Quasi-Static Thermal Stresses in a Robin's Thin Hollow Cylinder with Internal Moving Heat Source

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1.Abstract:

This paper is a transient non-homogeneous thermoelastic problem with Robin's boundary condition in thin hollow cylinder of isotropic material of inner radius a, outer radius b and height h occupying region R: $a \le r \le b, 0 \le z \le h, 0 \le \phi \le 2\pi$ having initial temperature $f(r,\phi,z)$ placed in an ambient temperature zero. The cylinder is subjected to the activity of moving heat source along circular trajectory of radius r_1 where $a < r_1 < b$ around the centre of the cylinder with constant angular velocity ω. The heat conduction equation containing heat generation term is solved by applying integral transform technique and Green's theorem is adopted in deriving the solution of the heat conduction equation. The solution is obtained in a series form of Bessel function and trigonometric function and thermal stresses are derived.

Keywords:Robint's Thin Hollow Cylinder, Moving Heat Source, Thermal Stresses, Green's Theorem

2.Introduction: During the second half of 20^{th} century, non-isothermal problems of the theory of elasticity became increasingly important. This is due to their wide application in diverse fields. The high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses that reduce the strength of aircraft structure.

Recently D. T. Solanke, M. H. Durge have studied the thermal stresses in thin cylinder with Dirichlet's, Neumann's, Robin's boundary condition and in thin hollow cylinder with Dirichlet's, Neumann's boundary condition. Now authors in this present paper determine temperature, thermal stresses, in thin hollow cylinder with Robis's boundary condition, determined by $a \le r \le b, 0 \le z \le h, 0 \le \phi \le 2\pi$ with internal moving heat source. Heat conduction equation with heat generation term is solved by applying integral transform technique and Green's theorem. Solution is obtained in series form of Bessel function and trigonometric function. The direct problems is very important in view of its relevance to various industrial mechanics subjected to heating such as main shaft of lathe, turbines and the role of rolling mill for base of furnace boiler of thermal power plant, gas power plant and measurement of aerodynamic heating.

3.Formulation of the Heat conduction problem :

Consider a thin hollow cylinder of isotropic material of inner radius a, outer radius b and thickness h occupying the space R: $a \le r \le b, 0 \le z \le h, 0 \le \phi \le 2\pi$ having initial temperature $f(r,\phi,z)$ placed in an ambient temperature zero. The cylinder is subjected to the activity of moving heat source which changes its place along circular trajectory of radius r_1, where $a < r_1 < b$, around the centre of the cylinder with constant angular velocity ω. The activity of moving heat source and initial temperature of the cylinder may cause the generation of heat due to nuclear interaction that may be a function of position and time in the form $g(r,\phi,z,t)$ w/s^3. The temperature distribution of the thin hollow cylinder is described by the differential equation of heat conduction with heat generation term as in [6] page no.8 is given by

$$\nabla^2 T + \frac{1}{k} g = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Where $T = T(r,\phi,z,t)$ is temperature distribution, k is thermal conductivity of the material of the cylinder, $\alpha = \dfrac{k}{\rho C_p}$ is thermal diffusivity, ρ is density , C_p is specific heat of the material and g is the volumetric energy generation term in the cylinder. Where ∇^2 is Laplacian operator in cylindrical coordinates and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Now consider an instantaneously moving heat source $g_s{}^i$ located at a point (r_1,ϕ',ξ) and releasing its energy spontaneously at time τ . Such volumetric heat source in cylindrical coordinates is

given by

$$g(r,\phi,z,t) = g_s{}^i \frac{1}{2\pi r}\delta(r-r_1)\delta(\phi-\phi')\delta(z-\xi)\delta(t-\tau)$$

