Construction of PBIB designs through chosen lines and Triangles of graphs

Davinder Kumar Garg [#1], Syed Aadil Farooq [*2]

[#] *Associate Professor, Department of Statistics,*
Punjabi University, Patiala, India.

***Abstract*— In this paper, we have constructed partially balanced incomplete block (PBIB) designs with two and three associate classes by establishing a link between PBIB designs and graphs through the chosen lines and number of triangles. We have considered six configurations for this purpose and constructed four two associate class PBIB designs and two three class PBIB designs.**

***Keywords*— PBIB designs, Graphs, paths, points, lines, triangles**

I. INTRODUCTION

The concept of constructing and establishing relationship between partially balanced incomplete block (PBIB) designs and graphs is not new in the literature. Walikar *et.al* [1,3] have established a relationship between minimum dominating and maximum independent sets of a graph with the blocks of PBIB designs. Alwardi and Soner [5] have obtained a relation between the dominating sets of strongly regular graphs and PBIB designs. Kumar *et.al* [6] have studied the relationship between minimum perfect dominating sets of Clebesch graph with PBIB designs. Kumar and Soner [4] have constructed designs associated with maximum independent sets of cubic graphs. Sharada and Soner [2] introduced PBIB arising from minimum efficient dominating sets of a graph. Recently Shrikol *et.al* [7] have made exhaustive study and established the relation between total minimum dominating sets with PBIB designs. Throughout this paper, G= (V, E) is a graph with vertex set V and edge set E, stands for a finite connected undirected graph, two vertices are adjacent if there is an edge between them otherwise non-adjacent. Here, we have shown that geometrical configuration of graphs consisting of certain suitably chosen lines and triangles.

II. DEFINITIONS AND PRELIMINARY RESULTS

Definition 2.1: Given v symbols {1, 2, 3, . . . , v} a relation satisfies the following conditions is said to form an association scheme with m associate classes.
(i) Any two symbols α, β are either I^{st}, 2^{nd}, . . . ,m^{th} associates for some i, with $1 \le i \le m$ and this relation of being i^{th} associates is symmetric i.e. if the symbol α is i^{th} associate of symbol β, then β is i^{th} associate of α.
(ii) The number of i^{th} associates of each symbol is n_i, the number n_i being independent.
(iii) If α and β are two symbols which are i^{th} associates , then number of symbols which are j^{th} associates of α and k^{th} associates of β is p^i_{jk} , and is independent of the pair of i^{th} associates α and β.

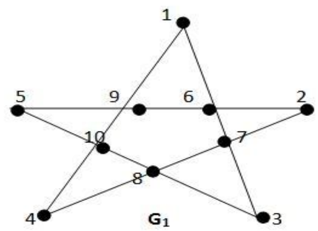
Definition 2.2: Consider a set of symbols S= {1,2,3,. . .,v} and an association scheme with m classes, we have a partially balanced incomplete block designs (PBIB) if v symbols are arranged in b blocks of size k(<v) such that
1. Every symbol occurs exactly in r blocks.
2. Every symbol occurs at most once in a block.
3. If two symbols α and β are i^{th} associates, then they occur together λ_i blocks, the number λ_i being independent of the particular pair of the i^{th} associates α and β

The numbers v, b, r, k, λ_i (i=1,2,3,. . .,m) are called parameters of the first kind, whereas the numbers n_i and p^i_{jk} are called parameters of the second kind.
Definition 2.3*:* A graph is called a PBIB Graph if all possible triangles in G_1, G_3, G_4, G_6 forms blocks of PBIB designs with two and three association scheme.

Definition 2.4: A graph is called a PBIB Graph if the chosen lines inside the graph G_2 and lines between two vertices in G_5 forms blocks of PBIB designs with 2- association scheme.

III. CONSTRUCTION OF DESIGNS

Theorem 3.1: The set of all possible triangles in graph G_1 shown below forms blocks of PBIB design with two associate class association scheme.



Proof: We see that all possible triangle triplets of graph G_1 are given by
1. {1, 6, 9} 2. {2, 6, 7} 3. {3, 7, 8} 4. {4, 8, 10} 5. {5, 9, 10}
6. {1, 3,10} 7. {1, 4, 7} 8. {2, 5, 8} 9. {5, 6, 3} 10. {4, 9, 2}

We observe that each of the symbol in the above triangle triplets occurs in exactly r sets. Also, we can verify the conditions vr=bk, as v=10, b=10, r=3. k=3. By considering all possible triangle triplets as blocks, a PBIB designs with two associate class association scheme can be constructed. This association scheme is defined as under:
"Consider those blocks which have exactly one treatment in common, namely θ. Treatments which occur with treatment θ are first associates and all the remaining treatments are 2^{nd} associates of θ".

