owner indices $i_p, j_p \in \{1, \dots, k\}$ satisfying $q_p \mid n + i_p$ and $q_p \mid m + j_p$, and set

$$\delta_p = j_p - i_p, \quad Q_\delta = \prod_{\delta_p = \delta} q_p \quad (-(k-1) \leq \delta \leq k-1), \quad (52)$$

with an empty product interpreted as 1.

Owner indices are selected once for each prime and then kept fixed. A maximal prime power exceeding the block length has a unique owner in each block; smaller powers may have several possible owners. The packet identities hold for any fixed choice, because each prime base is assigned to exactly one edge.

Proposition 3 (Diagonal packet factorization). The integers Q_δ are pairwise coprime and satisfy

$$L = \prod_{\delta=-(k-1)}^{k-1} Q_\delta, \quad Q_\delta \mid d + \delta. \quad (53)$$

If

$$\begin{aligned} S_0 &= Q_0, & T &= L/S_0, \\ K_k^*(d) &= \text{lcm}(d - k + 1, \dots, d - 1, \\ &\quad d + 1, \dots, d + k - 1). \end{aligned} \quad (54)$$

then

$$S_0 \mid d, \quad T \mid K_k^*(d), \quad (S_0, T) = 1, \quad T > 1. \quad (55)$$

Proof. Every prime occurs in exactly one packet, so the packets are pairwise coprime and their product is L . The owner congruences give

$$q_p \mid (m + j_p) - (n + i_p) = d + \delta_p. \quad (56)$$

Within a fixed packet, the factors q_p have distinct prime bases; therefore, their product divides $d + \delta$, proving (53) and the first three assertions of (55). If $T = 1$, then $L = S_0 \mid d$. But $m + 1 \mid L$ and $m + 1 = n + d + 1 > d$, an impossibility. $\square$

The last assertion is small but qualitative: no collision can be built entirely from index-preserving transports. At least one maximal prime-power packet must genuinely change its owner index.

The same construction can be refined from diagonals to individual owner pairs. Define

$$W_{ij} = \prod_{(i_p, j_p) = (i, j)} q_p, \quad 1 \leq i, j \leq k, \quad (57)$$

again allowing empty products.

Proposition 4 (Transport matrix with residual budget). The nontrivial entries W_{ij} are pairwise coprime and

$$\prod_{i,j} W_{ij} = L, \quad W_{ij} \mid d + j - i. \quad (58)$$

There are positive integers $x_1, \dots, x_k, y_1, \dots, y_k$ such that

$$n + i = x_i \prod_{j=1}^k W_{ij}, \quad m + j = y_j \prod_{i=1}^k W_{ij}, \quad (59)$$

and the two residual budgets are exactly

$$\prod_{i=1}^k x_i = D_k(n) = B_k e, \quad \prod_{j=1}^k y_j = D_k(m) = B_k f. \quad (60)$$

In addition, every vertex residual satisfies

$$x_i \mid (i-1)!(k-i)!, \quad y_j \mid (j-1)!(k-j)!. \quad (61)$$

In particular, both products in (60) are at most $(k-1)!$.

Proof. Distinct matrix entries contain disjoint sets of prime bases, which proves coprimality, and the owner congruence proves (58). The product of the entries in row i divides $n + i$, while the product in column j divides $m + j$; this defines the integers in (59). Multiplying all row identities and then all column identities gives (60), because the product of all matrix entries is L .

For (61), fix i . If a prime p divides x_i , its selected lower owner $h = i_p$ is different from i ; otherwise, the full maximal power q_p would have been removed from $n + i$. The power $p^{v_p(x_i)}$ divides both $n + i$ and $n + h$, and therefore divides $|i - h|$. Grouping the prime powers according to h , and using their coprimality gives

$$x_i \mid \prod_{h \neq i} |i - h| = (i-1)!(k-i)!. \quad (62)$$

Prime powers assigned to the same lower owner all divide the same nonzero index difference. Their prime bases are distinct, so their product divides that difference. Multiplication over the other owners gives (62), without counting any prime base twice.

The column argument is identical.

Corollary 5 (Vertex-LCM compression). Put

$$U_i = \prod_{j=1}^k W_{ij}, \quad V_j = \prod_{i=1}^k W_{ij}. \quad (63)$$

Then

$$U_i \mid \text{lcm}(d + 1 - i, \dots, d + k - i), \quad (64)$$

$$V_j \mid \text{lcm}(d + j - k, \dots, d + j - 1). \quad (65)$$

Consequently

$$n + i \mid (i-1)!(k-i)! \text{lcm}(d + 1 - i, \dots, d + k - i), \quad (66)$$

$$m + j \mid (j-1)!(k-j)! \text{lcm}(d + j - k, \dots, d + j - 1). \quad (67)$$

Here juxtaposition between the factorial factor and the LCM means multiplication.

Proof. In a fixed row or column, the nontrivial W_{ij} are pairwise coprime, and each divides its corresponding integer $d + j - i$. Their product, therefore, divides the LCM of those k integers. Combining this with (59) and the residual divisibilities (61) proves (66)–(67).

This is the most economical form of the transport method: all of the large arithmetic is stored in pairwise coprime edge weights constrained by the short differences $d + j - i$, while every factor not transported at maximal prime-power level is charged to one of two factorial-size residual budgets.

Figure 1 illustrates how selected owner positions determine diagonal and moved transport packets.

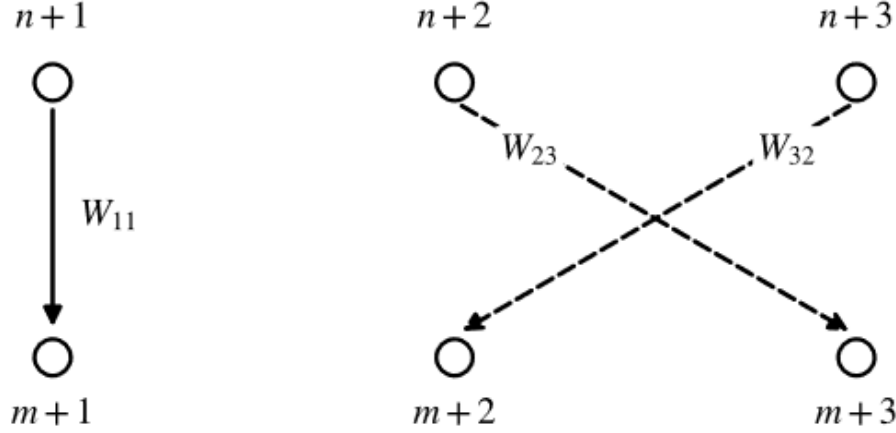


Fig. 1 Schematic transport between selected block positions. (Solid and dashed edges show diagonal and moved packets. The diagram represents the construction, not a numerical equal-LCM collision)

There is a second-order constraint on every large-prime-power edge. It is useful to record first a divisibility coming directly from the normalized product identity. Put

$$D = b - a, \quad h = \gcd(d, D), \quad d_0 = \frac{d}{h}. \quad (68)$$

Since $P_k(m) \equiv P_k(n) \pmod{d}$, Equation (15) gives $d \mid DP_k(n)$. Writing $D = hD_0$ one first obtains $d_0 \mid P_k(n)$. If $P_k(n) = d_0Z$, division of

$$DP_k(n) = a(P_k(m) - P_k(n)) \quad (69)$$

by hd_0 , together with $(a, D_0) = 1$, shows that $a \mid Z$. Consequently

$$d_0 \mid R, \quad ad_0 \mid P_k(n), \quad bd_0 \mid P_k(m). \quad (70)$$

Proposition 6 (Large-prime-power quadratic lift). Define the signed endpoint weights.

$$A_i = (-1)^{i-1} (i-1)! (k-i)!. \quad (71)$$

For every $p \mid L$, retain $q_p = p^{v_p(L)}$. Whenever $q_p > k$, its owner in each block is unique. For an owner pair (i, j) , put

$$H_{ij} = \prod_{\substack{q_p > k \\ (i_p, j_p) = (i, j)}} q_p. \quad (72)$$

Also put

$$L_{\leq k} = \prod_{q_p \leq k} q_p. \quad (73)$$

Then

$$L = L_{\leq k} \prod_{i, j} H_{ij}, \quad L_{\leq k} \mid \lambda_k, \quad (74)$$

and every edge packet satisfies

$$H_{ij}^2 \mid bA_i(n+i) - aA_j(m+j). \quad (75)$$

There is an exact primitive normalization, including when the prime base is at most k . Put

$$\begin{aligned} C_i &= \binom{k-1}{i-1}, \\ \gamma_{ij} &= \gcd(bC_j, aC_i), \\ \text{you}_{ij} &= \frac{bC_j}{\gamma_{ij}}, \quad v_{ij} = \frac{aC_i}{\gamma_{ij}}, \end{aligned} \quad (76)$$

and, for $p \mid H_{ij}$, define the integral coefficient charge

$$\chi_{ij}(p) = v_p((k-1)! \gamma_{ij}) - v_p(C_i C_j). \quad (77)$$

Then $(\text{you}_{ij}, v_{ij}) = 1$, and

$$\begin{aligned} \mathfrak{H}_{ij} &:= \prod_{p \mid H_{ij}} p^{\max\{0, 2v_p(H_{ij}) - \chi_{ij}(p)\}} \\ &\mid (-1)^{i-1} \text{you}_{ij}(n+i) - (-1)^{j-1} v_{ij}(m+j). \end{aligned} \quad (78)$$

For the subpacket

$$W_{ij}^{>k} = \prod_{\substack{p > k \\ (i_p, j_p) = (i, j)}} q_p, \quad (79)$$

one has $\chi_{ij}(p) = 0$, so (78) recovers the coefficient-free square divisibility

$$(W_{ij}^{>k})^2 \mid (-1)^{i-1} \text{you}_{ij}(n+i) - (-1)^{j-1} v_{ij}(m+j). \quad (80)$$

In particular, if $bA_i = aA_j$, then

$$(W_{ij}^{>k})^2 \mid d + j - i. \quad (81)$$

If the primitive linear form in (78) is nonzero, then

$$\begin{aligned} \mathcal{H}_{ij} &\leq (\text{you}_{ij} + v_{ij})(m+k) \\ &\leq (a+b) \binom{k-1}{\lfloor (k-1)/2 \rfloor} (m+k). \end{aligned} \quad (82)$$

Proof. Fix $q = q_p > k$ in the packet. A block of length $k < q$ contains at most one multiple of q , so write its unique owner vertices as $n+i = qx$ and $m+j = qy$. The polynomial centred at its i -th root has the exact expansion.

$$F_i(z) := \prod_{t=1}^k (z+t-i) = A_i z + z^2 K_i(z), \quad K_i \in \mathbb{Z}[z]. \quad (83)$$

Thus $P_k(n) = F_i(qx)$ and $P_k(m) = F_j(qy)$. Substitution in $aP_k(m) = bP_k(n)$, followed by division by q , gives

$$bA_i x - aA_j y \equiv 0 \pmod{q}. \quad (84)$$

Multiplication by q proves the q^2 -divisibility in (75). Distinct prime bases multiply, proving the packet assertion. Every omitted maximal prime power is at most k , so their product divides λ_k , proving (74).