Hence above equation reduces to

$$\nabla^2 T + g_s{}^i \frac{1}{2\pi kr}\delta(r-r_1)\delta(\phi-\phi')\delta(z-\xi)\delta(t-\tau) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3.1}$$

$$\phi' = \omega t \tag{3.2}$$

With initial and homogeneous boundary conditions,

$$k\frac{\partial T}{\partial r} - h_1 T = 0 \quad \text{at} \quad r = a \tag{3.3}$$

$$k\frac{\partial T}{\partial r} + h_1 T = 0 \quad \text{at} \quad r = b \tag{3.4}$$

$$k\frac{\partial T}{\partial z} - h_1 T = 0 \quad \text{at} \quad z = 0 \tag{3.5}$$

$$k\frac{\partial T}{\partial z} + h_1 T = 0 \quad \text{at} \quad z = h \tag{3.6}$$

$T = f(r,\phi,z)$ at $t = 0$, $\tau = -\infty$

Where h_1 is heat transfer coefficient of the cylinder (3.7)

4. Formulation of thermoelastic Problem:

Let us introduce a thermal stress function χ related to component of stress in the cylindrical coordinates system as in [7]

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \chi}{\partial r^2} \tag{4.1}$$

$$\sigma_{\phi\phi} = \frac{\partial^2 \chi}{\partial r^2} \tag{4.2}$$

$$\sigma_{r\phi} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial \phi}\right) \tag{4.3}$$

The boundary condition for a traction free body are

$\sigma_{rr} = 0, \sigma_{r\phi} = 0$ at $r = a$ or $r = b$ (4.4)

Where $\chi = \chi_c + \chi_p$ (4.5)

Where χ_c is complementary solution and χ_p is particular solution and

χ_c satisfies the equation $\nabla^4 \chi_c = 0$ (4.6)

χ_p satisfies the equation $\nabla^4 \chi_p = -\lambda E \nabla^2 \Gamma$ (4.7)

Where Γ is temperature change $\Gamma = T - T_i$ where T_i is initial temperature

$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$ since cylinder is thin, z component is negligible

5. Solution:

We define integral transform of $T(r,\phi,z,t)$ by

$$\overline{T}(\beta_m, v, \alpha_n, t) = \int_R T(r,\phi,z,t) R_v(\beta_m r)\cos v(\phi - \phi')Z(\alpha_n z)dv \tag{5.1}$$

And its inverse by $T(r,\phi,z,t) = \sum_{m,n,v=0}^{\infty} \frac{\overline{T}(\beta_m, v, \alpha_n, t) R_v(\beta_m r)\cos v(\phi - \phi')Z(\alpha_n z)}{N(\beta_m)N(v)N(\alpha_n)}$ (5.2)

$$R_v(\beta_m r) = J_v(\beta_m r)\left[\beta_m Y'_v(\beta_m b) + HY_v(\beta_m b)\right] - \left[\beta_m J'_v(\beta_m b) + HJ_v(\beta_m b)\right]Y_v(\beta_m r) \tag{5.3}$$

Where $H = \frac{h_1}{k}$

$$Z(\alpha_n z) = \alpha_n \cos(\alpha_n z) + H\sin(\alpha_n z) \tag{5.4}$$

$$N(\beta_m) = \int_a^b r[Rv(\beta_m r)]^2 dr = \text{Normalization integral}$$

$$N(\beta_m) = \frac{\frac{2}{\pi^2}\left\{\left[\frac{b^2(H^2+\beta_m{}^2)-v^2}{\beta_m{}^2 b^2}\right]\left[\beta_m J'_v(\beta_m a)+HJ_v(\beta_m a)\right]^2 - \left[\frac{a^2(H^2+\beta_m{}^2)-v^2}{\beta_m{}^2 a^2}\right]\left[\beta_m J'_v(\beta_m b)+HJ_v(\beta_m b)\right]^2\right\}}{\left[\beta_m J'_v(\beta_m a)+HJ_v(\beta_m a)\right]^2} \quad (5.5)$$

$$N(v) = \int_0^{2\pi} [\cos v(\phi - \phi')]^2 d\phi = \pi \quad (5.6)$$

$$N(\alpha_n) = \int_0^h [Z(\alpha_n z)]^2 dz = \left(\alpha_n{}^2 + H^2\right)\tfrac{h}{2} + H \quad (5.7)$$