The following table shows the association scheme with two associate classes.

Symbols	*1^{st} associates*	*2^{nd} associates*
1.	2,5,8	3,4,6,7,9,10
2.	1,3,10	4,5,6,7,8,9
3.	2,3,9	1,5,6,7,8,10
4.	3,5,6	1,2,7,8,9,10
5.	1,4,5	2,3,6,8,9,10
6.	4,6,8	1,2,3,5,7,9
7.	5,9,10	1,2,3,4,6,8
8.	1,6,9	2,3,4,5,7,10
9.	3,7,8	1,2,4,5,6,10
10.	2,6,7	1,3,4,5,8,9

Thus the graph G_1 is a PBIB –graph with the parameters of first kind (v, b, r, k, λ_1, λ_2) are given by v=10, b=10, r=3, k=3, λ_1 =0, λ_2 =1 and parameters of second kind(n_1,n_2) are given by n_1=3,n_2=6 .

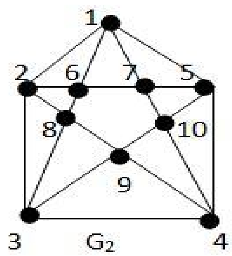
Also the P-matrices of the association scheme are

$$P_1 = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

E_1=65.07%, E_2=75.92%, E=71.92%

Theorem 3.2: The set of all possible quadruplets in Pentagon graph G_2 shown below is a PBIB design with two associate class association scheme.

Proof: By taking points of the graph G_2 shown below as the treatments, and lines inside the Pentagram as the set of blocks of a PBIB design. The blocks of G_2 are given by



1. {1, 6, 8, 3} 2.{1, 4, 7, 10} 3.{2, 6, 7, 5} 4. {2, 8, 9, 4} 5.{3, 9, 10, 5}

We observe that each of the symbol in the above quadruplets occurs in exactly r sets. Also, we can verify the conditions vr=bk, as v=10, b=5, r=2, k=4. By considering quadruplets as blocks, a PBIB designs with two associate class association scheme can be constructed . This association scheme is defined as under:

"Consider those blocks which have exactly one treatment in common namely θ. Treatment which occur with θ are first associates and all the remaining treatments are 2nd associates of θ".

The following table explains the association scheme with two associate classes:

Symbols	*1st associate*	*2nd associate*
1.	2,5,9	3,4,6,7,8,10
2.	1,3,10	4,5,6,7,8,9
3.	2,4,7	1,5,6,8,9,10
4.	3,5,6	1,2,7,8,9,10
5.	1,4,8	2,3,6,7,9,10
6.	4,9,10	1,2,3,5,7,8
7.	2,4,6	1,4 10,3,5,9
8.	5,7,10	1,2,3,4,6,9
9.	1,6,7	2,4,8,3,5,10
10.	2,6,8	1,4,7,3,5,9

The parameters of first kind (v, b, r, k, λ_1, λ_2) are given by v=10, b=5, r=2, k=4, $\lambda_1 = 0$, $\lambda_2 = 1$ and parameters of second kind (n_1, n_2) are given by $n_1=3$, $n_2=6$.

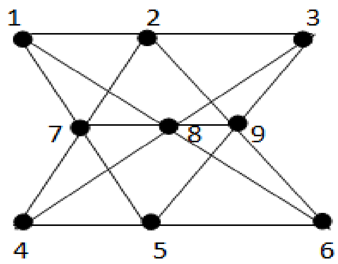
Also the P-matrices of the association scheme are

$$P_1 = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

E_1=71.42%, E_2=83.33%, E=78. 94%

Theorem 3.3: The set of all possible triangles in pappus graph G_3 forms blocks of PBIB designs with two associate class association scheme.

Proof: By taking points of the graph G_3 shown below as the treatments, and blocks of the designs are all possible triangles in a pappus graph.



G_3

The blocks of the graph G_3 are given by

1. {1,2,7} 2. {1,8,7} 3.{1,5,3} 4.{1,6,5} 5. {2,7,9} 6.{2,3,9} 7.{3,8,9} 8.{4,7,8} 9.{4,5,7} 10.{4,8,6} 11.{5,7,9} 12.{6,8,9} 13.{5,6,9} 14.{1,3,8} 15.{2,3,4} 16.{3,4,5} 17.{2,4,6} 18.{1,2,6}

We observe each of the above symbol in the above triangle triplets sets occurs exactly in r sets. Also, we can verify the conditions vr=bk, as v=9, b=18, r=6, k=3. By considering all possible triangles triplets as blocks a PBIB designs with two associate class association scheme can be constructed. This association scheme is defined as under:

"A pair of treatments, say u and v together are either λ_1 times or λ_2 times. Those occurring together λ_1 =0 times are first associates and all the remaining treatments λ_2=2 are second associates of each other".