Only the linear Taylor term remains after division by q and reduction modulo q : every higher term originally contains q^2 . This is why the resulting divisibility has exponent two. The passage from a single prime power to H_{ij} uses coprimality of the prime-power factors, rather than a separate congruence assumption.

Finally,

$$\begin{aligned} &C_i C_j (bA_i(n+i) - aA_j(m+j)) \\ &= (k-1)! \gamma_{ij} \left((-1)^{i-1} u_{ij}(n+i) - (-1)^{j-1} v_{ij}(m+j) \right). \end{aligned} \quad (85)$$

Taking p -adic valuations gives (77)–(78). When $p > k$, none of $C_i, C_j, (k-1)!$, or γ_{ij} is divisible by p , which proves (80). The cancellation case and the explicit size bound then follow exactly as stated. $\square$

The charge in (77) is not merely an upper bound. On every occupied large-prime-power edge, it is determined exactly by the two endpoint factorials, and the quotient of the central difference satisfies an exact saturation law.

Proposition 7 (Endpoint balance and edge saturation). Fix $p \mid H_{ij}$, write

$$q_p = p^e, \quad \alpha_i = v_p(|A_i|), \quad \alpha_j = v_p(|A_j|), \quad (86)$$

and retain $C_0 = R/L$. Then

$$\begin{aligned} v_p(C_0) &= \min(\alpha_i, \alpha_j), \\ v_p(a) &= \max(0, \alpha_i - \alpha_j), \\ v_p(b) &= \max(0, \alpha_j - \alpha_i), \\ \chi_{ij}(p) &= \max(\alpha_i, \alpha_j). \end{aligned} \quad (87)$$

In particular $p \nmid \text{you}_{ij} v_{ij}$. Define the saturation packet.

$$\mathcal{S}_{ij} = \prod_{p \mid H_{ij}} p^{\max\{0, v_p(H_{ij}) - \chi_{ij}(p)\}}, \quad (88)$$

put $\varepsilon_t = (-1)^{t-1}$, and set

$$c_{ij} = \varepsilon_i \text{you}_{ij} - \varepsilon_j v_{ij}. \quad (89)$$

Then the divisor in (78) can be strengthened to

$$H_{ij} \mathcal{S}_{ij} \mid \varepsilon_i \text{you}_{ij}(n+i) - \varepsilon_j v_{ij}(m+j), \quad (90)$$

and the cofactor of the transported central difference obeys the exact identity

$$\gcd\left(\mathcal{S}_{ij}, \frac{d+j-i}{H_{ij}}\right) = \gcd(\mathcal{S}_{ij}, |c_{ij}|). \quad (91)$$

For the rough packet $W_{ij}^{>k}$, where every charge vanishes, this specializes to

$$\gcd\left(W_{ij}^{>k}, \frac{d+j-i}{W_{ij}^{>k}}\right) = \gcd(W_{ij}^{>k}, |bA_i - aA_j|). \quad (92)$$

Thus, the exponent by which a central difference overlaps its transported maximal power is not free: it is exactly the truncated exponent of a fixed edge coefficient. In particular $(W_{ij}^{>k})^2 \mid d+j-i$ if and only if $W_{ij}^{>k} \mid bA_i - aA_j$.

Proof. The unique lower owner contributes e to the common LCM and α_i to the lower defect; the corresponding upper contribution is α_j . Since

$$D_k(n) = C_0 a, \quad D_k(m) = C_0 b, \quad (a, b) = 1, \quad (93)$$

Comparison of p -adic valuations gives the first three identities in (87). Substitution in (77) gives the fourth and shows that u_{ij}, v_{ij} are p -adic units.

To justify these defect valuations explicitly, write $q_p = p^e > k$. For $t \neq i$, the inequality $|t-i| < p^e$ gives $v_p(n+t) = v_p(t-i)$. Summing over $t \neq i$ yields α_i ; the upper block gives α_j . Since a and b are coprime, subtracting the smaller of these valuations gives the first three identities in (87).

Write $n+i = q_p x$, $m+j = q_p y$, and $d+j-i = q_p z$. Exact maximality of q_p in both blocks gives $p \nmid xy$. The normalized linear form divided by q_p is

$$c_{ij}x - \varepsilon_j v_{ij}z. \quad (94)$$

The charged lift (78), combined with the automatic factor q_p coming from both owner terms, shows that this expression vanishes modulo

$$p^{\max\{0, e - \chi_{ij}(p)\}}. \quad (95)$$

Both x and $\varepsilon_j v_{ij}$ are units. Two congruent integers have the same valuation truncated at the modulus exponent, so

$$\begin{aligned} & \min \left(v_p(z), \max \left(0, e - \chi_{ij}(p) \right) \right) \\ &= \min \left(v_p(c_{ij}), \max \left(0, e - \chi_{ij}(p) \right) \right). \end{aligned} \quad (96)$$

Multiplication over the distinct prime bases in the packet proves (90)–(91). When $p > k$, all endpoint and primitive charges vanish; rescaling c_{ij} by a p -adic unit gives (92). $\square$

There is also a completion statement for the entire large diagonal packet, not only for the part forced by the reduced gap. Put

$$\Sigma_i = H_{ii}, \quad \Sigma = \prod_{i=1}^k \Sigma_i, \quad \bar{\Sigma}_i = \frac{\Sigma_i}{\gcd(\Sigma_i, |A_i|)}. \quad (97)$$

Corollary 8 (Whole diagonal packet). One has

$$\frac{\Sigma_i^2}{\gcd(\Sigma_i, |A_i|)} = \Sigma_i \bar{\Sigma}_i \mid \ell_i, \quad (98)$$

$$\gcd \left(\bar{\Sigma}_i, \frac{d}{\Sigma_i} \right) = \gcd(\bar{\Sigma}_i, D), \quad (99)$$

and

$$\Sigma^2 \leq C_0 \prod_{\Sigma_i > 1} \ell_i. \quad (100)$$

Proof. On a diagonal edge $u_{ii} = b$, $v_{ii} = a$, and $c_{ii} = \varepsilon_i D$. Equations (90)–(91) therefore give (98)–(99). Formula (87) gives $v_p(C_0) = v_p(|A_i|)$ at every prime assigned to Σ_i , so the pairwise-disjoint local charges satisfy

$$\prod_i \gcd(\Sigma_i, |A_i|) \mid C_0. \quad (101)$$

Multiplying (98) and using the positivity of the ℓ_i proves (100). $\square$

The essential threshold is therefore the size of the transported prime power, not the size of its prime base. Formula (74) shows that these packets contain the whole common LCM except for one factor. λ_k .

Proposition 9 (Forced diagonal completion of the gap). Retain the collision notation, and remove the ambiguity between the two gcds by writing.

$$\begin{aligned} C_0 &= B_k \gcd(e, f) = \frac{R}{L}, \\ D &= b - a, \\ h &= \gcd(d, D), \\ d_0 &= \frac{d}{h}. \end{aligned} \quad (102)$$

Define

$$\begin{aligned} \eta &= \frac{d_0}{\gcd(d_0, C_0)}, \\ \eta_{lo} &= \prod_{p^{v_p(\eta)} \leq k} p^{v_p(\eta)}, \\ \eta_{hi} &= \frac{\eta}{\eta_{lo}}. \end{aligned} \quad (103)$$

Then $\eta \mid L$ and $\eta_{l_0} \mid \lambda_k$. For every prime $p \mid \eta_{hi}$, the entire maximal power $q_p = p^{v_p(L)}$ lies on a diagonal edge and divides d . Allocate these maximal powers by their diagonal owner:

$$\Theta_i = \prod_{p \mid \eta_{hi}} q_p, \quad \Theta = \prod_{i=1}^k \Theta_i, \quad r = \#\{i: \Theta_i > 1\}. \quad (104)$$

Thus $\eta_{hi} \mid \Theta \mid d$. With

$$\ell_i = D(n+i) - ad = b(n+i) - a(m+i), \quad (105)$$

one has

$$J_i := \frac{\Theta_i^2}{\gcd(\Theta_i, |A_i|)} \mid \ell_i, \quad 0 < \ell_i < b(k-1)d. \quad (106)$$

Moreover, the product of all local coefficient charges divides the already present common defect factor C_0 . Consequently

$$\eta_{hi}^2 \leq \Theta^2 \leq C_0 \prod_{\Theta_i > 1} \ell_i. \quad (107)$$

If $r > 0$, the last member is strictly smaller than $C_0(b(k-1)d)^r$; if $r = 0$, then $\eta_{hi} = \Theta = 1$.

Prime by prime, if $p \mid \eta_{hi}$ has owner i , then

$$q_p \mid d, \quad p^{v_p(L) + \max\{0, v_p(L) - v_p(C_0)\}} \mid \ell_i. \quad (108)$$

Proof. Equation (70) gives $d_0 \mid R = C_0 L$, hence $\eta \mid L$. If $p^s \parallel \eta$ and $p^s \leq k$, then $s \leq \lfloor \log_p k \rfloor$, which proves $\eta_{l_0} \mid \lambda_k$.

Now suppose $p^s \parallel \eta$ and $p^s > k$. Since $p^s \mid d$ and $p^s \mid q_p$, while $q_p \mid d + j_p - i_p$, one has $p^s \mid j_p - i_p$. The strict inequality $|j_p - i_p| < k < p^s$ forces $j_p = i_p$. Hence the full q_p is diagonal, and then $q_p \mid d$. Hence $\Theta_i \mid \ell_i$ automatically, while the diagonal case of (75) gives $\Theta_i^2 \mid A_i \ell_i$. Taking the stronger of these two divisibilities, prime by prime, proves (106).

There is no new factorial loss here. For an activated prime with owner i , $q_p > k$, and for every $t \neq i$, the ultrametric inequality gives

$$v_p(n+t) = v_p(t-i). \quad (109)$$

The same holds in the upper block. Removing the unique maximal valuation, therefore, gives

$$v_p(D_k(n)) = v_p(D_k(m)) = \sum_{t \neq i} v_p(t-i) = v_p(|A_i|). \quad (110)$$

Since $D_k(n) = C_0 a$, $D_k(m) = C_0 b$, and $(a, b) = 1$, it follows that $p \nmid ab$ and

$$v_p(C_0) = v_p(|A_i|). \quad (111)$$

The Θ_i have disjoint prime bases. Thus

$$\prod_i \gcd(\Theta_i, |A_i|) \mid C_0. \quad (112)$$

Multiplying the divisibilities $J_i \mid \ell_i$ proves the middle inequality in (107).

