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Numerical Solution of Fuzzy Differential Equations by Adams fifth order predictor-corrector method

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Abstract

In this paper, an Adams fifth order predictor-corrector (AFPC) method is developed to solve the fuzzy initial value problems(IVPs). The AFPC method is generated by compining an explicit five step method and implicit four step method. The convergence and stability of the proposed methods are also presented in detail. These methods are illustrated by solving some examples.

Keywords: Fuzzy differential equations; Adams fifth order predictor-corrector method.

1. Introduction

The concept of fuzzy derivative was first introduced by Chang, Zadeh in [6] it was followed up by Dubois, Prede in [7], who defined and used the extension principle. The fuzzy differential equation and the intial value problem where regularly treated by Kaleva in [18, 19] and by Seikkal in [20]. The numerical method for solving fuzzy differential equations is introduced by Ma, Friedmen and Kandedl in [22] by the standard Eular method and [1, 2] by Taylor method. In the last few years many works have been performed by several authours in numerical solutions of fuzzy differential equations [1, 2, 3, 4, 5, 11, 12, 20]. Recently, the nemerical solution by predictor-corrector method has been stiuded in[5]. In this work we replace the fuzzy differential equation by its parametric form and then solve numerically the new system. Which consider the two classic ordinarry differintial equations with initial condition.

In this paper we develop numerical solution of fuzzy differential equation by an application of the fifth order predictor-corrector method. In Section 2, some basic definitions and results are brought. In Section 3 we define the problem, this is a fuzzy Cauchy problem. In Section 4 Adams-Bashforth five step method for solving fuzzy differential equations are introduced. In Section 5, Adams-Moulton four step methods for solving fuzzy differential equations are proposed. Adams fifth order predictor-corrector algorithm is discussed in Section 6 convergence and stability of the mentioned methods are proved in Section 7. Two examples are presented in Section 8.

2. Preliminaries

2.1 Multistep methods

Definition 2.1.

An m-step method for solving the initial-value problem is one whose difference equation for finding the approximation $y(t_{i+1})$ at the mesh point t_{i+1} can be represented by the following equation:

$$y(t_{i+1}) = a_{m-1}y(t_i) + a_{m-2}y(t_{i-1}) + ... + a_0 y(t_{i+1-m})$$
$$+h\{b_m f(t_{i+1}, y_{i+1}) + b_{m-1}f(t_i, y_i) + ... + b_0 f(t_{i+1-m}, y_{i+1-m})\}, \tag{1}$$

for $i = m-1, m, ..., N-1$, such that $a = t_0 \le t_1 \le ... \le t_N = b$,
$h = \frac{(b-a)}{N} = t_{i+1} - t_i$, and $a_0, a_1, ..., a_{m-1}$, $b_0, b_1, ..., b_m$ are constants with the starting values

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, ..., y_{m-1} = \alpha_{m-1}$$

. when $b_m = 0$, the method is known as explicit, since (1) gives y_{i+1} explicit in terms of previously determined values. When $b_m \neq 0$, the method is know as implicit , since y_{i+1} occurs on both sides of (1) and is specified only implicitly.

With consideration Definition 2.1, several multi step methods are as follows:

Adams-Bashforth five-Step method:

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_3 = \alpha_3 \quad y_4 = \alpha_4,$$

$$y_{i+1} = y_i + \frac{h}{720}[1901f(t_i, y_i) - 2774f(t_{i-1}, y_{i-1}) + 2664f(t_{i-2}, y_{i-2}) - 1274f(t_{i-3}, y_{i-3})$$
$$+251f(t_{i-4}, y_{i-4}], \qquad where \;\; i = 4, \dots, N-1.$$

Adams-Moulton four-Step method:

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_2 = \alpha_3,$$

$$y_{i+1} = y_i + \frac{h}{720}[251f(t_{i+1}, y_{i+1}) + 646f(t_i, y_i) - 264f(t_{i-1}, y_{i-1}) + 106f(t_{i-2}, y_{i-2})$$
$$-19f(t_{i-3}, y_{i-3})],$$
$$where \;\; i = 4, \dots, N-1.$$

Definition 2.2.

Associated with the difference equation

$$\left.\begin{aligned} &y_{i+1} = a_{m-1}y_i + a_{m-2}y_{i-1} + \dots + a_0 y_{i+1-m} + hF(t_i, h, y_{i+1}, y_i, \dots, y_{i+1-m}), \\ &y_0 = \alpha, \quad y_1 = \alpha_1, \dots, y_{m-1} = \alpha_{m-1}, \end{aligned}\right\} \tag{2}$$

is a polynomiyal, called the characteristic polynomial of the method given by

$$p(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \dots - a_1\lambda - a_0.$$

If $|\lambda_i| \le 1$ for each $i = 1, 2, \dots, m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Theorem 2.1

A multistep method of the form (2) is stable if and only if it satisfies the root condition.
Proof. See [10].

Notations used in this chapter is as follows:
A tilde is placed over a symbol to denote a fuzzy set so $\widetilde{\alpha_1}, \widetilde{f}(t), \ldots$
An arbitrary fuzzy number with an ordered pair of functions $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \le \alpha \le 1$, which satisfy the following requirements is represented.
1. $\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function over [0,1],
2. $\overline{u}(\alpha)$ is a bounded left continuous nonincreasing function over [0,1],
3. $\underline{u}(\alpha) \le \overline{u}(\alpha), 0 \le \alpha \le 1$.
Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded α-level intervals. It means if $v \in E$ then the α-level set

$$[v]^\alpha = \{s|v(s) \ge \alpha\}, \quad 0 < \alpha \le 1,$$

is a closed bounded intervel which is denoted by

$$[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha].$$

Let I be a real interval. A mapping $y : I \to E$ is called a fuzzy process and its α-leval set is denoted by

$$[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)], \quad t \in I, \quad \alpha \in (0,1].$$

Triangular fuzzy numbers are those fuzzy sets in E which are characterized by an ordered triple $(x^l, x^c, x^r) \in R^3$ with $x^l \le x^c \le x^r$ such that $[U]^0 = [x^l, x^r]$ and $[U]^1 = \{x^c\}$ then

$$[U]^\alpha = [x^c - (1-\alpha)(x^c - x^l), x^c + (1-\alpha)(x^r - x^c)], \tag{3}$$

for any $\alpha \in I$.

Definition 2.3.

The supremum metric d_∞ on E is defined by

$$d_\infty(U, V) = sup\{d_H([U]^\alpha, [V]^\alpha) : \alpha \in I\},$$

and (E, d_∞) is a complete metric space.

Definition 2.4.

A mapping $F : I \to E$ is Hukuhara differentiable at $t_0 \in T \subseteq R$ if for some $h_0 > 0$ the Hukuhara difference

$$F(t_0 + \Delta t) \sim_h F(t_0), \quad F(t_0) \sim_h F(t_0 - \Delta t),$$

exist in E for all $0 < \Delta t < h_0$ and if there exists an $F'(t_0) \in E$ such that

$$\lim_{\Delta t \to 0+} d_\infty \frac{(F(t_0 + \Delta t) \sim_h F(t_0))}{\Delta t} - F'(t_0) = 0$$

and

$$\lim_{\Delta t \to 0+} d_\infty \frac{(F(t_0) \sim_h F(t_0 - \Delta t))}{\Delta t} - F'(t_0) = 0,$$

the fuzzy set $F'(t_0)$ is called the Hukuhara derivative of F at t_0.