Where β_m is roots of the transcendental equation

$$\left[\beta_m J'_v(\beta_m a) - HJ_v(\beta_m a)\right]\left[\beta_m Y'_v(\beta_m b) + HY_v(\beta_m b)\right] - \left[\beta_m J'_v(\beta_m b) + HJ_v(\beta_m b)\right]\left[\beta_m Y'_v(\beta_m a) - HY_v(\beta_m a)\right] = 0 \quad (5.8)$$

α_n is roots of the transcendental equation $\tan \alpha_n h = \dfrac{2\alpha_n H}{\alpha_n{}^2 - H^2}$ (5.9)

$v \in W$ where W is a set of whole numbers

By taking integral transform of equation (3.1) and using following Green's theorem given in [6] page no.506 we obtain

$$\int_R \nabla^2 T\psi_k dv = \int_R T\nabla^2\psi_k dv + \sum_{i=1}^{N}\int_{s_i}\left[\psi_k \frac{\partial T}{\partial n_i} - T\frac{\partial \psi_k}{\partial n_i}\right]ds_i \quad (5.10)$$

$$\frac{d\overline{T}}{dt} + \alpha\left(\beta_m{}^2 + \alpha_n{}^2\right)\overline{T} = \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi)\delta(t-\tau)$$

This is linear differential equation of first order whose solution by applying initial condition (3.7) we obtain

$$\overline{T} = \left[\overline{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi) e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right] e^{-(\beta_m{}^2+\alpha_n{}^2)t}$$

By taking inverse integral transform we obtain

$$T = \sum_{m,n,v=0}^{\infty} \frac{R_v(\beta_m r)\cos v(\phi-\phi')Z(\alpha_n z)}{N(\beta_m)N(v)N(\alpha_n)}\left[\overline{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi) e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right] e^{-(\beta_m{}^2+\alpha_n{}^2)t} \quad (5.11)$$

$$\Gamma = \sum_{m,n,v=0}^{\infty} \frac{R_v(\beta_m r)\cos v(\phi-\phi')Z(\alpha_n z)}{N(\beta_m)N(v)N(\alpha_n)}\left[\overline{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi) e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right][e^{-(\beta_m{}^2+\alpha_n{}^2)t} - 1] \quad (5.12)$$

6.Solution of Thermoelastic Problem:

Let the suitable form of χ_c satisfying equation (4.6) be

$$\chi_c = \sum_{v=0}^{\infty}\left(Ar^{v+2} + Br^{-v+2}\right)\cos v(\phi-\phi') + \left(Cr^{v+2} + Dr^{-v+2}\right)\sin v(\phi-\phi') \tag{6.1}$$

Let suitable form of χ_p satisfying (4.7) be

$$\chi_p = \lambda E\sum_{m,n,v=0}^{\infty}\frac{R_v(\beta_m r)\cos v(\phi-\phi')Z(\alpha_n z)}{\beta_m^{\,2}N(\beta_m)N(v)N(\alpha_n)}\left[\overline{f}(\beta_m,v,\alpha_n) + \frac{\alpha}{2\pi k}g_s{}^i R_v(\beta_m r_1)Z(\alpha_n\xi)e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right][e^{-(\beta_m{}^2+\alpha_n{}^2)t}-1] \tag{6.2}$$

From (4.5) , (6.1) and (6.2) we obtain

$$\chi = \sum_{v=0}^{\infty}\left(Ar^{v+2} + Br^{-v+2}\right)\cos v(\phi-\phi') + \left(Cr^{v+2} + Dr^{-v+2}\right)\sin v(\phi-\phi') +$$

$$\sum_{m,n,v=0}^{\infty}\frac{R_v(\beta_m r)\cos v(\phi-\phi')Z(\alpha_n z)}{N(\beta_m)N(v)N(\alpha_n)}\left[\overline{f}(\beta_m,v,\alpha_n) + \frac{\alpha}{2\pi k}g_s{}^i R_v(\beta_m r_1)Z(\alpha_n\xi)e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right][e^{-(\beta_m{}^2+\alpha_n{}^2)t}-1] \tag{6.3}$$