For each activated prime, the two available exponents are e and $2e - v_p(A_i)$. Their maximum is $2e - \min\{e, v_p(A_i)\}$, exactly the exponent in J_i . Equation (112) then bounds the product of the local losses by the single common factor. C_0 .

For positivity and size, put

$$\rho_t = \frac{m+t}{n+t} > 1, \quad \prod_{t=1}^k \rho_t = \frac{b}{a}. \quad (113)$$

The product is strictly larger than each factor, so $\ell_i > 0$. Moreover

$$\ell_i = a(m+i)(\prod_{t \neq i} \rho_t - 1). \quad (114)$$

Applying

$$\prod_t (1 + x_t) - 1 \leq \prod_t (1 + x_t) \sum_t \frac{x_t}{1+x_t} \quad (115)$$

with $x_t = d/(n + t)$, and using $n + i \leq n + k \leq m$, gives

$$\ell_i \leq b(n + i) \sum_{t \neq i} \frac{d}{m+t} < b(k - 1)d. \quad (116)$$

This completes (106)–(108). $\square$

Corollary 10 (Explicit two-site threshold). If $r = 0$, then

$$d \leq hC_0\lambda_k. \quad (117)$$

If $r = 1$, then

$$d < h^2 C_0^3 b(k - 1)\lambda_k^2. \quad (118)$$

In particular, since $h < b \leq Q_k$, $C_0 \leq (k - 1)!$, and $C_0 b = B_k f \leq B_k Q_k = (k - 1)!$, every hypothetical counterexample with

$$d \geq \mathcal{B}_k := (k - 1)Q_k^2((k - 1)!)^3\lambda_k^2 \quad (119)$$

has $r \geq 2$. It therefore has full maximal prime powers on at least two distinct diagonal owner positions, and their charged squares divide two distinct terms of the progression (ℓ_i) .

Proof. For $r = 0$, one has $\eta = \eta_{l_0} \leq \lambda_k$, while $d_0 \leq C_0\eta$, giving the first assertion. If $r = 1$, then (107) gives

$$\eta_{hi}^2 < C_0 b(k - 1)d. \quad (120)$$

On the other hand,

$$d = hd_0 \leq hC_0\eta \leq hC_0\lambda_k\eta_{hi}. \quad (121)$$

Squaring and cancelling $d > 0$ proves (118). The displayed bounds on $h, C_0, C_0 b$ give (119). $\square$

The preceding completion begins with the common LCM. There is a second, owner-free completion which begins directly with the normalized product identity and captures *every* prime of the reduced gap.

Put

$$D_0 = \frac{b-a}{h}, \quad d_0 = \frac{d}{h}, \quad \mu_i = \frac{\ell_i}{h} = D_0(n + i) - ad_0. \quad (122)$$

Thus $(D_0, d_0) = 1$. For $s \geq 0$, write

$$\mathcal{E}_s(\mu) = \sum_{1 \leq i_1 < \dots < i_s \leq k} \mu_{i_1} \cdots \mu_{i_s}, \quad \mathcal{E}_0 = 1, \quad (123)$$

and put

$$\mathcal{S}_s(a, b) = \sum_{v=0}^s a^v b^{s-v} = \frac{b^{s+1} - a^{s+1}}{b - a}. \quad (124)$$

Proposition 11 (Primitive gap progression). The integers $\mu_1, \dots, \mu_k$ are positive and satisfy

$$\begin{aligned} \mu_j - \mu_i &= D_0(j - i), \\ \gcd(\mu_i, D_0) &= 1, \\ \gcd(\mu_i, \mu_j) &\mid |j - i|. \end{aligned} \quad (125)$$

They obey the exact identity

$$\prod_{i=1}^k \mu_i = ab d_0^2 \sum_{s=0}^{k-2} \mathcal{S}_s(a, b) d_0^s \mathcal{E}_{k-2-s}(\mu). \quad (126)$$

In particular,

$$ab d_0^2 \mid \prod_{i=1}^k \mu_i. \quad (127)$$

When $d \geq k$, one also has

$$0 < \mu_i < b(k-1)d_0. \quad (128)$$

Proof. Positivity and (128) are (106) divided by h . The progression relation is immediate. Since $(a, D_0) = (d_0, D_0) = 1$,

$$\mu_i \equiv -ad_0 \pmod{D_0} \quad (129)$$

is a unit. A common divisor of μ_i, μ_j therefore divides $D_0(j-i)$ and is coprime to D_0 , proving (125).

Solving (122) for the two block terms gives

$$D_0(n+i) = \mu_i + ad_0, \quad D_0(m+i) = \mu_i + bd_0. \quad (130)$$

The product equation becomes

$$a \prod_i (\mu_i + bd_0) = b \prod_i (\mu_i + ad_0). \quad (131)$$

In its elementary-symmetric expansion, the terms of degree zero in d_0 have coefficient $a-b$, the terms of degree one cancel, and the coefficient at degree $r \geq 2$ is

$$ab(b^{r-1} - a^{r-1}) = ab(b-a)S_{r-2}(a, b). \quad (132)$$

More explicitly, the coefficient of $d_0^r E_{k-r}(\mu)$ in the left side of (131), after moving the right side across, is $ab^r - ba^r$. It equals $a-b$ for $r=0$, is zero for $r=1$, and equals $ab(b-a)S_{r-2}(a, b)$ for $r \geq 2$. Moving the constant term to the other side and dividing by $b-a$ gives (126).

Cancelling $b-a$ proves (126), hence (127). $\square$

The divisibility (127) has a canonical packet extraction with only one factorial charge. Put

$$N_0 = ab d_0^2, \quad F_k = (k-1)!. \quad (133)$$

For every $p \mid N_0$, choose an index $\iota(p)$ at which $v_p(\mu_i)$ is maximal, and set

$$v_p = v_p(N_0), \quad \alpha_p = v_p(|A_{\iota(p)}|). \quad (134)$$

Define pairwise-coprime site packets and their total charge by

$$\mathcal{J}_i = \prod_{\iota(p)=i} p^{\max(0, v_p - \alpha_p)}, \quad \mathcal{C} = \prod_{p \mid N_0} p^{\min(v_p, \alpha_p)}. \quad (135)$$

Proposition 12 (Maximal-valuation extraction). One has

$$\mathcal{J}_i \mid \mu_i, \quad N_0 = \mathcal{C} \prod_i \mathcal{J}_i, \quad \mathcal{C} \mid F_k. \quad (136)$$

If

$$q = \#\{i: \mathcal{J}_i > 1\}, \quad (137)$$

then

$$ab d_0^2 \leq F_k \prod_{\mathcal{J}_i > 1} \mu_i. \quad (138)$$

Consequently, if $d \geq k$, then

$$\begin{aligned} q = 0 &\Rightarrow ab d_0^2 \leq F_k, \\ q = 1 &\Rightarrow a d_0 < F_k(k-1). \end{aligned} \quad (139)$$

Proof. Fix p and put $i = \iota(p)$. Maximality and (125) imply, for $j \neq i$,

$$v_p(\mu_j) = v_p(\gcd(\mu_i, \mu_j)) \leq v_p(|j-i|). \quad (140)$$

Therefore

$$\sum_{j \neq i} v_p(\mu_j) \leq v_p((i-1)!(k-i)!) = \alpha_p. \quad (141)$$

The factorization in (136) is exact because $\min\{v_p, \alpha_p\} + \max\{0, v_p - \alpha_p\} = v_p$ for every prime. Each prime is assigned to one site only, so the nontrivial site packets are pairwise coprime. The factorial charge therefore occurs once in (138), not once per active site.

Since (127) gives $\sum_j v_p(\mu_j) \geq v_p$, the owner contains at least $\max(0, v_p - \alpha_p)$ copies of p . This proves the first divisibility in (136). The remaining identities follow prime by prime; in particular $\alpha_p \leq v_p(F_k)$, so $\mathcal{C} \mid F_k$. Multiplication gives (138). If there is one active site, (128) gives

$$ab d_0^2 \leq F_k \mu_i < F_k b(k-1)d_0, \quad (142)$$

which proves (139). $\square$

For rough primes, the square packet has a further exact saturation law. Write

$$\prod_{t=1}^k (z + t - i) = A_i z + B_i z^2 + C_i^{(3)} z^3 + \dots, \quad (143)$$

so

$$B_i = A_i \sum_{t \neq i} \frac{1}{t-i} \in \mathbb{Z}. \quad (144)$$

For every $p > k$ dividing d_0 , put $q_p = p^{v_p(d_0)}$. Equation (70), together with $p \nmid a$, gives a unique owner. i_p with $q_p \mid n + i_p$. Allocate these powers by

$$Z_i = \prod_{\substack{p > k, p \mid d_0 \\ i_p = i}} q_p, \quad d_{0, > k} = \prod_i Z_i. \quad (145)$$

Proposition 13 (Second Taylor saturation). For every i ,

$$Z_i^2 \mid \mu_i, \quad \gcd\left(Z_i, \frac{\mu_i}{Z_i^2}\right) = \gcd(Z_i, |B_i|). \quad (146)$$

If $k = 2r + 1$, then $B_{r+1} = 0$, and only that middle coefficient vanishes. Hence, the central rough packet satisfies

$$\begin{aligned} Z_{r+1}^3 \mid \mu_{r+1}, \\ \gcd\left(Z_{r+1}, \frac{\mu_{r+1}}{Z_{r+1}^3}\right) = \gcd(Z_{r+1}, |(a+b)C_{r+1}^{(3)}|). \end{aligned} \quad (147)$$

In particular,

$$Z_{r+1}^3 < b(k-1)d_0. \quad (148)$$

Proof. Fix $q = q_p$ with owner i . Since $p > k$, the other $\mu_j = \mu_i + D_0(j-i)$ are p -adic units. Divisibility (127), therefore, forces $q^2 \mid \mu_i$. Let

$$\Psi = \sum_{s=0}^{k-2} \mathcal{S}_s(a, b) d_0^s \mathcal{E}_{k-2-s}(\mu) = \frac{\prod_j \mu_j}{ab d_0^2}. \quad (149)$$

All factors outside μ_i/q^2 in the last quotient are p -adic units. Moreover, modulo q ,

$$\Psi \equiv \mathcal{E}_{k-2}(\mu) \equiv D_0^{k-2} B_i \pmod{q}. \quad (150)$$

In $E_{k-2}(\mu)$, every summand containing μ_i vanishes modulo q . The surviving summands are obtained by omitting i and one further index. Their sum is $D_0^{k-2} A_i \sum_{t \neq i} (t-i)^{-1} = D_0^{k-2} B_i$, which proves (150). All denominators here are units at $p > k$.