Recall that $U \sim_h V = W \in E$ are defined on level sets, where $[U]^\alpha \sim_h [V]^\alpha = [W]^\alpha$ for all $\alpha \in I$. By consideration of definition of the metric d_∞, all the level set mappings $[F(t_0)]^\alpha$ are Hukuhara differentiable at t_0 with Hukuhara derivatives $[F'(t_0)]^\alpha$ for each $\alpha \in I$ when $F : I \to E$ is Hukuhara differentiable at t_0 with Hukuhara derivative $F'(t_0)$.

Definition 2.5.

The fuzzy integral

$$\int_a^b y(t)dt, \quad 0 \le a \le b \le 1,$$

is defined by

$$\left[\int_a^b y(t)dt\right]^\alpha = \left[\int_a^b \underline{y}^\alpha(t)dt, \int_a^b \overline{y}^\alpha(t)dt\right],$$

provided the Lebesgue integrals on the right exist.

Remark 2.1.

If $F : I \to E$ is Hukuhara differentiable and its Hukuhara derivative F' is integrable over [0,1], then

$$F(t) = F(t_0) + \int_{t_0}^t F'(s)ds.$$

for all values of t_0, t where $0 \le t_0 \le t \le 1$.

Definiton 2.6.

A mapping $y : I \to E$ is called a fuzzy process. We denote

$$[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)], \quad t \in I, \quad 0 < \alpha \le 1.$$

The Seikkala derivative $y'(t)$ of a fuzzy process y is defined by

$$[y'(t)]^\alpha = [(\underline{y}^\alpha)'(t), (\overline{y}^\alpha)'(t)], \qquad 0 < \alpha \le 1.$$

provided that this equation defines a fuzzy number $y'(t) \in E$.

Remark 2.2.

If $y : I \to E$ is Seikkala differentiable and its Seikkala derivative y' is integrable over [0,1], then

$$y(t) = y(t_0) + \int_{t_0}^t y'(s)ds,$$

for all values of t_0, t where $t_0, t \in I$.

3. A fuzzy Cauchy problem

Consider the first-order fuzzy differential equation $y' = f(t, y)$ where y is a fuzzy function of $t, f(t, y)$ is a fuzzy function of crisp variable t and fuzzy variable y, and y' is Hukuhara or Seikkala fuzzy derivative of y. If an initial value $\widetilde{y}(t_0) = \widetilde{\alpha}_0$ is given, a fuzzy Cauchy problem of first order will be obtained as follows:

$$\left.\begin{array}{l} y'(t) = f(t, y(t)), \quad t_0 \le t \le T, \\ \widetilde{y}(t_0) = \widetilde{\alpha}_0, \end{array}\right\} \tag{4}$$

Sufficient conditions for the existence of a unique solution to equation (2.4)are

(i) Continuity of f,

(ii) Lipschitz condition $d_\infty(f(t,x),(f(t,y)) \leq Ld_\infty(x,y), \quad L>0.$

3.1. Interpolation of fuzzy number

The problem of interpolation for fuzzy sets is as follows:
Suppose that at various time instant t information $f(t)$ is presented as fuzzy set. The aim is to approximate the function $f(t)$, for all t in the domine of f.
Let $t_0 < t_1 < \ldots < t_n$ be $n+1$ distinct points in R and let $\widetilde{u_0}, \widetilde{u_1}, \ldots, \widetilde{u_n}$ be $n+1$ fuzzy sets in E.
A fuzzy polynomial interpolation of the data is a fuzzy value continuous function $f: R \to E$ satisfying:
(i) $f(t_i) = \widetilde{u}_i, \quad i = 1, \ldots, n.$
(ii) If the data is cricp, then the interpolation f is a crisp polynomial.
A function f which fulfilling these condition may be constructed as follows.
Let $C^i_\alpha = [\widetilde{u}_i]^\alpha$ for any $\alpha \in [0,1], \quad i = 0,1,2,\ldots,n$. For each $x = (x_0, x_1, \ldots, x_n) \in R^{n+1}$, the unique polynomial of degree $\leq n$ denoted by P_X such that

$$P_X(t_i) = x_i, \qquad i = 0,1,2,\ldots,n,$$

$$P_X(t) = \sum_{i=0}^{n} x_i \left(\prod_{i \neq j} \frac{t - t_j}{t_i - t_j} \right).$$

Finally, for each $t \in R$ and all $\xi \in R$ is defined by $f(t) \in E$ by

$$(f(t))(\xi) = sup\{\alpha \in [0,1] : \exists X \in C^0_\alpha \times \ldots \times C^n_\alpha \quad \text{such that} \quad P_X(t) = \xi\}.$$

The interpolation polynomial can be written level set wise as

$$[f(t)]^\alpha = \{y \in R : y = P_X(t), \quad x \in [\widetilde{u}_i]^\alpha, \ i = 1,2,...,n\}, \qquad \text{for} \quad 0 \leq \alpha \leq 1.$$

When the data $\widetilde{u}_i$ presents as triangular fuzzy numbers, values of the interpolation polynomial are also triangular fuzzy numbers. Then $f(t)$ has a particular simple form that is well situated to computation.

Theorem 3.1.

Let $(t_i, \widetilde{u}_i), \quad i = 0,1,2,\ldots,n$ be the observed data and suppose that each of the $\widetilde{u}_i = (u^l_i, u^c_i, u^r_i)$ is an element of E. Then for each $t \in [t_0, t_n]$,
$\widetilde{f}(t) = (f^l(t), f^c(t), f^r(t)) \in E,$

$$\begin{aligned} f^l(t) &= \sum_{l_i(t)\geq 0} l_i(t)u^l_i + \sum_{l_i(t)<0} l_i(t)u^r_i, \\ f^c(t) &= \sum_{l=0}^{n} l_i(t)u^c_i, \\ f^r(t) &= \sum_{l_i(t)\geq 0} l_i(t)u^r_i + \sum_{l_i(t)<0} l_i(t)u^l_i, \end{aligned}$$

such that $l_i(t) = \prod_{j\neq i}^{n} \frac{t-t_j}{t_i-t_j}$.

Proof. See[13].