From (4.1) and (6.3) we obtain

$$\sigma_{rr} = \sum_{v=0}^{\infty}\left[A(2+v-v^2)r^{v} + B(2-v-v^2)r^{-v}\right]\cos v(\phi-\phi') + \left[C(2+v-v^2)r^{v} + D(2-v-v^2)r^{-v}\right]\sin v(\phi-\phi') +$$

$$\lambda E\sum_{m,n,v=0}^{\infty}\left[\tfrac{\beta_m}{r}R_v^{'}(\beta_m r) - \tfrac{v^2}{r^2}Rv(\beta_m r)\right]\frac{\cos v(\phi-\phi')Z(\alpha_n z)}{\beta_m^{\,2}N(\beta_m)N(v)N(\alpha_n)}\left[\overline{f}(\beta_m,v,\alpha_n) + \frac{\alpha}{2\pi k}g_s{}^i R_v(\beta_m r_1)Z(\alpha_n\xi)e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right]$$

$$\left(e^{-(\beta_m{}^2+\alpha_n{}^2)t}-1\right) \tag{6.4}$$

From (4.2) and (6.3) we obtain

$$\sigma_{\phi\phi} = \sum_{v=0}^{\infty}\left[A(v^2+3v+2)r^{v} + B(v^2-3v+2)r^{-v}\right]\cos v(\phi-\phi') + \left[C(v^2+3v+2)r^{v} + D(v^2-3v+2)r^{-v}\right]\sin v(\phi-\phi') +$$

$$\lambda E\sum_{m,n,v=0}^{\infty}\left[\tfrac{\beta_m}{r}R_{v+1}(\beta_m r) - \beta_m{}^2 R_v(\beta_m r)\right]\frac{\cos v(\phi-\phi')Z(\alpha_n z)}{\beta_m^{\,2}N(\beta_m)N(v)N(\alpha_n)}\left[\overline{f}(\beta_m,v,\alpha_n) + \frac{\alpha}{2\pi k}g_s{}^i R_v(\beta_m r_1)Z(\alpha_n\xi)e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right]$$

$$\left(e^{-(\beta_m{}^2+\alpha_n{}^2)t}-1\right) \tag{6.5}$$

From (4.3) and (6.3) we obtain

$$\sigma_{r\phi} = \sum_{v=0}^{\infty}\left[A(v^2+v)r^{v} + B(v-v^2)r^{-v}\right]\sin v(\phi-\phi') + \left[C(-v^2-v)r^{v} + D(v^2-v)r^{-v}\right]\cos v(\phi-\phi') +$$

$$\lambda E v \sum_{m,n,v=0}^{\infty} \left[\tfrac{\beta_m}{r} R'_v(\beta_m r) - \tfrac{1}{r^2} Rv(\beta_m r)\right] \frac{\sin v(\phi-\phi')Z(\alpha_n z)}{\beta_m{}^2 N(\beta_m)N(v)N(\alpha_n)} \left[\bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi) e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right]$$

$$\left(e^{-(\beta_m{}^2+\alpha_n{}^2)t} - 1\right) \tag{6.6}$$

Applying condition (4.4) to (6.4) and (6.6) we obtain

$$A = \lambda E \sum_{m,n,v=0}^{\infty} \frac{a^{-v-2} Z(\alpha_n z)}{\beta_m{}^2 N(\beta_m)N(v)N(\alpha_n)} \left\{\left(\frac{v+2v^2-2-v^3}{2v(v^2-1)}\right) R_v(\beta_m a) - \left(\frac{1}{2v}\right)\beta_m a R'_v(\beta_m a)\right\} \left[\bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi) e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right]$$

$$\left(e^{-(\beta_m{}^2+\alpha_n{}^2)t} - 1\right) \tag{6.7}$$

$$B = \lambda E \sum_{m,n,v=0}^{\infty} \frac{a^{v-2} Z(\alpha_n z)}{\beta_m{}^2 N(\beta_m)N(v)N(\alpha_n)} \left\{\left(\frac{2+v-2v^2-v^3}{2v(v^2-1)}\right) R_v(\beta_m a) + \left(\frac{1}{2v}\right)\beta_m a R'_v(\beta_m a)\right\} \left[\bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi) e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right]$$