Thus

$$\min(v_p(\mu_i) - 2v_p(q), v_p(q)) = \min(v_p(B_i), v_p(q)), \quad (151)$$

which multiplies over distinct rough primes to give (146).

The equality $B_i = 0$ is equivalent to

$$\sum_{t \neq i} \frac{1}{t-i} = H_{k-i} - H_{i-1} = 0, \quad (152)$$

which occurs exactly at the middle index of an odd block. The first (146) gives $q^3 \mid \mu_i$. Dividing Ψ by q and reducing once more modulo q leaves only.

$$(a+b) \frac{d_0}{q} D_0^{k-3} C_i^{(3)} \quad (153)$$

At the middle index, the constant term of $E_{k-2}(\mu)$ vanishes because $B_i = 0$, while its remaining terms are multiples of μ_i and hence of q^3 . After division by q , these terms vanish modulo q . Terms of degree of at least two in d_0 also vanish modulo q , leaving precisely the term displayed in (153). This proves (147), and (148) follows from (128). $\square$

The decisive feature of (138) is that the exponent of d_0 is exactly two. When only two sites are active, the remaining cofactors therefore range over a finite spectrum.

Let $\zeta = \zeta_{k,a,b} > 0$ be the unique positive solution of

$$a(\zeta + b)^k = b(\zeta + a)^k. \quad (154)$$

For fixed k, a, b , define the finite set

$$Z_{k,a,b} = \left\{ \sqrt{\frac{abt}{c}} : C \mid F_k, \quad t \in \mathbb{Z}_{>0}, \quad at < Cb(k-1)^2 \right\}, \quad (155)$$

and its separation from the slope root

$$\Delta_{k,a,b} = \min_{z \in Z_{k,a,b}} |\zeta - z|. \quad (156)$$

Theorem 14 (Finite two-site spectrum). For $k \geq 3$,

$$\zeta_{k,a,b}^2 \notin \mathbb{Q}, \quad (157)$$

so $\Delta_{k,a,b} > 0$. If exactly two packets in (135) are nontrivial, then

$$d_0 \leq \frac{D_0(k-1)}{\Delta_{k,a,b}} \leq \frac{(b-a)(k-1)}{\Delta_{k,a,b}}. \quad (158)$$

Consequently, for every fixed admissible pair $a < b$, all identities with

$$d_0 > \max \left\{ \sqrt{\frac{F_k}{ab}}, \frac{F_k(k-1)}{a}, \frac{(b-a)(k-1)}{\Delta_{k,a,b}} \right\} \quad (159)$$

have at least three active primitive packets. Since the collision condition $ab \mid Q_k$ leaves only finitely many pairs, the maximum of (159) is an explicit three-site threshold depending only on k .

The hypothesis $ab \mid Q_k$ Moreover, the coprimality of a, b remains in force here. They are essential to the irrationality argument in Section 3. The active-site count in this theorem is q from (137), rather than the count r of forced diagonal owners in (104).

Proof. Suppose the active sites are $r < s$, and write.

$$\mu_r = \mathcal{J}_r x, \quad \mu_s = \mathcal{J}_s y. \quad (160)$$

Equations (136) and (128) give

$$xy < \frac{cb(k-1)^2}{a}, \quad (161)$$

and, exactly,

$$\frac{\sqrt{\mu_r \mu_s}}{d_0} = \sqrt{\frac{abxy}{c}} \in Z_{k,a,b}. \quad (162)$$

For completeness, the pair also has a finite generalized-Pell description. If c_r, c_s are the squarefree kernels of $\mathcal{J}_r, \mathcal{J}_s$, and $\mathcal{J}_r = c_r u^2, \mathcal{J}_s = c_s v^2$, then

$$c_s y v^2 - c_r x u^2 = D_0(s - r), \quad (163)$$

where xy is bounded by (161) and every prime in $c_r c_s$ is at most k .

Put $z_i = \mu_i/d_0$ and $\phi(z) = \log((z+b)/(z+a))$. Dividing (131) by d_0^k gives

$$\frac{1}{k} \sum_{i=1}^k \phi(z_i) = \frac{1}{k} \log \frac{b}{a} = \phi(\zeta). \quad (164)$$

The function ϕ is strictly decreasing, so ζ lies between the least and greatest z_i . The geometric mean in (162) also lies in that interval. Since the full progression has a width $D_0(k-1)/d_0$,

$$\left| \zeta - \frac{\sqrt{\mu_r \mu_s}}{d_0} \right| \leq \frac{D_0(k-1)}{d_0}. \quad (165)$$

This proves (158) once (157) is known.

Put $\alpha = (b/a)^{1/k}$. Then

$$\zeta = \frac{b-a\alpha}{\alpha-1}. \quad (166)$$

Section 3 proves $\alpha \notin \mathbb{Q}$, hence $\zeta \notin \mathbb{Q}$. If $\zeta^2 \in \mathbb{Q}$, the quadratic conjugate $-\zeta$ would also be a root of

$$a(z+b)^k - b(z+a)^k. \quad (167)$$

Indeed, if ζ were rational, then $\alpha = (\zeta+b)/(\zeta+a)$ would be rational. If ζ^2 were rational, its irrational square root ζ would have a minimal polynomial $z^2 - \zeta^2$ over $\mathbb{Q}$. Every rational polynomial vanishing at ζ would therefore also vanish at $-\zeta$, justifying the conjugate-root step.

For odd k , the equation has only the real ratio. $(z+b)/(z+a) = \alpha$, whose Möbius map is injective. For even k , the only alternative is the ratio $-\alpha$. Requiring the corresponding root to equal $-\zeta$ gives

$$\frac{b+a\alpha}{\alpha+1} = \frac{b-a\alpha}{\alpha-1}, \quad (168)$$

and hence $\alpha^2 = b/a = \alpha^k$. Since $k > 2$ and $\alpha > 1$, this is impossible. Thus ζ^2 is irrational, the finite separation is positive, and (159) follows from (139) and (158). $\square$

Finite Exact Certificate. The finite spectrum is effectively certifiable without floating point. The companion program `verify_erdos677_three_site.py` isolates ζ and the neighboring square roots by rational bisection, then exhausts the complete range (158). The certified bounds are reported in Table 1.

Table 1. Exact bounds for the at-most-two-site regime.

k	b/a	certified d_0 bound	exact reduced-gap tests
5	2	1,768	1,764
5	3	3,271	4,903
5	4	4,079	6,795
5	6	4,131	7,432
5	12	7,158	13,662
5	3/2	1,889	1,885
5	4/3	544	540
6	2	51,930	51,925
6	5	51,101	102,198
6	10	343,813	802,226
6	5/2	11,945	19,904

No disjoint normalized product identity occurs in any of these ranges. Together with Theorem 14, this proves computationally, with exact rational and integer arithmetic, that every still-possible collision for $k = 5$ or $k = 6$ has at least three active primitive packets. It does not exclude the three-site regime.

Restricting Proposition 9 to prime bases $p > k$ recovers the simpler coefficient-free rough-gap lock. If

$$d_{0,>k} = \prod_{p>k} p^{v_p(d_0)}, \quad (169)$$

and its factors are allocated to diagonal packets Z_i , then

$$\begin{aligned} d_{0,>k} &= \prod_i Z_i, \\ Z_i^2 &\mid \mu_i, \\ d_{0,>k} &< (b(k-1)d_0)^{k/2}, \end{aligned} \quad (170)$$

and $p^{2v_p(d_0)} < b(k-1)d_0$ for every $p > k$. Proposition 9 closes the previously untreated high-power/small-base loophole, while Propositions 11–14 subsume the rough lock into a primitive all-prime extraction. The remaining case is a packing in three or more terms of (μ_i) , together with at least one moved transport packet; any directed transport cycle contains a nonvanishing edge by Proposition 15. The two-site Pell alternative is bounded and effectively enumerable rather than an unbounded obstruction.

There is one more rigidity in the lifted graph.

Proposition 15 (Vanishing edges are directed paths). Among the occupied packets H_{ij} , call an edge vanishing if

$$bA_i(n+i) - aA_j(m+j) = 0. \quad (171)$$

The vanishing edges have distinct initial and terminal vertices, contain no directed cycle, and therefore form vertex-disjoint directed paths. In particular, there are at most $k-1$ of them. Hence, every directed transport cycle contains an edge on which the charged divisor in (78) is bounded by the nonzero linear form in (82).

Proof. Proposition 22 gives $n \geq k > 0$, and of course $m > 0$. Put $w_i = |A_i| = (k-1)!/C_i$. For every fixed integer $x > 0$, the numbers $w_i(x+i)$ are pairwise distinct. Indeed, if $r < s$ and $w_r(x+r) = w_s(x+s)$, then

$$1 < \frac{C_s}{C_r} = \frac{x+s}{x+r} < \frac{s}{r}. \quad (172)$$

The first inequality and strict unimodality of the binomial coefficients in row $k-1$ give $s \leq k-r$. The last inequality, together with $C_t/t = \binom{k}{t}/k$, gives $\binom{k}{s} < \binom{k}{r}$, hence $s > k-r$, a contradiction. Thus (171) gives at most one terminal vertex for a fixed initial vertex and, symmetrically, at most one initial vertex for a fixed terminal vertex.

The strict inequalities use $x > 0$ and $r < s$. Symmetry and monotonicity of the binomial row show that. $C_s > C_r$ requires $s \leq k-r$, whereas $\binom{k}{s} < \binom{k}{r}$ requires $s > k-r$. This contradiction proves distinctness even when two unsigned endpoint factorials coincide.

Suppose vanishing edges formed a directed cycle. $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_s \rightarrow i_1$. Multiplication of their equations cancels the endpoint weights and gives

$$\prod_{t=1}^s \frac{m+i_t}{n+i_t} = \left(\frac{b}{a}\right)^s. \quad (173)$$

However, the product over all k indices is $b/a > 1$. A proper subproduct is strictly smaller than b/a , whereas the right side is at least b/a , with strict inequality for $s > 1$. The full cycle is equally impossible. Thus, no directed cycle exists; a finite partial matching without cycles is a disjoint union of paths and has at most $k-1$ edges. $\square$

There is a complementary owner-free formulation which will be useful in any attempt to close the transport argument. It is stronger than merely reducing the opposite block product modulo a vertex.