4. Adams-Bashforth methods

Now we are going to solve fuzzy initial value problem $y'(t) = f(t, y(t))$ by Adams-Bashforth five-step method.
Let the fuzzy initial values be $\widetilde{y}(t_{i-1}),\ \widetilde{y}(t_i),\ \widetilde{y}(t_{i+1}),\ \widetilde{y}(t_{i+2}),\ \widetilde{y}(t_{i+2}),\ \widetilde{y}(t_{i+3})$

$$i.e.\ \widetilde{f}(t_{i-1}, y(t_{i-1})),\ \widetilde{f}(t_i, y(t_i)),\ \widetilde{f}(t_{i+1}, y(t_{i+1}))\ ,\ \widetilde{f}(t_{i+2}, y(t_{i+2})),\ \widetilde{f}(t_{i+3}, y(t_{i+3})),$$

which are triangular fuzzy numbers and are shown by

$$\begin{array}{r} \left\{f^l(t_{i-1}, y(t_{i-1})), f^c(t_{i-1}, y(t_{i-1})), f^r(t_{i-1}, y(t_{i-1}))\right\}, \\ \left\{f^l(t_i, y(t_i)), f^c(t_i, y(t_i)), f^r(t_i, y(t_i))\right\}, \\ \left\{f^l(t_{i+1}, y(t_{i+1})), f^c(t_{i+1}, y(t_{i+1})), f^r(t_{i+1}, y(t_{i+1}))\right\}, \\ \left\{f^l(t_{i+2}, y(t_{i+2})), f^c(t_{i+2}, y(t_{i+2})), f^r(t_{i+2}, y(t_{i+2}))\right\}, \\ \left\{f^l(t_{i+3}, y(t_{i+3})), f^c(t_{i+3}, y(t_{i+3})), f^r(t_{i+3}, y(t_{i+3}))\right\}, \end{array}$$

also

$$\widetilde{y}(t_{i+4}) = \widetilde{y}(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}} \widetilde{f}(t, y(t))dt. \tag{5}$$

By fuzzy interpolation of $\widetilde{f}(t_{i-1}, (t_{i-1})),\ \widetilde{f}(t_i, (t_i)),\ \widetilde{f}(t_{i+1}, (t_{i+1})),\ \widetilde{f}(t_{i+2}, (t_{i+2})),\ \widetilde{f}(t_{i+3}, (t_{i+3}))$, we have:

$$\begin{aligned} f^l(t, y(t)) &= \sum_{j=i-1, l_j(t)\geq 0}^{i+3} l_j(t) f^l(t_j, y(t_j)) + \sum_{j=i-1, l_i(t)<0}^{i+3} l_j(t) f^r(t_j, y(t_j)), \\ f^c(t, y(t)) &= \sum_{j=i-1}^{i+3} l_j(t) f^c(t_j, y(t_j)), \\ f^r(t, y(t)) &= \sum_{j=i-1, l_j(t)\geq 0}^{i+3} l_j(t) f^r(t_j, y(t_j)) + \sum_{j=i-1, l_j(t)<0}^{i+3} l_j(t) f^l(t_j, y(t_j)), \end{aligned}$$

for $t_{i+3} \leq t \leq t_{i+4}$:

$$\begin{aligned} l_{i-1}(t) &= \frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})(t_{i-1}-t_{i+3})} \geq 0, \\ l_i(t) &= \frac{(t-t_{i-1})(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})}{(t_i-t_{i-1})(t_i-t_{i+1})(t_i-t_{i+2})(t_{i-1}-t_{i+3})} \leq 0, \\ l_{i+1}(t) &= \frac{(t-t_{i-1})(t-t_i)(t-t_{i+2})(t-t_{i+3})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})} \geq 0, \\ l_{i+2}(t) &= \frac{(t-t_{i-1})(t-t_i)(t-t_{i+1})(t-t_{i+3})}{(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})} \leq 0, \\ l_{i+3}(t) &= \frac{(t-t_{i-1})(t-t_i)(t-t_{i+1})(t-t_{i+2})}{(t_{i+3}-t_{i-1})(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} \geq 0, \end{aligned}$$

therefore the following results will be obtained:

$$f^l(t,y(t)) = l_{i-1}(t)f^l(t_{i-1},y(t_{i-1})) + l_i(t)f^r(t_i,y(t_i)) + l_{i+1}(t)f^l(t_{i+1},y(t_{i+1}))$$
$$+l_{i+2}(t)f^r(t_{i+2},y(t_{i+2})) + l_{i+3}(t)f^l(t_{i+3},y(t_{i+3})), \qquad (6)$$
$$f^c(t,y(t)) = l_{i-1}(t)f^c(t_{i-1},y(t_{i-1})) + l_i(t)f^c(t_i,y(t_i)) + l_{i+1}(t)f^c(t_{i+1},y(t_{i+1}))$$
$$+l_{i+2}(t)f^c(t_{i+2},y(t_{i+2})) + l_{i+3}f^c(t_{i+3},y(t_{i+3})), \qquad (7)$$
$$f^r(t,y(t)) = l_{i-1}(t)f^r(t_{i-1},y(t_{i-1})) + l_i(t)f^l(t_i,y(t_i)) + l_{i+1}(t)f^r(t_{i+1},y(t_{i+1}))$$
$$+l_{i+2}(t)f^l(t_{i+2},y(t_{i+2})) + l_{i+3}f^r(t_{i+3},y(t_{i+3})). \qquad (8)$$

From (3) and (5) it follows that:

$$\widetilde{y}^\alpha(t_{i+4}) = [\underline{y}^\alpha(t_{i+4}), \quad \overline{y}^\alpha(t_{i+4})],$$

where

$$\underline{y}^\alpha(t_{i+4}) = \underline{y}^\alpha(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}} \left\{\alpha f^c(t,y(t)) + (1-\alpha)f^l(t,y(t))\right\} dt \qquad (9)$$

and

$$\overline{y}^\alpha(t_{i+4}) = \overline{y}^\alpha(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}} \{\alpha f^c(t,y(t)) + (1-\alpha)f^r(t,y(t))\}\, dt, \qquad (10)$$

If (6) and (7) are in (9) and (7),(8) in (10):

$$\underline{y}^\alpha(t_{i+4}) = \underline{y}^\alpha(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}}\{\alpha(l_{i-1}(t)f^c(t_{i-1},y(t_{i-1})) + l_i(t)f^c(t_i,y(t_i))$$
$$+l_{i+1}(t)f^c(t_{i+1},y(t_{i+1})) + l_{i+2}(t)f^c(t_{i+2},y(t_{i+2})) + l_{i+3}((t)f^c(t_{i+3},y(t_{i+3}))$$
$$+(1-\alpha)l_{i-1}(t)f^l(t_{i-1},y(t_{i-1})) + l_i(t)f^r(t_i,y(t_i)) + l_{i+1}(t)f^l(t_{i+1},y(t_{i+1}))$$
$$+l_{i+2}(t)f^r(t_{i+2},y(t_{i+2})) + l_{i+3}(t)f^l(t_{i+3},y(t_{i+3}))\}dt,$$

$$\overline{y}^\alpha(t_{i+4}) = \overline{y}^\alpha(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}}\{\alpha(l_{i-1}(t)f^c(t_{i-1},y(t_{i-1})) + l_i(t)f^c(t_i,y(t_i))$$
$$+l_{i+1}(t)f^c(t_{i+1},y(t_{i+1})) + l_{i+2}(t)f^c(t_{i+2},y(t_{i+2})) + l_{i+3}((t)f^c(t_{i+3},y(t_{i+3}))$$
$$+(1-\alpha)l_{i-1}(t)f^r(t_{i-1},y(t_{i-1})) + l_i(t)f^l(t_i,y(t_i)) + l_{i+1}(t)f^r(t_{i+1},y(t_{i+1}))$$
$$+l_{i+2}(t)f^l(t_{i+2},y(t_{i+2})) + l_{i+3}((t)f^r(t_{i+3},y(t_{i+3})\}dt.$$