The following table explains the association scheme with two associate classes

Symbols	*I^st associates*	*2^nd associates*
1.	4,9	2,3,5,6,7,8
2.	5,8	1,3,4,6,7,9
3.	6,7	1,2,4,5,8,9
4.	1,9	2,3,5,6,7,8
5.	2,8	1,3,4,6,7,9
6.	3,7	1,2,4,5,8,9
7.	3,6	1,2,4,5,8,9
8.	2,5	1,3,4,6,7,9
9.	1,4	2,3,5,6,7,8

Thus the graph G_3 is a PBIB-graph with the parameters of first kind (v, b, r, k, λ_1, λ_2) are given by V=9, b=18, r=6, k=3, λ_1 =0, λ_2 =2 and parameters of second kind (n_1, n_2) are given by n_1=2, n_2=6.

Also the P-matrices of the association scheme are

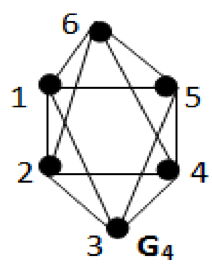
$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$$

E_1=66.67%, E_2=75%, E=72. 92%

Theorem 3.4: The set of all possible triangles in Hexagon graph G_4 shown below forms blocks of PBIB designs with two associate class association scheme.

Proof: By taking points of the graph G_4 shown below as the treatments and blocks are all possible triangles in a Hexagon graph. We see that all possible triangles of the graph G_5 are given by

1.{1,3,5} 2.{1,6,5} 3.{2,4,6} 4.{2,3,4}



We observe that each of the symbol in the above triangle triplets occurs in exactly r sets. Also, we can verify the conditions vr=bk, as v=6, b=4, r=2, k=3. By considering all possible triangles triplets as blocks, a PBIB designs with two associate class association scheme can be constructed. This association scheme can be defined as under:

"Consider two sets, say P and Q not having any symbol common between them. Also consider two sets, say R and S not having any symbol common between them. Then symbols common between either P with R and S or Q with R and S are first associate of each other and all the remaining treatments are 2^{nd} associates of each other".

The following table explains the association scheme with two associate classes:

Symbols	*I^{st} associates*	*2^{nd} associates*
1.	2,4,6	3,5
2.	1,3,5	4,6
3.	2,4,6	1,5
4.	1,3,5	2,6
5.	2,4,6	1,3
6.	1,3,5	2,4

The parameters of first kind (v, b, r, k, λ_1, λ_2) are given by v=6, b=4, r=2, k=3, λ_1=0, λ_2 =1 and parameters of second kind (n_1, n_2) are given by n_1=3, n_2=2.

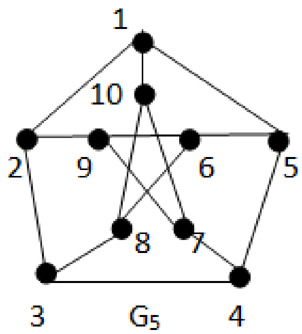
Also the P-matrices of the association scheme are

$$P_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

E_1=55.55%, E_2=83.33%, E=64.10%

Theorem 3.5: The set of all quadruplets in a Petersen graph G_5 is a PBIB design with three associate class association scheme.

Proof: By taking Points of the graph G_5 shown below as the treatments and the lines inside the Petersen Graph as the set of blocks of a PBIB design. The blocks of the graph G_5 are given by



1. (1,10,8,3) 2.(1,10,7,4) 3.(2,9,6,5) 4.(2,9,7,4) 5.(3,8,6,5)

We observe that each of the symbol in the above quadruplets occurs in exactly r sets. Also, we can verify the conditions vr=bk, as v=10, b=5, r=2, k=4. By considering quadruplets as blocks, a PBIB designs with three associate class association scheme can be constructed. This association scheme is defined as under:

" A pair of treatments say u and v together occur either λ_1 times or λ_2 or λ_3 times .Those occurring together λ_1=0 times are first associates. Those occurring together λ_2 = 1 times are first associates and all the remaining treatments are third associates of each other".

The following table explains the association scheme with three associate classes:

Symbols	*Ist associates*	*2nd associates*	*3rd associates*
1.	2,5,6,9	3,4,7,8	10
2	1,3,8,10	4,5,6,7	9
3.	2,4,7,9	1,5,6,10	8
4.	3,5,6,8	1,2,9,10	7
5.	1,4,7,10	2,3,8,9,	6
6.	1,4,7,10	2,3,8,9,	5
7.	3,5,6,8	1,2,9,10	4
8.	2,4,7,9	1,5,6,10	3
9.	1,3,8,10	4,5,6,7	2
10.	2,5,6,9	3,4,7,8	1

Thus the graph G_6 is a PBIB –graph with the parameters of first kind (v, b, r, k, λ_1, λ_2, λ_3) are given by v=10, b=5, r=2, k=4, λ_1 =0, λ_2 =1, λ_3 =2 and parameters of second kind(n_1,n_2, n_3) are given by n_1=4, n_2=4, n_3=1.