Proposition 16 (Sliding-LCM reflection). For arbitrary $n < m$, put $d = m - n \geq k$. For each $1 \leq j \leq k$, one has the exact equivalence

$$\Leftrightarrow \begin{aligned} & m + j \mid M_k(n) \\ & m + j \mid \text{lcm}(d + j - k, \dots, d + j - 1). \end{aligned} \quad (174)$$

There is symmetrically an equivalence

$$\Leftrightarrow \begin{aligned} & n + i \mid M_k(m) \\ & n + i \mid \text{lcm}(d + 1 - i, \dots, d + k - i). \end{aligned} \quad (175)$$

Hence, under a collision, all of the divisibilities on the right of (174)–(175) hold. If

$$\begin{aligned} A_i(d) &= \text{lcm}(d + 1 - i, \dots, d + k - i), \\ \Omega_k(d) &= \prod_{i=1}^k A_i(d), \end{aligned} \quad (176)$$

Then the two collections of sliding LCMs in (174) and (175) are the same in reverse order, and

$$\text{lcm}(P_k(n), P_k(m)) = abR \mid \Omega_k(d). \quad (177)$$

Moreover,

$$\begin{aligned} \Omega_k(d) \mid \Delta_k(d) &:= \prod_{i=1}^k \prod_{j=1}^k (d + j - i) \\ &= \prod_{\delta=-(k-1)}^{k-1} (d + \delta)^{k-|\delta|}. \end{aligned} \quad (178)$$

Proof. Let $q = m + j$, and fix a prime power $p^v \parallel q$. The condition $q \mid M_k(n)$ is equivalent to the assertion that, for every such p^v , at least one $n + i$ is divisible by p^v . Since

$$n + i \equiv -(d + j - i) \pmod{q}, \quad (179)$$

This is, in turn, equivalent to requiring that p^v divide one of $d + j - k, \dots, d + j - 1$. Taking all maximal prime powers of q proves (174). The proof of (175) is identical.

Under a collision, multiplying (175) over i gives $P_k(n) \mid \Omega_k(d)$. Multiplying (174) over j , and observing that its sliding windows are the same windows in reverse order, gives $P_k(m) \mid \Omega_k(d)$. Their LCM is abR by (15), proving (177). Finally, each $A_i(d)$ divides the product of the entries in its window; multiplication proves (178). $\square$

Unlike (66)–(67), the reflection has no auxiliary factorial multiplier and no owner choices. Unlike the packet matrix, it does not separate prime bases into coprime edges. The two formulations, therefore, retain genuinely different information. Notice also that (174) exactly encodes the one-way divisibility. $M_k(m) \mid M_k(n)$, vertex by vertex; it is not, by itself, a proof that this divisibility is impossible.

4.2. A Quadratic Lower Bound for the Gap

The binomial bridge already yields a uniform gap theorem.

Theorem 17 (Quadratic lower bound for the gap). Every hypothetical counterexample satisfies

$$d > \frac{1}{6} \sqrt{k(m+1)} > \frac{1}{6} \sqrt{kn}. \quad (180)$$

Proof. Put $r = 2k - 1$. Applying the defect formula (10) to the central block of length r gives

$$K_k(d) \leq \lambda_r \binom{d+k-1}{r}. \quad (181)$$

By (50), $kC_k(m) \leq K_k(d)$, and hence

$$\begin{aligned} \frac{k(m+1)^k}{k!} &< kC_k(m) \leq K_k(d) \\ &\leq \lambda_{2k-1} \frac{(d+k-1)^{2k-1}}{(2k-1)!}. \end{aligned} \quad (182)$$

Equation (5) gives $\lambda_{2k-1} < 9^k$, while

$$\frac{(2k-1)!}{k!} = \prod_{j=k+1}^{2k-1} j > k^{k-1}. \quad (183)$$

Before taking roots, (182) gives $(m+1)^k < \lambda_{2k-1} k! (d+k-1)^{2k-1}/[k(2k-1)!]$. The factorial ratio in (183) makes the denominator exceed k^k , while $\lambda_{2k-1} < 9^k$ and $d+k-1 < 2d$. Thus $(m+1)^k < 9^k (2d)^{2k-1}/k^k$, which yields (184). The cases $k = 1, 2$ are already impossible because $Q_k = 1$.

Since $d \geq k$, also $d+k-1 < 2d$. Taking k -th roots in (182) now yields

$$m+1 < 36 \frac{d^{2-1/k}}{k} \leq 36 \frac{d^2}{k}, \quad (184)$$

which is equivalent to the first inequality in (180). The second follows from $m+1 > n$. $\square$

5. Prime-Power-Free Blocks and Adjacent Exclusion

The second block of a collision cannot contain a positive prime power. Indeed, if

$$q = p^a = m+j \quad (a \geq 1, 1 \leq j \leq k) \quad (185)$$

then $q > m \geq n+k$. Hence q divides the LCM of the second block, while no smaller positive integer $n+i < q$ is divisible by q . Thus q does not divide the first LCM, contradicting equality. In particular, the second block contains no prime.

This elementary observation and Nagura's theorem give a uniform exclusion.

Proposition 18 (Prime-power-free obstruction). Every hypothetical counterexample satisfies

$$m > 5k. \quad (186)$$

Proof. Suppose first that $m \geq 25$ and $m \leq 5k$. Nagura's theorem [3] supplies a prime

$$m < p < \frac{6}{5}m \leq m+k. \quad (187)$$

Thus p belongs to the second block, contrary to the preceding observation.

It remains to inspect $m < 25$. The constraints $1 \leq k \leq m$, $m \leq 5k$, and the absence of a prime from $\{m+1, \dots, m+k\}$ leave, apart from the immediate case $k = 1$, only

$$(m, k) = (7, 2), (8, 2), (7, 3), (13, 3), (23, 5). \quad (188)$$

The direct LCM calculations for the allowed range $0 \leq n \leq m-k$ are summarized in Table 2.

Table 2. Exact elimination of the residual cases with $m < 25$.

(m, k)	$M_k(m)$	allowed n	result
(7,2)	72	$0 \leq n \leq 5$	no $M_2(n) = 72$
(8,2)	90	$0 \leq n \leq 6$	no $M_2(n) = 90$
(7,3)	360	$0 \leq n \leq 4$	no $M_3(n) = 360$
(13,3)	1680	$0 \leq n \leq 10$	no $M_3(n) = 1680$
(23,5)	491400	$0 \leq n \leq 18$	no $M_5(n) = 491400$

This exhausts $m < 25$ and proves (186).

6. Fixed-Length Results

6.1. Adjacent Disjoint Blocks

Theorem 19 (Adjacent disjoint blocks). For every $k \geq 1$ and $n \geq 0$,

$$M_k(n) \neq M_k(n+k). \quad (189)$$

Proof. The case $k = 1$ is immediate. Let $k \geq 2$. If $n < 2k$, then $m = n + k < 3k \leq 5k$, so Proposition 18 excludes equality. If $n \geq 2k$, apply the Sylvester–Schur theorem [1] to the $2k$ consecutive integers

$$n + 1, n + 2, \dots, n + 2k. \quad (190)$$

Their first entry is greater than $2k$, so their product has a prime divisor $p > 2k$. Such a prime divides at most one of the $2k$ entries, because the difference between any two entries is at most $2k - 1 < p$. It therefore occurs in exactly one of the two adjacent k -blocks, and their LCMs cannot be equal. $\square$

6.2. Complete exclusion for length three

Theorem 20 (Complete exclusion for length three). There is no counterexample with $k = 3$.

Proof. Here $\lambda_3 = 6$, $B_3 = 1$, and $Q_3 = 2$. Thus $e, f \mid 2$ and $e < f$ force $(e, f) = (1, 2)$. Put

$$X = n + 2, \quad Y = m + 2. \quad (191)$$

Equation (12) becomes

$$Y^3 - Y = 2(X^3 - X). \quad (192)$$

Since $Y > X \geq 2$, equation (192) also forces $Y < 2X$. Define

$$s = Y - X > 0, \quad r = 2X - Y > 0, \quad (193)$$

so that $X = r + s$ and $Y = r + 2s$. Expanding (192) gives the homogeneous cubic identity with a linear right side

$$F(r, s) := r^3 - 6rs^2 - 6s^3 = r. \quad (194)$$

Let $g = \gcd(r, s)$, $r = gR$, and $s = gS$, where $(R, S) = 1$. Homogeneity in (194) gives

$$g^2 F(R, S) = R. \quad (195)$$

Hence $g^2 \mid R$. Write $R = g^2 t$. Then $F(R, S) = t$, and reducing this equality modulo t gives $t \mid 6S^3$. Since $(t, S) = 1$, one has

$$t \in \{1, 2, 3, 6\}. \quad (196)$$

With $U = g^2$, the remaining equation is one of only four Thue equations:

$$t^3 U^3 - 6tUS^2 - 6S^3 = t, \quad t \in \{1, 2, 3, 6\}. \quad (197)$$

Table 3 gives the complete integral solution sets of (197).

Table 3. Certified integral solutions of the four cubic Thue equations.

t	all integer pairs (U, S)
1	$(1, -1), (1, 0)$
2	$(1, -1)$
3	$(1, -2), (1, 1)$
6	$(0, -1)$

For reproducibility, these lists were certified unconditionally with PARI/GP 2.17.2 [20] by `thue(thueinit(t^3*x^3 - 6*t*x - 6, 1), t)`

For the four values of t . The flag 1 requests unconditional class-group certification; no generalized Riemann hypothesis is used. The accompanying program `verify_erdos677_k3_thue.py` checks the lists and the inverse substitution.

The restrictions $U = g^2 > 0$ and $S > 0$ retain only $(t, U, S) = (3, 1, 1)$. It gives

$$(r, s) = (3, 1), \quad (X, Y) = (4, 5). \quad (198)$$

Thus, the only positive solution relevant to increasing blocks has $m - n = Y - X = 1$. For disjoint 3-term blocks, one needs $m - n \geq 3$, so no counterexample exists. $\square$

This gcd-compression step is more than a fixed-box computation: it turns the entire infinite $k = 3$ problem into four completely solved binary cubic equations.

6.3. Complete Exclusion for Length Four

Theorem 21 (Complete exclusion for length four). There is no counterexample with $k = 4$.