The following results will be obtained by integration:

$$\underline{y}^\alpha(t_{i+4}) = \underline{y}^\alpha(t_{i+3}) + \frac{1901h}{720}[\alpha f^c(t_{i+3},y(t_{i+3})) + (1-\alpha)f^l(t_{i+3},y(t_{i+3}))]$$
$$-\frac{2774h}{720}[\alpha f^c(t_{i+2},y(t_{i+2})) + (1-\alpha)f^r(t_{i+2},y(t_{i+2}))]$$
$$+\frac{2616h}{720}[\alpha f^c(t_{i+1},y(t_{i+1})) + (1-\alpha)f^l(t_{i+1},y(t_{i+1}))]$$
$$-\frac{1274h}{720}[\alpha f^c(t_i,y(t_i)) + (1-\alpha)f^r(t_i,y(t_i))]$$
$$+\frac{251h}{720}[\alpha f^c(t_{i-1},y(t_{i-1})) + (1-\alpha)f^l(t_{i-1},y(t_{i-1}))], \quad \text{and}$$

$$\overline{y}^{\alpha}(t_{i+4}) = \overline{y}^{\alpha}(t_{i+3}) + \frac{1901h}{720}[\alpha f^{c}(t_{i+3}, y(t_{i+3})) + (1-\alpha) f^{r}(t_{i+3}, y(t_{i+3}))]$$
$$-\frac{2774h}{720}[\alpha f^{c}(t_{i+2}, y(t_{i+2})) + (1-\alpha) f^{l}(t_{i+2}, y(t_{i+2}))]$$
$$+\frac{2616h}{720}[\alpha f^{c}(t_{i+1}, y(t_{i+1})) + (1-\alpha) f^{r}(t_{i+1}, y(t_{i+1}))]$$
$$-\frac{1274h}{720}[\alpha f^{c}(t_{i}, y(t_{i})) + (1-\alpha) f^{l}(t_{i}, y(t_{i}))]$$
$$+\frac{251h}{720}[\alpha f^{c}(t_{i-1}, y(t_{i-1})) + (1-\alpha) f^{r}(t_{i-1}, y(t_{i-1}))],$$

Thus

$$\underline{y}^{\alpha}(t_{i+4}) = \underline{y}^{\alpha}(t_{i+3}) + \frac{h}{720}[1901\underline{f}^{\alpha}(t_{i+3}, y(t_{i+3})) - 2774\overline{f}^{\alpha}(t_{i+2} y(t_{i+2}))$$
$$+2616\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) - 1274\overline{f}^{\alpha}(t_{i}, y(t_{i})) + 251\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1}))],$$

$$\overline{y}^{\alpha}(t_{i+4}) = \overline{y}^{\alpha}(t_{i+3}) + \frac{h}{720}[1901\overline{f}^{\alpha}(t_{i+3}, y(t_{i+3})) - 2774\underline{f}^{\alpha}(t_{i+2}, y(t_{i+2}))$$
$$+2616\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) - 1274\underline{f}^{\alpha} t_{i}, y(t_{i}) + 251\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1}))].$$

Therefore Adams-Basforth five-step method is obtained as follows:

$$\left.\begin{aligned}
&\underline{y}^{\alpha}(t_{i+4}) = \underline{y}^{\alpha}(t_{i+3}) + \frac{h}{720}[1901\underline{f}^{\alpha}(t_{i+3}, y(t_{i+3})) - 2774\overline{f}^{\alpha}(t_{i+2}, y(t_{i+2}))\\
&\qquad +2616\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) - 1274\overline{f}^{\alpha}(t_{i}, y(t_{i})) + 251\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1}))],\\
&\overline{y}^{\alpha}(t_{i+4}) = \overline{y}^{\alpha}(t_{i+3}) + \frac{h}{720}[1901\overline{f}^{\alpha}(t_{i+3}, y(t_{i+3})) - 2774\underline{f}^{\alpha}(t_{i+2}, y(t_{i+2}))\\
&\qquad +2616\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) - 1274\underline{f}^{\alpha}(t_{i}, y(t_{i})) + 251\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})],\\
&\underline{y}^{\alpha}(t_{i-1}) = \alpha_0,\ \underline{y}^{\alpha}(t_{i}) = \alpha_1,\ \underline{y}^{\alpha}(t_{i+1}) = \alpha_2,\ \underline{y}^{\alpha}(t_{i+2}) = \alpha_3,\ \underline{y}^{\alpha}(t_{i+3}) = \alpha_4\\
&\overline{y}^{\alpha}(t_{i-1}) = \alpha_5,\ \overline{y}^{\alpha}(t_{i}) = \alpha_6,\ \overline{y}^{\alpha}(t_{i+1}) = \alpha_7\ , \overline{y}^{\alpha}(t_{i+2}) = \alpha_8, \overline{y}^{\alpha}(t_{i+3}) = \alpha_9.
\end{aligned}\right\} \quad (11)$$

5. Adams-Moulton methods

Fix $k \in Z^{+}$. The fuzzy initial value problem (4) can be solved by Adams-Moulton five-step method. The Adams-Moulton five step method is obtained as follows:

$$\left.\begin{aligned}
&\underline{y}^{\alpha}(t_{i+3}) = \underline{y}^{\alpha}(t_{i+2}) + \frac{h}{720}[251\underline{f}^{\alpha}(t_{i+3}, y(t_{i+3})) + 646\underline{f}^{\alpha}(t_{i+2}, y(t_{i+2}))\\
&\qquad -264\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) + 106\underline{f}^{\alpha}(t_{i}, y(t_{i})) - 19\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1}))],\\
&\overline{y}^{\alpha}(t_{i+3}) = \overline{y}^{\alpha}(t_{i+2}) + \frac{h}{720}[251\overline{f}^{\alpha}(t_{i+3}, y(t_{i+3})) + 646\overline{f}^{\alpha}(t_{i+2}, y(t_{i+2}))\\
&\qquad -264\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1})) + 106\overline{f}^{\alpha}(t_{i}, y(t_{i})) - 19\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1}))],
\end{aligned}\right\} \quad (12)$$

$$\left.\begin{array}{l}\underline{y}^{\alpha}(t_{i-1})=\alpha_0,\underline{y}^{\alpha}(t_i)=\alpha_1,\underline{y}^{\alpha}(t_{i+1})=\alpha_2,\underline{y}^{\alpha}(t_{i+2})=\alpha_3,\underline{y}^{\alpha}(t_{i+3})=\alpha_4,\\ \overline{y}^{\alpha}(t_{i-1})=\alpha_5,\overline{y}^{\alpha}(t_i)=\alpha_6,\overline{y}^{\alpha}(t_{i+1})=\alpha_7,\overline{y}^{\alpha}(t_{i+2})=\alpha_8,\overline{y}^{\alpha}(t_{i+3})=\alpha_9.\end{array}\right\}.$$

6. Algorithm

The following algorithm is based on Adams-Bashforth five step method as a predictor and also an iteration of Adams-Moulton four-step method as a corrector.
To approximate the solution of following fuzzy initial value problem.

$$y'(t)=f(t,y(t)),\quad t_0\le t\le T,$$

$$\underline{y}^{\alpha}(t_0)=\underline{\alpha}_0,\ \underline{y}^{\alpha}(t_1)=\underline{\alpha}_1,\ \underline{y}^{\alpha}(t_2)=\underline{\alpha}_2,\ \underline{y}^{\alpha}(t_3)=\underline{\alpha}_3,\ \underline{y}^{\alpha}(t_4)=\underline{\alpha}_4,$$

$$\overline{y}^{\alpha}(t_0)=\overline{\alpha}_5,\ \overline{y}^{\alpha}(t_1)=\overline{\alpha}_6,\ \overline{y}^{\alpha}(t_2)=\overline{\alpha}_7,\ \overline{y}^{\alpha}(t_3)=\overline{\alpha}_8,\ \overline{y}^{\alpha}(t_4)=\overline{\alpha}_9.$$

positive integer N is chosen.