Also the P-matrices of the association scheme are

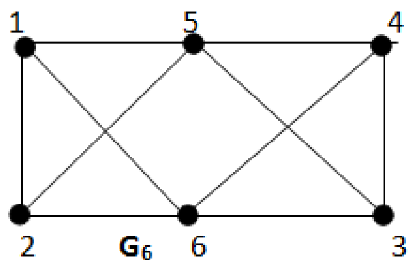
$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

E_1=62.06%, E_2=78.26%, E_3=1, E=71. 68%

Theorem 3.6: The graph G_6 shown below is a PBIB design with three associate class association scheme.

Proof: By taking points of the graph G_6 shown below as the treatments and blocks are all possible triangles in the graph G_6. The blocks of the graph G_6 are given by

1.{1,2,6} 2.{1,6,4} 3.{3,4,6} 4.{1,2,5} 5.{2,5,3} 6.{3,5,4}



We observe that each of the above triangle triplets occurs in exactly r sets. Also, we can verify the conditions vr=bk, as v=6, b=6, r=3, k=3. By considering all possible triangle triplets as blocks, a PBIB designs with three associate class association scheme can be constructed. This association scheme can be defined as under:

"A pair of treatments says u and v together occur either λ_1 times or λ_2 or λ_3 times. Those occurring together λ_1 times are first associates. Those occurring together λ_2 times are first associates and all the remaining treatments are third associates of each other".

The following table explains the association scheme with two associate classes:

Symbols	*I^st associates*	*2^nd associates*	*3^rd associates*
1.	3	4,5	2,6
2	4	3,6	1,5
3.	1	2,6	4,5
4.	2	1,5	3,6
5.	6	1,4	2,3
6.	5	2,3	1,4

Thus the graph G_6 is a PBIB-graph with the parameters of first kind (v, b, r, k, λ_1, λ_2, λ_3) are given by v=6, b=6, r=3, k=3, λ_1 =0, λ_2 =1, λ_3 =2 and parameters of second kind (n_1, n_2, n_3) are given by n_1=1, n_2=2, n_3=2.

Also the P-matrices of the association scheme are

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

E_1=63.33%, E_2=68.77%, E_3=81.40%, E=72. 00%

IV. CONCLUSIONS

In this paper, we consider six configurations through chosen lines and triangles of graphs, as a result we get new PBIB designs with two and three associate class PBIB designs. Efficiencies of the new designs are also computed for the purpose of comparison. Some designs are new with parameters given in theorem II, and some newly constructed designs with parameters given in theorem I are more efficient, as compared to the existing designs with same parameters as listed in the tables of Clatworthy [1971].

REFERENCES

[1] H.B Walikar, H.S Ramane, B.D Acharya, H.S Shekhareppa, S.Arumugum, "Partially balanced incomplete block design arising from minimum dominating sets of paths and cycles", *AKCE J. Graphs Combin.* **4**(2), pp. 223-232(2007).

[2] B Sharada, and N.D Soner ," Partially Balanced Incomplete Block designs arising from minimum efficient dominating sets of a graphs", *Bull.Pure.Appl.Math.*,2(1).pp.47-56(2008)

[3] H.B Wailkar B.D Acharaya, S Shrikol Shailaja, "*Designs associated with maximum independent sets of a graph", Des. Codes. Cryptography., Springer, New york* (2009).

[4] P.N. Vinay Kumar and D.Soner Nandappa, "Designs associated with maximum independent sets of cubic graphs", *Journal of computer and Mathematical Science* Vol. 1(3), pp.300-309 (2010)

[5] Anwar Alwardi and N.D.Soner," Partially Balanced Incomplete Block designs arising from some minimal dominating sets of SRNT graphs", *International Journal of Mathematical Archive*-2(2), pp.233-235 (2011).

[6] P.N.Vinay Kumar, D.S. Nandappa and U.S. Mahabaleswar, "Partially Balanced incomplete Block design with minimum perfect dominating sets of Clebesch graph", *International Journal of Applied Mathematics and computation* Vol.4(1), pp.39-48 (2012).

[7] S.Shrikol Shailaja, C. Gudageri Manjula and H.G.Shekharappa, "Partially balanced incomplete block design arising from minimum total dominating sets in a graph", *International Journal Mathematics Research*. Vol.5(1), pp.119-126(2013).