Proof. Here $Q_4 = 3$, so $e, f \mid 3$ and $e < f$ force $(e, f) = (1, 3)$. Hence, a collision would satisfy.

$$P_4(m) = 3P_4(n). \quad (199)$$

Ponnudurai proved [6] that the only positive-integer solutions of

$$y(y+1)(y+2)(y+3) = 3x(x+1)(x+2)(x+3) \quad (200)$$

are $(x, y) = (2, 3)$ and $(5, 7)$. Taking $x = n + 1$ and $y = m + 1$, these give respectively

$$(n, m) = (1, 2), \quad (n, m) = (4, 6). \quad (201)$$

Their gaps are 1 and 2, whereas disjoint 4-term blocks require $m - n \geq 4$. Thus, neither is a counterexample. $\square$
For comparison, the centred substitution

$$X = 2n + 5, \quad Y = 2m + 5 \quad (202)$$

gives the exact biquadratic equation

$$(Y^2 - 1)(Y^2 - 9) = 3(X^2 - 1)(X^2 - 9). \quad (203)$$

Equivalently, with $U = X^2 - 5$ and $V = Y^2 - 5$,

$$V^2 - 3U^2 = -32, \quad U + 5 = X^2, \quad V + 5 = Y^2, \quad (204)$$

where X, Y are odd, and disjointness requires $Y - X \geq 8$. Ponnudurai's two solutions become $(X, Y) = (7, 9)$ and $(13, 17)$, illustrating how the quartic product equation is a Pell orbit with two simultaneous square constraints. This centred bridge is useful structurally, although the complete resolution of that orbit is the cited 1975 result rather than a new claim here.

6.4. A Seven-Curve Genus-Two Frontier for Length Five

For $k = 5$, one has $Q_5 = 12$. The coprime pairs (a, b) allowed by $a < b$ and $ab \mid 12$ are

$$(1, 2), (1, 3), (1, 4), (1, 6), (1, 12), (2, 3), (3, 4). \quad (205)$$

Put

$$F(z) = z(z^2 - 1)(z^2 - 4), \quad X = n + 3, \quad Y = m + 3. \quad (206)$$

The product equation in (15) is

$$aF(Y) = bF(X). \quad (207)$$

The factor $(1 - z^{-2})(1 - 4z^{-2})$ is increasing for $z \geq 3$, so (207) implies $Y/X < (b/a)^{1/5} < b/a$. Thus, the slope-adapted variables

$$s = Y - X > 0, \quad r = bX - aY > 0, \quad D = b - a \quad (208)$$

are positive, and

$$X = \frac{r+as}{D}, \quad Y = \frac{r+bs}{D}. \quad (209)$$

For odd j , define the integral binary form

$$G_j(r, s) = \frac{b(r+as)^j - a(r+bs)^j}{D}. \quad (210)$$

Substitution in (207) gives the exact mixed-degree identity

$$G_5(r, s) - 5D^2G_3(r, s) + 4D^4r = 0. \quad (211)$$

Now write $r = hR$, $s = hS$, with $(R, S) = 1$. Dividing (211) by h , and setting $q = h^2$, yields a quadratic equation in the single square variable q :

$$G_5(R, S)q^2 - 5D^2G_3(R, S)q + 4D^4R = 0. \quad (212)$$

Two exact scale locks survive this compression. First, reduction of (212) modulo q gives

$$q \mid 4D^4R, \quad \frac{q}{\gcd(q, R)} \mid 4D^4. \quad (213)$$

Second, put

$$c = \gcd(X, Y), \quad X = cU, \quad Y = cV, \quad w = bU - aV. \quad (214)$$

Then $(U, V) = 1$ and $w > 0$. Since $c \mid h$, while $h \mid r + as = DX$ and $h \mid r + bs = DY$, there is a divisor $\eta \mid D$ such that

$$h = c\eta. \quad (215)$$

Cancelling c directly in $aF(cV) = bF(cU)$ gives

$$c^4(aV^5 - bU^5) - 5c^2(aV^3 - bU^3) - 4w = 0. \quad (216)$$

Consequently

$$c^2 \mid 4w, \quad \gcd(w, U) \mid a, \quad \gcd(w, V) \mid b. \quad (217)$$

The last two divisibilities follow by reducing $w = bU - aV$ modulo U and V . Since $0 < w < DU$, (217) also gives the explicit bound

$$c^3 < 4DX. \quad (218)$$

Thus, neither square scale in the genus-two reduction is free: after a primitive direction is selected, all but a divisor of a fixed small coefficient must be absorbed by that direction. These locks are suitable for local conditions for a future Mordell–Weil sieve.

The discriminant of the quadratic (212) must be a square. After removing the square factor D^4 , every solution therefore gives a rational point on

$$\mathcal{C}_{a,b}: \quad Z^2 = 25G_3(R, S)^2 - 16RG_5(R, S). \quad (219)$$

The right side is a homogeneous sextic. Exact discriminant calculation shows that it is squarefree for every pair in (205); hence, the affine models.

$$z^2 = 25G_3(x, 1)^2 - 16xG_5(x, 1) \quad (220)$$

are seven nonsingular genus-two curves. Conversely, a point relevant to (207) must also make one root of (212) a positive integral square and satisfy both integrality conditions in (209). Thus, this is a strict compression, not merely a reformulation of the original quintic curve.

The carry formula supplies a more exact sieve. In this case

$$E_5(x) = 2^{\mathbf{1}_{x \equiv 1 \pmod{2}} + \mathbf{1}_{x \equiv 3 \pmod{4}}} 3^{\mathbf{1}_{x \not\equiv 0 \pmod{3}}}, \quad (221)$$

So a reconstructed product solution must also satisfy

$$\frac{E_5(m)}{E_5(n)} = \frac{b}{a}. \quad (222)$$

Computational Reconnaissance. PARI/GP's bounded-height rational-point search, followed by the square, integrality, and carry filters, gives the reconnaissance in Table 4 through naive height 100,000.

Table 4. Bounded-height reconnaissance on the seven length-five curves

b/a	integral product solutions $(n, m, m - n)$	carry-compatible
2	(4,5,1)	(4,5,1)
3	none	none
4	none	none
6	(0,1,1), (2,4,2)	(0,1,1)
12	(2,5,3)	none
3/2	(9,10,1)	(9,10,1)
4/3	(14,15,1)	(14,15,1)

Every carry-compatible survivor has a gap of less than 5. A certified determination of $\mathcal{C}_{a,b}(\mathbb{Q})$ For the seven curves, for example, by a valid Mordell–Weil and Chabauty computation, is the exact remaining fixed-length task. The accompanying program `verify_erdos677_k5_genus2.py` checks the polynomial identity symbolically, verifies squarefreeness, performs the bounded search, and applies the inverse and carry filters. The separate Magma driver `certify_erdos677_k5.m` is fail-closed: it announces a complete $k = 5$ certificate only if PointsGenus2 proves completeness on all seven curves. No successful run of that driver is claimed in this paper.

7. Global Restrictions on Hypothetical Collisions

The uniform estimate (5) sharpens the location of any collision.

Proposition 22 (Location of a collision). Every hypothetical counterexample satisfies

$$n \geq k, \quad d < 2(n + k), \quad m < 5n. \quad (223)$$

Proof. By (186), $m > 5k$. If $n < k$, then for every $1 \leq i \leq k$,

$$m + i > 3(n + i), \quad (224)$$

and consequently $P_k(m)/P_k(n) > 3^k$. But (12), (10), and (5) give

$$\frac{P_k(m)}{P_k(n)} = \frac{f}{e} \leq Q_k = \frac{\lambda_k}{k} < 3^k, \quad (225)$$

a contradiction. Hence $n \geq k$. Also, (29) and (5) give $\alpha < 3$. The left-hand inequality in the corridor preceding (32) now yields

$$1 + \frac{d}{n+k} < 3, \quad (226)$$

so $d < 2(n + k)$. Finally $m = n + d < 3n + 2k \leq 5n$. $\square$

There is also an asymptotic lower bound for the gap. Let

$$H = d + k - 1. \quad (227)$$

By the collision bridge (43), every prime factor of every entry in either block is at most H . On the other hand, Erdős proved [4] that there is an absolute constant $C > 0$ such that every interval of at least

$$C \frac{H}{\log H} \quad (228)$$

Consecutive integers, all greater than H , have a prime divisor exceeding H . Since $n \geq k$, the first entry $m + 1 = n + d + 1$ of the second block is greater than H . Therefore, a collision forces

$$k < C \frac{H}{\log H}. \quad (229)$$

In particular, as $k \rightarrow \infty$,

$$d \gg k \log k, \quad n \gg k \log k. \quad (230)$$

Thus, any still-possible counterexample is not near the disjointness boundary: it must have a superlinear gap and block length $k = o(n)$. This is a rigorous reduction, not the missing contradiction.

The constant in Theorem 17 also has a clean asymptotic form. The prime number theorem gives

$$\log \lambda_{2k-1} \sim 2k. \quad (231)$$

Applying this and Stirling's formula directly to (182), while using $d/k \rightarrow \infty$ from (230), gives

$$d \geq \left(\frac{2}{e^{3/2}} - o(1) \right) \sqrt{km}. \quad (232)$$

The constant $2e^{-3/2} \approx 0.44626$ replaces the deliberately elementary constant $1/6$ in the asymptotic regime.

7.1. An Asymptotic Gap Bound

The quadratic bridge can be combined with two theorems on primes in short intervals to improve (230).

Theorem 23 (Asymptotic gap bound). For every fixed $\eta > 0$, there are constants $c_\eta > 0$ and k_η such that every hypothetical counterexample with $k \geq k_\eta$ satisfies

$$d \geq c_\eta k^{3/2-\eta}. \quad (233)$$

Equivalently, along any sequence of hypothetical counterexamples with $k \rightarrow \infty$,

$$d \geq k^{3/2-o(1)}. \quad (234)$$

In fact, the prime-gap input and (223) give the stronger location bounds

$$n \gg k^{40/21}, \quad m \gg k^{40/21}. \quad (235)$$

Proof. It is enough to take $0 < \eta < 1/2$, because $d \geq k$ handles larger η . Baker, Harman, and Pintz proved [11] that consecutive prime gaps are at most $x^{21/40}$ for sufficiently large x . The prime-free second block, therefore, implies

$$k \ll m^{21/40}, \quad m \gg k^{40/21}. \quad (236)$$

Together with $m < 5n$, this proves (235).

If $m \geq k^{2-2\eta}$, Theorem 17 gives

$$d > \frac{1}{6} \sqrt{k(m+1)} \gg k^{3/2-\eta}. \quad (237)$$

Suppose instead that $m < k^{2-2\eta}$. Then

$$k > m^{1/2+\theta}, \quad \theta = \frac{\eta}{2(1-\eta)} > 0. \quad (238)$$

Choose a sufficiently small $\varepsilon \in (0, \theta)$ for which

$$\frac{40}{21} \left(\frac{25}{26} - \varepsilon \right) > \frac{3}{2} - \eta. \quad (239)$$

This choice is possible because $(40/21)(25/26) = 500/273 > 3/2$. After fixing η , one first chooses a positive $\varepsilon < \theta$ small enough to preserve (239), and then takes the maximum of the thresholds in the two cited prime-distribution results. The resulting constants are independent of the hypothetical collision.