$$\left(e^{-(\beta_m{}^2+\alpha_n{}^2)t} - 1\right) \tag{6.8}$$

Substituting the above values in (6.4), (6.5) and (6.6) we obtain

$$\sigma_{rr} = \lambda E \sum_{m,n,v=0}^{\infty} \left\{ \begin{array}{l} a^{-v-2} r^v \left[\frac{7v^2-3v^4-v^3+v^5-4}{2v(v^2-1)} R_v(\beta_m a) - \frac{2+v-v2}{2v} \beta_m a R'_v(\beta_m a)\right] + a^{v-2} r^{-v} \left[\frac{-7v^2-v^3+3v^4+v^5+4}{2v(v^2-1)} R_v(\beta_m a) + \frac{2-v-v^2}{2v} \beta_m a R'_v(\beta_m a)\right] + \\ \left[\frac{\beta_m}{r} R'_v(\beta_m r) - \frac{v^2}{r^2} R_v(\beta_m r)\right] \end{array} \right\}.$$

$$\frac{\cos v(\phi-\phi')Z(\alpha_n z)}{\beta_m{}^2 N(\beta_m)N(v)N(\alpha_n)} \left[\bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi) e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right] \left(e^{-(\beta_m{}^2+\alpha_n{}^2)t} - 1\right) \tag{6.9}$$

$$\sigma_{\phi\phi} = \lambda E \sum_{m,n,v=0}^{\infty} \left\{ \begin{array}{l} a^{-v-2} r^v \left[\frac{(v^2+3v+2)(-v^3+2v^2+v-2)}{2v(v^2-1)} R_v(\beta_m a) - \frac{v^2+3v+2}{2v} \beta_m a R'_v(\beta_m a)\right] + a^{v-2} r^{-v} \left[\frac{(v^2-3v+2)(-v^3-2v^2+v+2)}{2v(v^2-1)} R_v(\beta_m a) + \frac{v^2-3v+2}{2v} \beta_m a R'_v(\beta_m a)\right] + \\ \left[\frac{\beta_m}{r} R_{v+1}(\beta_m r) - \beta_m{}^2 R_v(\beta_m r)\right] \end{array} \right\}.$$

$$\frac{\cos v(\phi-\phi')Z(\alpha_n z)}{\beta_m{}^2 N(\beta_m)N(v)N(\alpha_n)} \left[\bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi) e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right] \left(e^{-(\beta_m{}^2+\alpha_n{}^2)t} - 1\right) \tag{6.10}$$

$$\sigma_{r\phi} = \lambda E \sum_{m,n,v=0}^{\infty} \left\{ \begin{array}{l} a^{-v-2} r^v \left[\frac{-v^5+v^4+3v^3-v^2-2v}{2v(v^2-1)} R_v(\beta_m a) - \frac{v^2+v}{2v} \beta_m a R'_v(\beta_m a)\right] + a^{v-2} r^{-v} \left[\frac{v^5+v^4-3v^3-v^2+2v}{2v(v^2-1)} R_v(\beta_m a) + \frac{v-v2}{2v} \beta_m a R'_v(\beta_m a)\right] + \\ v\left[\frac{\beta_m}{r} R'_v(\beta_m r) - \frac{1}{r^2} R_v(\beta_m r)\right] \end{array} \right\}.$$

$$\frac{\cos v(\phi-\phi')Z(\alpha_n z)}{\beta_m{}^2 N(\beta_m)N(v)N(\alpha_n)} \left[\bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s{}^i R_v(\beta_m r_1) Z(\alpha_n \xi) e^{(\beta_m{}^2+\alpha_n{}^2)\tau}\right] \left(e^{-(\beta_m{}^2+\alpha_n{}^2)t} - 1\right) \tag{6.11}$$

7. Conclusion :

In this paper we determined the time dependent temperature distribution in three dimensions and thermal stresses in a Robin's thin hollow cylinder with moving heat source with analytical approach on the surface is established. By giving particular values to the parameters one can obtain their desired results by putting values of the parameters in the equations (5.11), (6.9), (6.10), (6.11). From these equations we observe that initially ($t = 0$) all stresses vanishes.

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