Step 1.
Let $h=\dfrac{T-t_0}{N}$,

$$\left\{\begin{array}{l}\underline{w}^{\alpha}(t_0)=\underline{\alpha}_0,\underline{w}^{\alpha}(t_1)=\underline{\alpha}_1,\underline{w}^{\alpha}(t_2)=\underline{\alpha}_2,\underline{w}^{\alpha}(t_3)=\underline{\alpha_3},\underline{w^{\alpha}}(t_4)=\underline{\alpha_4},\\ \overline{w}^{\alpha}(t_0)=\overline{\alpha}_5,\overline{w}^{\alpha}(t_1)=\overline{\alpha}_6,\overline{w}^{\alpha}(t_2)=\overline{\alpha}_7,\overline{w}^{\alpha}(t_3)=\overline{\alpha}_8,\overline{w}^{\alpha}(t_4)=\overline{\alpha}_9.\end{array}\right.$$

Step 2.
Let $i=1$.
Step 3. Let

$$\left\{\begin{array}{l}\underline{w}^{(0)\alpha}(t_{i+4})=\underline{w}^{\alpha}(t_{i+3})+\dfrac{h}{720}[1901\underline{f}^{\alpha}(t_{i+3},w(t_{i+3}))-2774\overline{f}^{\alpha}(t_{i+2},w(t_{i+2}))\\ \qquad+2616\underline{f}^{\alpha}(t_{i+1},w(t_{i+1}))-1274\overline{f}^{\alpha}(t_i,w(t_i))+251\underline{f}^{\alpha}(t_{i-1},w(t_{i-1}))],\\ \overline{w}^{(0)\alpha}(t_{i+4})=\overline{w}^{\alpha}(t_{i+3})+\dfrac{h}{720}[1901\overline{f}^{\alpha}(t_{i+3},w(t_{i+3}))-2774\underline{f}^{\alpha}(t_{i+2},w(t_{i+2}))\\ \qquad+2616\overline{f}^{\alpha}(t_{i+1},w(t_{i+1}))-1274\underline{f}^{\alpha}(t_i,w(t_i))+251\overline{f}^{\alpha}(t_{i-1},w(t_{i-1}))].\end{array}\right.$$

Step 4. Let $t_{i+4}=t_0+(i+4)h$.
Step 5. Let

$$\left\{\begin{array}{l}\underline{w}^{\alpha}(t_{i+3})=\underline{y}^{\alpha}(t_{i+2})+\dfrac{h}{720}[251\underline{f}^{\alpha}(t_{i+3},w(t_{i+3}))+646\underline{f}^{\alpha}(t_{i+2},w(t_{i+2}))\\ \qquad-264\overline{f}^{\alpha}(t_{i+1},w(t_{i+1}))+106\underline{f}^{\alpha}(t_i,w(t_i))-19\overline{f}^{\alpha}(t_{i-1},w(t_{i-1}))],\\ \overline{w}^{\alpha}(t_{i+3})=\overline{w}^{\alpha}(t_{i+2})+\dfrac{h}{720}[251\overline{f}^{\alpha}(t_{i+3},w(t_{i+3}))+646\overline{f}^{\alpha}(t_{i+2},w(t_{i+2}))\\ \qquad-264\underline{f}^{\alpha}(t_{i+1},y(t_{i+1}))+106\overline{f}^{\alpha}(t_i,w(t_i))-19\underline{f}^{\alpha}(t_{i-1},w(t_{i-1}))].\end{array}\right.$$

Step 6. $i=i+1$.
Step 7. If $i\le N-3$ go to step 3.

Step 8. Algorithm will be completed and $(\underline{w}^{\alpha}(T), \overline{w}^{\alpha}(T)$ approximates real value of $(\underline{Y}^{\alpha}(T), \overline{Y}^{\alpha}(T)$.

7. Convergence and Stability

To integrate the system given in equation (12) from t_0 a prefixed $T > t_0$ the interval $[t_0, T]$ will be replaced by a set of discrete equally spaced grid point $t_0 < t_1 < t_2 < ... < t_N = T$ which the exact solution $(\underline{Y}(t,\alpha), \overline{Y}(t,\alpha))$ is approximated by some $(\underline{y}(t,\alpha), \overline{y}(t,\alpha))$. The exact and approximate solutions at t_n, $0 \le n \le N$ are denoded by $Y_n(t,\alpha) = (\underline{Y}_n(t,\alpha), \overline{Y}_n(t,\alpha))$, and $y_n(t,\alpha) = (\underline{y}_n(t,\alpha), \overline{y}_n(t,\alpha))$, respectively. The grid points which the solution is calculated are $t_n = t_0 + nh, h = \frac{(T-t_0)}{N}, 1 \le n \le N$.

From (11), the polygon curves

$$\underline{y}(t,h,\alpha) = \{[t_0, \underline{y_0}(\alpha)], [t_1, \underline{y_1}(\alpha)], \ldots, [t_N, \underline{y}_N(\alpha)]\}$$

$$\overline{y}(t,h,\alpha) = \{[t_0, \overline{y_0}(\alpha)], [t_1, \overline{y_1}(\alpha)], \ldots, [t_N, \overline{y}_N(\alpha)]\}$$

are the Adams-Moulton approximates to $\underline{Y}(t,\alpha)$ and $\overline{Y}(t,\alpha)$, respectively, over the interval $t_0 \le t \le t_N$. The following lemmas will be applied to show convergence of these apptoximates, i.e.

$$\lim_{h\to 0} \underline{y}(t,h,\alpha) = \underline{Y}(t,\alpha),$$

$$\lim_{h\to 0} \overline{y}(t,h,\alpha) = \overline{Y}(t,\alpha).$$

Lemma 7.1.