Jia and Liu proved [10] that the product of the integers in

$$m < x \leq m + m^{1/2+\varepsilon} \quad (240)$$

has a prime factor $p > m^{25/26-\varepsilon}$, once m is sufficiently large. By (238), this interval is contained in the second block. The central bridge forces every such prime to satisfy

$$p \leq d + k - 1 < 2d. \quad (241)$$

Combining this with (236) and (239) gives

$$d > \frac{1}{2} m^{25/26-\varepsilon} \gg k^{(40/21)(25/26-\varepsilon)} \gg k^{3/2-\eta}, \quad (242)$$

which completes the proof. $\square$

The exponent $3/2$ is therefore a genuine threshold emerging from two different mechanisms: the LCM-size bridge when m is at least nearly quadratic in k , and a large prime factor in a square-root-length subinterval when m is smaller.

7.2. Linear Occupation of Transport Diagonals

Let

$$t = \#\{\delta: Q_\delta > 1\} \quad (243)$$

for any packet factorization supplied by Proposition 3.

Proposition 24 (Occupation of transport diagonals). Every hypothetical counterexample satisfies the exact lower bound

$$t > \frac{k \log((m+1)/k)}{\log(2d)}. \quad (244)$$

Consequently, along any sequence of hypothetical counterexamples with $k \rightarrow \infty$,

$$t \geq \left(\frac{19}{40} - o(1)\right)k. \quad (245)$$

Proof. Every occupied packet divides an integer at most $d + k - 1 < 2d$. Equation (53), therefore, gives

$$L \leq (2d)^t. \quad (246)$$

On the other hand,

$$L = \frac{P_k(m)}{D_k(m)} \geq \frac{(m+1)^k}{(k-1)!} > \left(\frac{m+1}{k}\right)^k. \quad (247)$$

Taking logarithms proves (244). Since $d < m$, (236) gives

$$\frac{\log((m+1)/k)}{\log(2d)} \geq 1 - \frac{\log k}{\log m} - o(1) \geq \frac{19}{40} - o(1), \quad (248)$$

which proves (245).

At most one of these occupied diagonals is $\delta = 0$. Moreover, the packets are pairwise coprime, so selecting one prime from each packet gives t distinct primes; at most $\pi(k) = o(k)$ of them are at most k . Thus, a hypothetical large counterexample requires linearly many genuinely moved, distinct primes exceeding k , not merely one exceptional prime.

7.3. Odd-Parity Transports above the Quadratic Range

Mukhopadhyay and Shorey proved [13] that, apart from an explicit finite list, the product of a block of $k \geq 10$ consecutive integers beginning above k^2 has at least eight distinct prime divisors exceeding k that occur to odd exponent. Their Corollary 2 removes the exceptions whenever $k \geq 14$, or whenever $k \geq 10$ and the first integer is at least 5040.

Apply this to $P_k(n)$. In (16), the coefficient ab has no prime factor exceeding k . Hence, every $p > k$ occurring to an odd exponent in $P_k(n)$ must also occur to an odd exponent in $P_k(m)$. Because such a prime divides at most one member of either block, there are at least eight distinct parity-preserving transports. This independently corroborates the packet picture, although eight fixed transports alone do not close the problem.

The theorem of Saradha and Shorey [12] excluding squares formed from $k - 1$ members of a single k -term block does not close this case: here the odd large-prime valuations are paired between two different blocks rather than removed from one block. This is precisely the extra two-block structure retained by the transport matrix.

8. Centred scale locks and algebraic approximation

8.1. A Uniform Parity-Centred Gcd Descent

The scale lock found in the $k = 5$ calculation is a special case of a uniform elementary principle.

Proposition 25 (Parity-centred scale lock). Retain the coprime coefficients $a < b$ in (15).

If $k = 2\ell$, put

$$X = 2n + k + 1, \quad Y = 2m + k + 1, \quad c = \gcd(X, Y). \quad (249)$$

Then

$$c^2 \mid (b - a)((2\ell - 1)!!)^2. \quad (250)$$

If $k = 2\ell + 1$, put

$$\begin{aligned} X &= n + \ell + 1, \\ Y &= m + \ell + 1, \\ c &= \gcd(X, Y), \\ X &= cU, \quad Y = cV. \end{aligned} \quad (251)$$

Then

$$c^2 \mid (\ell!)^2(bU - aV), \quad (252)$$

and

$$\gcd(bU - aV, U) \mid a, \quad \gcd(bU - aV, V) \mid b. \quad (253)$$

In either parity, let $r = bX - aY$, $s = Y - X$, and $h = \gcd(r, s)$, using the corresponding centred X, Y above. Then

$$c \mid h \mid (b - a)c. \quad (254)$$

Moreover, $c \mid d$: this is immediate in the odd case, while in the even case, c is odd and $c \mid Y - X = 2d$.

Proof. For even $k = 2\ell$, define

$$H_\ell(z) = \prod_{t=1}^{\ell} (z^2 - (2t - 1)^2). \quad (255)$$

Pairing terms about the centre gives

$$2^k P_k(n) = H_\ell(X), \quad 2^k P_k(m) = H_\ell(Y). \quad (256)$$

Thus $aH_\ell(Y) = bH_\ell(X)$. After writing $X = cU$ and $Y = cV$, reduction modulo c^2 turns both polynomial values into the same constant

$$H_\ell(cz) \equiv (-1)^\ell ((2\ell - 1)!!)^2 \pmod{c^2}. \quad (257)$$

This proves (250).

For odd $k = 2\ell + 1$, define

$$F_\ell(z) = z \prod_{t=1}^{\ell} (z^2 - t^2). \quad (258)$$

Now $P_k(n) = F_\ell(X)$ and $P_k(m) = F_\ell(Y)$. Cancelling c from $aF_\ell(cV) = bF_\ell(cU)$, then reducing modulo c^2 , gives

$$(-1)^\ell (\ell!)^2 (aV - bU) \equiv 0 \pmod{c^2}, \quad (259)$$

which is (252). Finally, $(U, V) = 1$; reducing $bU - aV$ modulo U and V proves (253). For (254), c divides both r and s , whereas

$$r + as = (b - a)X, \quad r + bs = (b - a)Y. \quad (260)$$

Thus $c \mid h$ and $h \mid (b - a)\gcd(X, Y) = (b - a)c$.

For fixed k, a, b , (250) bounds the common centred scale outright in the even case. In the odd case, the only unbounded factor is the primitive slope numerator $bU - aV$, whose common factors with U and V are confined to the small coprime coefficients. This is not yet a contradiction, but it supplies arithmetic information absent from a bare Thue–Siegel finiteness argument.

8.2. The Universal Square-Slope Polynomial

The preceding gcd descent uses only the constant and linear terms of the centred products. Retaining every term produces a single integral polynomial in the square of the slope gcd.

For $j \geq 0$, define

$$\mathcal{T}_j(u, v) = \frac{b(u+av)^j - a(u+bv)^j}{D}, \quad D = b - a. \quad (261)$$

The apparent denominator cancels in $\mathbb{Z}[u, v]$: modulo D , one has $a \equiv b$ and $u + av \equiv u + bv$. Thus $\mathcal{T}_0 = 1$, $\mathcal{T}_1 = u$, and $\mathcal{T}_j$ is homogeneous of degree j .

Write the two-centred polynomials as

$$F_\ell(z) = z \prod_{t=1}^\ell (z^2 - t^2) = \sum_{u=0}^\ell \alpha_u z^{2u+1}, \quad (262)$$

$$H_\ell(z) = \prod_{t=1}^\ell (z^2 - (2t-1)^2) = \sum_{u=0}^\ell \beta_u z^{2u}. \quad (263)$$

Their extreme coefficients are

$$\begin{aligned} \alpha_0 &= (-1)^\ell (\ell!)^2, & \alpha_\ell &= 1, \\ \beta_0 &= (-1)^\ell ((2\ell-1)!!)^2, & \beta_\ell &= 1. \end{aligned} \quad (264)$$

Theorem 26 (Square-slope polynomial). Use the parity-appropriate centres X, Y from Proposition 25, and put

$$\begin{aligned} r &= bX - aY, & s &= Y - X, \\ h &= \gcd(|r|, s), & r &= hR, \quad s = hS, \quad q = h^2. \end{aligned} \quad (265)$$

If $k = 2\ell + 1$, every normalized product identity satisfies

$$\sum_{u=0}^\ell \alpha_{\text{you}} D^{2(\ell-u)} \mathcal{T}_{2u+1}(R, S) q^{\text{you}} = 0. \quad (266)$$

Consequently

$$q \mid (\ell!)^2 D^{2\ell} R, \quad \frac{q}{\gcd(q, R)} \mid (\ell!)^2 D^{2\ell}. \quad (267)$$

If $k = 2\ell$, every normalized product identity satisfies

$$\sum_{u=0}^\ell \beta_{\text{you}} D^{2(\ell-u)} \mathcal{T}_{2u}(R, S) q^{\text{you}} = 0. \quad (268)$$

Consequently

$$q \mid ((2\ell-1)!!)^2 D^{2\ell}, \quad h \mid (2\ell-1)!! D^\ell. \quad (269)$$

Proof. The definitions of r and s give

$$r + as = DX, \quad r + bs = DY, \quad (270)$$

and hence

$$\mathcal{T}_j(r, s) = D^{j-1} (bX^j - aY^j). \quad (271)$$

For odd length, $P_k(n) = F_\ell(X)$, $P_k(m) = F_\ell(Y)$, and $aF_\ell(Y) = bF_\ell(X)$. Substitution of (271), followed by multiplication by $D^{2\ell}$, gives

$$\sum_{u=0}^\ell \alpha_u D^{2(\ell-u)} \mathcal{T}_{2u+1}(r, s) = 0. \quad (272)$$

Homogeneity and division by h give (266). Every term except the constant term in q is divisible by q , while that constant term is $(-1)^\ell (\ell!)^2 D^{2\ell} R$. This proves (267).