Let a sequence of numbers $\{W_n\}_{n=0}^{N}$ satisfy

$$|w_{n+1}| \le A|w_n| + B|w_{n-1}| + C|w_{n-2}| + D|w_{n-3}| + E, \quad 0 \le n \le N-1$$

for some given positive constants A,B,C D and E. Then

$$|w_{n+1}| \le (A^{n-2} + \alpha_1 A^{n-4}B + \alpha_2 A^{n-5}C + ... + \alpha_w A D^{[\frac{n-5}{2}]})|w_3|$$

$$+(A^{n-3}B + \beta_1 A^{n-5}B^2 + ... + \beta_s B^{[\frac{n-5}{2}]} D^{[\frac{n-5}{2}]})|w_2|$$

$$+(A^{n-3}C + \gamma_1 A^{n-5}CB^2 + ... + \gamma_t C^{[\frac{n-5}{2}]} D^{[\frac{n-5}{2}]})|w_1|$$

$$+(A^{n-3}D + \eta_1 A^{n-5}CD^2 + ... + \eta_q B D^{[\frac{n-3}{2}]})|w_0|$$

$$+(A^{n-3} + A^{n-4} + .. + 1)E + (\delta_1 A^{n-5} + \delta_2 A^{n-6} + ... + \delta_m A + 1)BE$$

$$+(\varsigma_1 A^{n-6} + \varsigma_2 A^{n-7} + \varsigma_2 A^{n-8} + ... + \varsigma_l A + 1)CE + (\lambda_1 A^{n-7} + \lambda_2 A^{n-8} + ... + \lambda_p A + 1)DE$$

$$+(\mu_1 A^{n-8} + \mu_2 A^{n-9} + ... + \mu_r A + 1)B^2E + ..., n \text{ odd} \quad \text{and}$$

$$|w_{n+1}| \le (A^{n-2} + \alpha_1 A^{n-4}B + \alpha_2 A^{n-5}C + ... + \alpha_w BD^{[\frac{n-4}{2}]})|w_3|$$
$$+(A^{n-3}B + \beta_1 A^{n-5}B^2 + \beta_2 A^{n-7}CB^3 + ... + \beta_s AD^{[\frac{n-4}{2}]})|w_2|$$
$$+(A^{n-3}C + \gamma_1 A^{n-5}CB^2 + ... + \gamma_t B^{[\frac{n-4}{2}]}D^{[\frac{n-4}{2}]})|w_1|$$
$$+(A^{n-3}D + \eta_1 A^{n-5}CD^2 + ... + \eta_q C^{[\frac{n-4}{2}]-}D^{[\frac{n-4}{2}]})|w_0|$$
$$+(A^{n-3} + A^{n-4} + .. + 1)E + (\delta_1 A^{n-5} + \delta_2 A^{n-6} + ... + \delta_m A + 1)BE$$
$$+(\varsigma_1 A^{n-6} + \varsigma_2 A^{n-7} + \varsigma_2 A^{n-8} + ... + \varsigma_l A + 1)CE + (\lambda_1 A^{n-7} + \lambda_2 A^{n-8} + ... + \lambda_p A + 1)DE$$
$$+(\mu_1 A^{n-8} + \mu_2 A^{n-9} + ... + \mu_r A + 1)B^2E + ..., n\ even$$

where $\alpha_w, \beta_s, \gamma_t, \delta_m, \varsigma_l, \lambda_p, \eta_q$ are constants for all w, s, t, m, l, q and r.
The proof, by using mathematical induction is straightforward.

Theorem 7.1

For arbitrary fixed $\alpha : 0 \le \alpha \le 1$, the Adams Moultan four-step approximates of (12) converges to the exact solution $\underline{Y}(t,\alpha), \overline{Y}(t,\alpha)$ for $\underline{y}, \overline{y} \in c^5[t_0, T]$.
Proof.
It is sufficient to show

$$\lim_{h\to 0} \underline{y}_N(\alpha) = \underline{Y}(T,\alpha), \lim_{h\to 0} \overline{y}_N(\alpha) = \overline{Y}(T,\alpha).$$

By using exact value the following results will be obtained:

$$\underline{Y}_{n+1}(t;\alpha) = \underline{Y}_n(t;\alpha) + \frac{h}{720}[251f(t_{n+1}, \underline{Y}_{n+1}(t;\alpha)) + 646f(t_n, \underline{Y}_n(t;\alpha))$$
$$-264f(t_{n-1}, \overline{Y}_{n-1}(t;\alpha)) + 106f(t_{n-2}, \underline{Y}_{n-2}(t;\alpha)) - 19f(t_{n-3}, \underline{Y}_{n-3}(t;\alpha))] - \frac{3h^5}{160}Y^{('''''')}(\underline{\xi}_n),$$
$$\overline{Y}_{n+1}(t;\alpha) = \overline{Y}_n(t;\alpha) + \frac{h}{720}[251f(t_{n+1}, \overline{Y}_{n+1}(t;\alpha)) + 646f(t_n, \overline{Y}_n(t;\alpha))$$
$$-264f(t_{n-1}, \underline{Y}_{n-1}(t;\alpha)) + 106f(t_{n-2}, \overline{Y}_{n-2}(t;\alpha)) - 19f(t_{n-3}, \overline{Y}_{n-3}(t;\alpha))] - \frac{3h^5}{160}Y^{('''''')}(\overline{\xi}_n),$$

where $t_n < \underline{\xi}_n, \overline{\xi}_n < t_{n+1}$, Consequently

$$\underline{Y}_{n+1}(t;\alpha) - \underline{y}_{n+1}(t;\alpha) = \underline{Y}_n(t;\alpha) - \underline{y}_n(t;\alpha) + \frac{h}{720}[251(f(t_{n+1}, \underline{Y}_{n+1}(t;\alpha) - f(t_{n+1}, \underline{y}_{n+1})(t;\alpha)))$$
$$+646(f(t_n, \underline{Y}_n(t;\alpha)) - f(t_n, \underline{y}_n(t;\alpha))) - 264(f(t_{n-1}, \overline{Y}_{n-1}(t;\alpha))$$
$$-f(t_{n-1}, \overline{y}_{n-1}(t;\alpha))) + 106(f(t_{n-2}, \underline{Y}_{n-2}(t;\alpha)) - f(t_{n-2}, \underline{y}_{n-2}(t;\alpha)))$$
$$-19(f(t_{n-3}, \underline{Y}_{n-3}(t;\alpha)) - f(t_{n-3}, \underline{y}_{n-3}(t;\alpha)))] - \frac{3h^5}{160}Y^{('''''')}(\underline{\xi}_n),$$

$$\overline{Y}_{n+1}(t;\alpha)-\overline{y}_{n+1}(t;\alpha)=\overline{Y}_n(t;\alpha)-\overline{y}_n(t;\alpha)+\frac{h}{720}[251(f(t_{n+1},\overline{Y}_{n+1}(t;\alpha)-f(t_{n+1},\overline{y}_{n+1}(t;\alpha))))$$
$$+646(f(t_n,\overline{Y}_n(t;\alpha))-f(t_n,\overline{y}_n(t;\alpha)))-264(f(t_{n-1},\underline{Y}_{n-1}(t;\alpha))$$
$$-f(t_{n-1},\underline{y}_{n-1}(t;\alpha)))+106(f(t_{n-2},\overline{Y}_{n-2}(t;\alpha))-f(t_{n-2},\overline{y}_{n-2}(t;\alpha)))$$
$$-19(f(t_{n-3},\overline{Y}_{n-3}(t;\alpha))-f(t_{n-3},\overline{y}_{n-3}(t;\alpha)))]-\frac{3h^5}{160}Y^{(''''')}(\underline{\xi}_n).$$

Denote $w_n=\underline{Y}_n(t;\alpha)-\underline{y}_n(t;\alpha), v_n=\overline{Y}_n-\overline{y}_n(t;\alpha)$.
Then

$$|w_{n+1}|\leq\left(1+\frac{646hL_1}{720}\right)|w_n|+\left(\frac{264hL_2}{720}\right)|v_{n-1}|+\left(\frac{251hL_3}{720}\right)|w_{n+1}|$$
$$+\left(\frac{106hL_4}{720}\right)|w_{n-2}|+\left(\frac{19hL_5}{720}\right)|v_{n-3}|+\frac{646}{720}h^5\underline{M},$$
$$|v_{n+1}|\leq\left(1+\frac{646hL_6}{720}\right)|v_n|+\left(\frac{264hL_7}{720}\right)|w_{n-1}|+\left(\frac{251hL_8}{720}\right)|v_{n+1}|$$
$$+\left(\frac{106hL_9}{720}\right)|v_{n-2}|+\left(\frac{19hL_{10}}{720}\right)|w_{n-3}|+\frac{646}{720}h^5\overline{M},$$

where $\underline{M}=\max_{t_0\leq t\leq T}|\underline{Y}^{('''''')}(t,r)| and\ \ \overline{M}=\max_{t_0\leq t\leq T}|\overline{Y}^{('''''')}(t,r)|$ and is put
$L=max\{L_1,L_2,L_3,L_4L_5,L_6,L_7,L_8,L_9,L_{10}\}\leq\frac{720}{251h}$,
Then

$$|w_{n+1}|\leq\left(1+\frac{395hL}{720-251hL}\right)|w_n|+\left(\frac{264hL}{720-251hL}\right)|v_{n-1}|+\left(\frac{106hL}{720-251hL}\right)|w_{n-2}|$$
$$+\left(\frac{19hL}{720-251hL}\right)|v_{n-3}|+\left(\frac{27}{1440-502hL}h^5\overline{M}\right),$$
$$|v_{n+1}|\leq\left(1+\frac{395hL}{720-251hL}\right)|v_n|+\left(\frac{264hL}{720-251hL}\right)|w_{n-1}|+\left(\frac{106hL}{720-251hL}\right)|v_{n-2}|$$
$$+\left(\frac{19hL}{720-251hL}\right)|w_{n-3}|+\left(\frac{27}{1440-502}h^5\overline{M}\right),$$

are resulted,where $|u_n|=|w_n|+|v_n|$,then by Lemma 7.1 and $w_0=v_0=0$(also with $w_1=v_1=0$):

$$|u_n|\leq\frac{\left(1+\frac{395hL}{720-251hL}\right)^{n-1}-1}{\frac{395hL}{720-251hL}}\times\frac{27}{1440-502hL}h^6(\underline{M}+\overline{M})$$
$$+\left\{\delta_1\left(1+\frac{395hL}{720-251hL}\right)^{n-5}+\delta_2\left(1+\frac{395hL}{720-251hL}\right)^{n-6}+.......+\delta_m\left(1+\frac{395hL}{720-251hL}\right)+1\right\}$$

$$\times\left(\frac{264hL}{720-251hL}\right)\left(\frac{27}{1440-502hL}h^6(\underline{M}+\overline{M})\right)$$

$$+\left\{\zeta_1\left(1+\frac{395hL}{720-251hL}\right)^{n-6}+\zeta_2\left(1+\frac{395hL}{720-251hL}\right)^{n-7}+.......+\zeta_p\left(1+\frac{395hL}{720-251hL}\right)+1\right\}$$

$$\times\left(\frac{106hL}{720-251hL}\right)\left(\frac{27}{1440-502hL}h^6(\underline{M}+\overline{M})\right)$$

$$+\left\{\lambda_1\left(1+\frac{395hL}{720-251hL}\right)^{n-7}+\lambda_2\left(1+\frac{395hL}{720-251hL}\right)^{n-8}+........+\lambda_q\left(1+\frac{10hL}{24-9hL}\right)+1\right\}$$

$$\times\left(\frac{19hL}{720-251hL}\right)\left(\frac{27}{1440-502hL}h^6(\underline{M}+\overline{M})\right)+$$

$$+\left\{\xi_1\left(1+\frac{395hL}{720-251hL}\right)^{n-8}+\xi_2\left(1+\frac{395hL}{720-251hL}\right)^{n-9}+......+\xi_q\left(1+\frac{10hL}{24-9hL}\right)+1\right\}$$

$$\times\left(\frac{264hL}{720-251hL}\right)^2\left(\frac{27}{1440-502hL}h^6(\underline{M}+\overline{M})\right)+........$$

are obtained. If $h\to 0$, then $w_n\to 0, v_n\to 0$ which concludes the proof.

Remark 7.1.
Above theorem results that convergence order is $O(h^5)$

Theorem 7.2.
For arbitrary fixed $r: 0\le r\le 1$ the Adams-Bashforth five-step approximates of Equation (11) converge to the exact solutions $\underline{Y}(t,\alpha),\overline{Y}(t,\alpha)$ for $\underline{Y},\overline{Y}\in C^3[t_0,T]$.

Proof.
Similar to Theorem 7.1

Remark 7.2.
It is easy to show that convergence order of Adams-Bashforth five-step method is $O(h^5)$.

Theorem 7.3.
Adams-Bashforth four-step and five-step methods are stable.
Proof.
For Adams-Bashforth three-step method, exist only one characteristic polynomial $p(\lambda)=\lambda^4-\lambda^3$ and it is clear that satisfies the root condition by Theorem 2.1; then the method is stable. Also, for Adams-Bashforth five-step method, there only one characteristic polynomial $p(\lambda)=\lambda^4-\lambda^3$ and it satisfies the root condition, therefore it is a stable .

Theorem 7.4.
Adams-Moulton four-step and five-step methods are stable.
Proof.
Similar to Theorem 7.3.

8. Numerical Examples

Example 8.1
Consider the fuzzy initial value problem,

$$y'(t) = y(t), \quad t \in I = [0,1],$$

$$y(0) = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], \quad 0 < \alpha \leq 1$$

$$y(0.1) = [(0.75 + 0.25\alpha)e^{0.1}, (1.125 - 0.125\alpha)e^{0.1}],$$

$$y(0.2) = \left[(0.75 + 0.25\alpha)e^{0.2}, (1.125 - 0.125\alpha)e^{0.2}\right],$$

$$y(0.3) = [(0.75 + 0.25\alpha)e^{0.3}, (1.125 - 0.125\alpha)e^{0.3}],$$

$$y(0.4) = [(0.75 + 0.25\alpha)e^{0.4}, (1.125 - 0.125\alpha)e^{0.4}],$$

The exact solution at $t = 1$ is given by

$$Y(1;\alpha) = [(0.75 + 0.25\alpha)e, \quad (1.125 - 0.125\alpha)e], \quad 0 < \alpha \leq 1.$$

By using the Adams-fifth order predictor-corrector method the following results are obtained:

Table 8.1

α	RK-order 4		Adams-5		Exact Solution	
	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{Y}(t_i;\alpha)$	$\overline{Y}_2(t_i;\alpha)$
0.1	2.106666802	3.024086215	2.106668505	3.024088534	2.106668417	3.024088534
0.2	2.174623755	2.990101718	2.174625553	2.990110011	2.174625463	2.990110011
0.3	2.242580782	2.956129222	2.242582602	2.956131488	2.242582508	2.956131488
0.4	2.310537782	2.922150725	2.310539650	2.922152966	2.310539554	2.922152965
0.5	2.378494776	2.888172228	2.378496699	2.888174443	2.378496599	2.888174443
0.6	2.446451770	2.854193731	2.446453748	2.854195919	2.446453645	2.854195920
0.7	2.514408763	2.820215234	2.514410796	2.820217397	2.514410691	2.820217398
0.8	2.582365757	2.786236738	2.582367845	2.786238990	2.582367737	2.786238874
0.9	2.650322750	2.752258241	2.650324893	2.752260466	2.650324783	2.752260351
1.0	2.718279744	2.718279744	2.718281942	2.718281942	2.718281828	2.718281828

Table 8.2.