For even length, $2^k P_k(n) = H_\ell(X)$ and $2^k P_k(m) = H_\ell(Y)$. The same substitution gives

$$\sum_{u=0}^\ell \beta_u D^{2(\ell-u)} \mathcal{T}_{2u}(r, s) = 0. \quad (273)$$

Now every degree is even, so homogeneity directly gives (268). Its constant term is $(-1)^\ell ((2\ell-1)!!)^2 D^{2\ell}$, proving the first divisibility in (269). Both sides are squares up to sign, so the asserted divisibility for h follows prime by prime. $\square$

For $k = 5$, the coefficient vector is $(4, -5, 1)$, and (266) is exactly the quadratic (212) . The first new even instance is $k = 6$, for which $(\beta_0, \beta_1, \beta_2, \beta_3) = (-225, 259, -35, 1)$, so

$$\begin{aligned} T_6(R, S)q^3 - 35D^2T_4(R, S)q^2 \\ + 259D^4T_2(R, S)q - 225D^6 = 0, \\ h \mid 15D^3. \end{aligned} \quad (274)$$

Here $Q_6 = 10$, so only

$$(a, b) = (1, 2), (1, 5), (1, 10), (2, 5) \quad (275)$$

can occur. With $X = 2n + 7$, $Y = 2m + 7$, $U = X^2$, $V = Y^2$, and

$$\Phi(T) = (T - 1)(T - 9)(T - 25), \quad (276)$$

The four remaining product equations become the four explicit plane cubics

$$a\Phi(V) = b\Phi(U), \quad (277)$$

together with the stringent conditions that U, V are odd squares, $X, Y \geq 7$, and $Y - X \geq 12$. This is a genus-one frontier, not a completed $k = 6$ proof: an exact iterated-resultant calculation confirms that all four projective cubics are nonsingular, but determining their integral points and then applying the square and carry filters remains necessary.

8.3. One-Sided Algebraic Approximation

Set

$$\begin{aligned} c &= \frac{k+1}{2}, \\ h &= \frac{k-1}{2}, \\ X &= n + c, \\ Y &= m + c = X + d. \end{aligned} \quad (278)$$

For $x > h$, define

$$G_k(x) = \prod_{i=1}^k \left(1 + \frac{i-c}{x}\right) = \prod_{\substack{1 \leq i \leq k \\ i-c > 0}} \left(1 - \frac{(i-c)^2}{x^2}\right). \quad (279)$$

The paired expression shows that $0 < G_k(x) < 1$ and that G_k is strictly increasing. Since

$$P_k(n) = X^k G_k(X), \quad P_k(m) = Y^k G_k(Y), \quad (280)$$

Equation (12) yields

$$\alpha = \frac{Y}{X} \left(\frac{G_k(Y)}{G_k(X)} \right)^{1/k} > \frac{Y}{X}. \quad (281)$$

Moreover,

$$0 < \log \left(\frac{\alpha X}{Y} \right) < \frac{k^2 - 1}{24(X^2 - h^2)}. \quad (282)$$

To prove (282), note that

$$\log \left(\frac{\alpha X}{Y} \right) = \frac{1}{k} \log \left(\frac{G_k(Y)}{G_k(X)} \right) < -\frac{1}{k} \log G_k(X). \quad (283)$$

For $0 < t < 1$, $-\log(1 - t) \leq t/(1 - t)$. Therefore

$$-\log G_k(X) \leq \frac{1}{X^2 - h^2} \sum_{i-c > 0} (i - c)^2 = \frac{k(k^2 - 1)}{24(X^2 - h^2)}, \quad (284)$$

which proves the stated bound.

The variation over the actual gap gives a second, sharper estimate. Indeed,

$$\frac{d}{dx} \log G_k(x) = \sum_{i-c > 0} \frac{2(i-c)^2}{x(x^2 - (i-c)^2)} \leq \frac{k(k^2 - 1)}{12X(X^2 - h^2)} \quad (285)$$

for $x \geq X$. Integrating only from X to $Y = X + d$, and then dividing by k , yields

$$0 < \log\left(\frac{\alpha X}{Y}\right) < \frac{(k^2-1)d}{12X(X^2-h^2)}. \quad (286)$$

Combining (282) and (286), one obtains the exact two-regime bound

$$0 < \log\left(\frac{\alpha X}{Y}\right) < \frac{k^2-1}{24(X^2-h^2)} \min\left\{1, \frac{2d}{X}\right\}. \quad (287)$$

Since $\alpha < 3$ by (223),

$$0 < \alpha - \frac{Y}{X} < \frac{k^2-1}{8(X^2-h^2)} \min\left\{1, \frac{2d}{X}\right\}. \quad (288)$$

Thus, the error is $O(k^2/X^2)$ in general, but improves to $O(k^2d/X^3)$ when $d < X/2$. If

$$(k^2 - 1)dX < 2(X^2 - h^2), \quad (289)$$

then (288) is less than $1/(2X^2)$. After reducing $Y/X = p/q$, one has $q \leq X$, so Legendre's criterion makes p/q a continued-fraction convergent of α .

Equations (281)–(289) show that every collision produces a one-sided rational approximation to the explicitly irrational algebraic number. $(f/e)^{1/k}$. The differentiated estimate is a genuine gain in the short-gap regime, but even a certified convergent is not by itself a contradiction.

9. Exact Computational Methods and Validation

9.1. Certification Protocol

The finite-spectrum certificate uses the following deterministic procedure. For a fixed length k , it first enumerates all coprime pairs $a < b$ satisfying $ab \mid Q_k$. For each pair, it enumerates the finite set. $Z_{k,a,b}$ from (155). The unique positive root $\zeta_{k,a,b}$ of (154) is bracketed by rational endpoints on which the defining polynomial has opposite signs. Rational bisection continues until every comparison with an element of $Z_{k,a,b}$ has a certified sign. The resulting positive separation gives the integer upper bound in (158). Finally, every reduced gap up to that bound is tested against the exact product identity, the carry formula, disjointness, and the packet count.

Termination follows because Theorem 14 proves that. $\zeta_{k,a,b}$ is distinct from every member of the finite spectrum. The convergence criterion is therefore separation of rational isolating intervals, not a floating-point tolerance. Integer factorization and rational comparisons are exact at every stage.

For each pair (a, b) and reduced the gap d_0 , the enumeration tests every divisor $D_0 \mid (b - a)$ with $\gcd(D_0, d_0) = 1$ and reconstructs $d = (b - a)d_0/D_0$. The ratio $P_k(n + d)/P_k(n) = \prod_{i=1}^k (1 + d/(n + i))$ is strictly decreasing in $n \geq 0$, so exact bisection finds every possible integral starting point. The rational corridor (34) supplies a finite upper endpoint. The Table 1 program actually excludes every disjoint normalized product identity in those ranges; no additional carry or packet-count filter is needed when none survives.

Computational Reconnaissance. An exact-integer search evaluated $M_k(n)$ directly for

$$1 \leq k \leq 150, \quad 0 \leq n < m \leq 150,000. \quad (290)$$

No pair with $m - n \geq k$ had equal least common multiples. The computation used integer gcd and lcm operations only. It is finite evidence, not a proof beyond the stated rectangle.

Regression checks. A separate exact regression program checked the carry formula (23), the binomial divisibility, the general common-LCM bridge (40), and the diagonal packet factorization. On the box

$$1 \leq k \leq 30, \quad 0 \leq n \leq 500, \quad k \leq d \leq 120, \quad (291)$$

It completed 18,630 carry/binomial checks and 1,585,665 bridge checks. A factorization-intensive sub-box completed 28,755 packet checks and 2,412 vertex-residual checks. All passed. The programs `search_erdos677.py` and `verify_erdos677_reductions.py` accompany this paper; once again, these checks detect errors but do not replace the proofs.

The sliding-LCM equivalence (174)–(175) was independently exercised on 46,903,620 individual vertices in the same exact run. All equivalence checks passed. The same run exercised the parity-centred scale locks on 1,402 nonvacuous overlapping collisions; all passed.

The independent program `verify_erdos677_k3_thue.py` invokes PARI/GP’s unconditional Thue mode and reproduces the four complete solution lists in Theorem 20. Unlike the rectangular reconnaissance, this is a finite certificate for the infinite $k = 3$ case after the proved gcd reduction. The program `verify_erdos677_k5_genus2.py` checks (211)–(221) and performs the explicitly bounded genus-two search reported in Subsection 6.4; it deliberately labels that output as non-complete.

The additional program `verify_erdos677_slope_scale.py` expands the two master identities of Theorem 26 symbolically through $\ell = 6$, checks them on 122,850 exact integer specializations, exercises the factorial and primitive-binomial quadratic transport lifts on 7,723 exact reduced product identities, checks the generalized diagonal rough-gap lock on 3,575 disjoint specializations, and exercises the large-prime-power completion on 12,555 common prime powers. It also checks the forced diagonal gap core and the primitive progression identity on 5,850 separated block pairs, assigning 34,064 exact maximal-valuation primes. The second Taylor saturation was exercised on 638 rough packets, including 45 central cubic packets. The endpoint-balance and edge-saturation laws were checked on 4,426 large prime powers and 4,422 assembled packets, while the whole diagonal theorem was checked on 786 nontrivial packets. Finally, the program checks the vanishing-edge path theorem on 5,616 configurations, including 169 genuinely vanishing edges, and verifies nonsingularity of the four $k = 6$ cubics. All checks passed. The symbolic expansion verifies an identity of polynomials; the finite specializations and transport exercises remain regression tests rather than an infinite search certificate.

10. Conclusion

Equality of least common multiples can be analyzed without erasing the positions at which the maximal prime powers occur. The normalized defect converts a hypothetical collision into a product identity with coprime coefficients supported on primes at most k , while the transport matrix records every maximal prime power on a unique owner-to-owner edge. Its vertex budgets, quadratic lifts, and diagonal completion lead to a primitive positive arithmetic progression with tightly controlled pairwise gcds. The finite-spectrum theorem is the strongest uniform consequence of this framework: zero, one, or two active primitive sites form an effectively bounded regime. Exact certification removes that entire regime for $k = 5$ and $k = 6$. The same machinery proves the adjacent-block theorem, completes $k = 3$ by four unconditional Thue certificates, recovers the complete $k = 4$ exclusion from Ponnudurai’s theorem, and establishes the stated gap and occupation bounds. The unresolved part is distinct from these proved reductions. No argument presented here rules out three or more active sites when their square-packing constraints coexist with at least one moved transport packet. Closing that coupled configuration, or certifying all rational points on the seven genus-two curves for $k = 5$, are the nearest well-defined next steps. For length five, the bounded genus-two search excludes no point beyond height 100,000 and does not establish a complete list of rational or integral points. Completion of the seven curve calculations and exclusion of the general regime with at least three active sites remain future tasks. The present contribution is the proved structural reduction and the explicitly delimited finite certificates.

Conflicts of Interest

The authors declare no conflict of interest.

Funding Statement

No external funding was declared for this research.

Data and Code Availability

No experimental data were generated. The computational supplement contains the seven original source files, reproduction commands, and the exact Table 1 rerun log. The bounded length-five calculation is not a complete rational-point certificate.

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