α	Error in RK-order 4		Error in Adams-5	
	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{Y}(t_i;\alpha)$	$\overline{Y}(t_i;\alpha)$
0.1	1.615×10^{-6}	2.319×10^{-6}	8.7741×10^{-8}	1.2595×10^{-7}
0.2	1.708×10^{-6}	2.193×10^{-6}	9.0571×10^{-8}	1.2453×10^{-7}
0.3	1.726×10^{-6}	2.266×10^{-6}	9.3401×10^{-8}	1.2312×10^{-7}
0.4	1.772×10^{-6}	2.241×10^{-6}	9.6231×10^{-8}	1.2171×10^{-7}
0.5	1.824×10^{-6}	2.215×10^{-6}	9.9062×10^{-8}	1.2028×10^{-7}
0.6	1.876×10^{-6}	2.189×10^{-6}	1.0189×10^{-7}	1.1887×10^{-7}
0.7	1.928×10^{-6}	2.163×10^{-6}	1.0472×10^{-7}	1.1745×10^{-7}
0.8	1.980×10^{-6}	2.136×10^{-6}	1.0755×10^{-7}	1.1604×10^{-7}
0.9	2.033×10^{-6}	2.110×10^{-6}	1.1038×10^{-7}	1.1462×10^{-7}
1.0	2.084×10^{-6}	2.084×10^{-6}	1.1321×10^{-7}	1.1321×10^{-7}

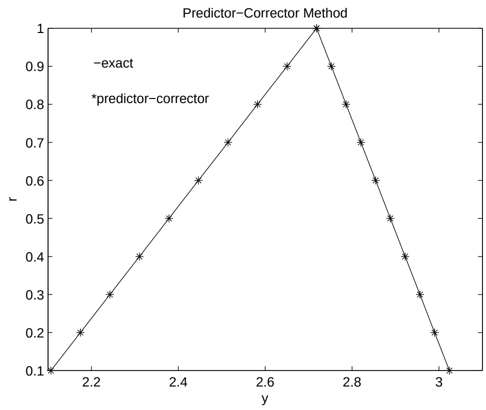

Figure 1: h=0.1

Example 8.2.

Consider the fuzzy initial value problem,[14],

$$y'(t) = -y(t), \qquad t \in I = [0,1],$$

$$y(0) = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], \qquad 0 < \alpha \leq 1$$

$$\underline{y}(0.01) = (0.9375 + 0.0625\alpha)e^{-0.01} - (1-\alpha)(0.1875)e^{0.01},$$

$$\overline{y}(0.01) = (0.9375 + 0.0625\alpha)e^{-0.01} + (1-\alpha)(0.1875)e^{0.01},$$

$$\underline{y}(0.02) = (0.9375 + 0.0625\alpha)e^{-0.02} - (1-\alpha)(0.1875)e^{0.02},$$

$$\overline{y}(0.02) = (0.9375 + 0.0625\alpha)e^{-0.02} + (1-\alpha)(0.1875)e^{0.02},$$

$$\underline{y}(0.03) = (0.9375 + 0.0625\alpha)e^{-0.03} - (1-\alpha)(0.1875)e^{0.03},$$

$$\overline{y}(0.03) = (0.9375 + 0.0625\alpha)e^{-0.03} + (1-\alpha)(0.1875)e^{0.03},$$

$$\underline{y}(0.04) = (0.9375 + 0.0625\alpha)e^{-0.04} - (1-\alpha)(0.1875)e^{0.04},$$

$$\overline{y}(0.04) = (0.9375 + 0.0625\alpha)e^{-0.04} + (1-\alpha)(0.1875)e^{0.04}.$$

The exact solution at $t = 0.1$ is given by

$$Y(0.1;\alpha) = [(0.9375 + 0.0625\alpha)e^{-0.1} - (1-\alpha)(0.1875)e^{0.1},$$

$$(0.9375 + 0.0625\alpha)e^{-0.1} + (1-\alpha)(0.1875)e^{0.1}]$$

By using the Adams fifth order predictor-corrector method the following results are obtained.

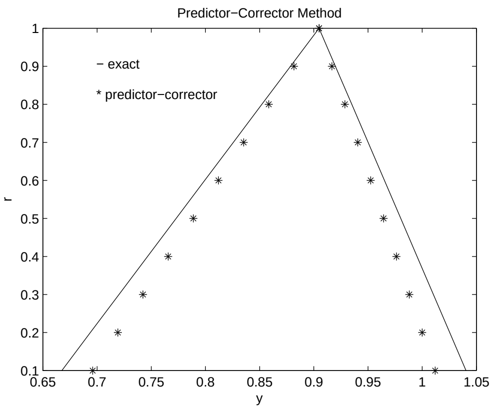

Figure 2: h=0.1

Table 8.3.

α	Adams-5		Exact Solution		Error in Adams-5	
	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{y}(t_i;\alpha)$	$\overline{y}(t_i;\alpha)$	$\underline{Y}(t_i;\alpha)$	$\overline{Y}(t_i;\alpha)$
0.1	0.695972086	1.011908540	0.667442721	1.040437906	2.852×10^{-2}	2.852×10^{-2}
0.2	0.719179346	1.000011749	0.693819909	1.025371185	2.535×10^{-2}	2.535×10^{-2}
0.3	0.742386605	0.988114957	0.720197098	1.010304464	2.219×10^{-2}	2.219×10^{-2}
0.4	0.765593864	0.976218166	0.746574287	0.995237743	1.902×10^{-2}	1.902×10^{-2}
0.5	0.788801123	0.964321375	0.772951475	0.980171022	1.585×10^{-2}	1.585×10^{-2}
0.6	0.812008382	0.952424583	0.799328664	0.965104301	1.268×10^{-2}	1.268×10^{-2}
0.7	0.835215641	0.940527792	0.825705852	0.950037581	9.510×10^{-3}	9.510×10^{-3}
0.8	0.858422900	0.928631007	0.852083041	0.934970860	6.340×10^{-3}	6.340×10^{-3}
0.9	0.881630159	0.916734209	0.878460229	0.919904139	3.170×10^{-3}	3.170×10^{-3}
1.0	0.904837418	0.904837418	0.904837418	0.904837418	1.092×10^{-13}	1.092×10^{-13}

9. Conclution

In this paper, we have applied itrative solution of Adam's predictor-corrector fifth order method for finding the numerical solution of fuzzy differential equations. Comparision of solution of Example (8.1) and (8.2) shows that our proposed method gives better solution then Runge-Kutta fourth order method.

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