Strongly θgs-Closed Functions and Strongly θgs-Homeomorphism

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Abstract— **The aim of this paper is to introduce and study of a new type of function called strongly θgs-closed function and new class of homeomorphism called strongly θgs-homeomorphism in topological spaces. Also, we obtain its characterizations and its basic properties.**

Key words— **θgs-closed set, strongly θgs-closed function, strongly θgs-homeomorphism.**

I. INTRODUCTION

Functions and of course closed functions stand among the most important notions in the whole of mathematical science. Many different forms of the closed functions have been introduced over the years. Recently in [5] the notion of θ-generalized semi closed (briefly, θgs-closed) set was introduced and studied. Many researchers have generalized the notion of homeomorphisms in topological spaces. Maki et al [4] introduced g-homeomorphisms in topological spaces. Using θgs-closed set the concept of θgs-homeomorphism was defined in [5]. In this paper we will continue the study of related functions involving θgs-open sets and generalization of homeomorphism.

II. PRELIMINARIES

Throughout this paper (X,τ) and (Y,σ) (or simply X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of space X, then Cl(A) and Int(A) denote the closure of A and the interior of A in X respectively.

The following definitions are useful in the sequel:

Definition 2.1: A subset A of space X is called
(i) a semi-open set [3] if $A \subseteq Cl(Int(A))$
(ii) a semi-closed set [1] if $Int(Cl(A)) \subseteq A$

Definition 2.2 [2]**:** A point $x \in X$ is called a semi-θ-cluster point of A if $A \cap sCl(U) \neq \emptyset$ for each semi-open set U containing x. The set of all semi-θ-cluster point of A is called semi- θ-closure of A and is denoted by $sCl_\theta(A)$. A subset A is called semi- θ-closed if $sCl_\theta(A) = A$. The complement of semi-θ-closed set is semi-θ-open set.

Definition 2.3 [5]**:** A subset A of a topological space X is called θ-generalized-semi closed (briefly, θgs-closed) if $sCl_\theta(A) \subset U$, whenever $A \subset U$ and U is open in X. The complement of θgs-closed set is θ-generalized-semi open (briefly, θgs-open). We denote the family of θgs-closed sets of X by $\theta GSC(X, \tau)$ and θgs-open sets by $\theta GSO(X, \tau)$.

Definition 2.4 [5]**:** (i) The intersection of all θgs-closed sets containing a set A is called θgs-closure of A and is denoted by θgsCl(A). A set A is θgs-closed if and only if θgsCl(A) = A.
(ii) The union of all θgs-open sets contained in A is called θgs-interior of A and is denoted by θgsInt(A). A set A is θgs-open if and only if θgsInt(A) = A.

Definition 2.5 [5]**:** A subset A of a topological space X is called θgs-neighbourhood of a point x of X if there exists θgs-open set G containing x such that $G \subset A$.

Definition 2.6 [7]**:** A space X is called $T_{\theta gs}$-space if every θgs-closed set in it is closed set.

Definition 2.7 [6]**:** A function f: X → Y is called
(i) θ-generalized semi-continuous (in briefly,θgs-continuous), if f^{-1} (F) is θgs-closed in X for every closed set F of Y.
(ii) θ-generalized semi-irresolute (in briefly, θgs-irresolute), if f^{-1}(F) is θgs-closed in X for every θgs-closed set F of Y.

Definition 2.8[8]: A function f: X →Y is said to be θgs-open (resp., θgs-closed) if f (V) is θgs-open (resp., θgs-closed) in Y for every open set (resp., closed) V in X.

Definition 2.9[8]: A function f: X →Y is said to be θ-generalized semi-homeomorphism (in briefly, θgs-homeomorphism) if f is both θgs-continuous and θgs-open

Remark 2.10 [8]: Every g-homeomorphism is θgs-homeomorphism**.**

Definition 2.11[11]: A function f: X→Y is said to be quasi θgs-closed if the image of each θgs-closed set in is closed set in Y.

III. Strongly θgs-Closed Functions

Definition 3.1 [9]: A function f: X→Y is said to be strongly θgs-closed (resp., strongly θgs-open) if f(A) is θgs-closed set (resp., θgs-open) in Y for every θgs-closed (resp., θgs-open) set A in X.

Remark 3.2: Every strongly θgs-closed function is θgs-closed function. But converse need not be true in general.

Example 3.3: Let X=Y= {a, b, c}, τ={X, Ø, {b}, {c}, {b, c}}, σ ={Y, Ø, {b}, {c}, {a, b}, {a, c}} be topologies on X and Y respectively. We have θGSC(X) = {X, Ø, {a}, {b}, {c}, {a, b}, {a, c}} and θGSC(Y) = {Y, Ø, {a}, {c}, {a, b}, {a, c}}. Define a function by f(a) = a, f(b) = b and f(c) = c. Then f is θgs-closed function but it is not strongly θgs-closed because for θgs-closed set {b} of X, f({b}) ={b} is not θgs-closed set in Y.

Remark 3.4: Every quasi θgs-closed function is strongly θgs-closed function. But converse need not be true in general.

Example 3.5: Let X=Y= {a, b, c}, τ={X, Ø, {a}, {b}, {a, b}}, σ ={Y, Ø, {b}, {c}, {a, b}, {a, c}} be topologies on X and Y respectively. We have θGSC(X) = {X, Ø, {a}, {a, b}, {a, c}} and θGSC(Y) = {Y, Ø, {a}, {c}, {a, b}, {a, c}}. Define a function by f(a) = b, f(b) = c and f(c) = a. Then f is strongly θgs-closed function but it is not quasi θgs-closed because for θgs-closed set {a, b} of X, f ({a, b}) = {b, c} is not closed set in Y.

Theorem 3.6: A surjective function f: X→Y is strongly θgs-closed (resp. strongly θgs-open), if and only if for any subset B of Y and each θgs-open (resp. θgs-closed) set U of X containing f^{-1}(B), there is an θgs-open (resp. θgs-closed) set V of Y containing B and $f^{-1}(V) \subset U$.

Proof: Assume that f is strongly θgs-closed function. Let S ⊂ Y and U be an open set of X containing f^{-1}(S). Since f is a strongly θgs-closed function and X-U is closed in X and hence θgs-closed, implies f(X-U) is θgs-closed set in Y. Then V = Y-f(X-U) is θgs-open set in Y such that S ⊂ V and $f^{-1}(V) \subset U$.
Conversely, let F be a closed set in X, then X-F is an open set in X and f^{-1}(Y- f(F)) ⊂ X-F. By hypothesis, there is a θgs-open set V of Y such that Y-f(F) ⊂ V and f^{-1}(V) ⊂ X-F. Therefore Y-V⊂ f(F) ⊂ f(X-f^{-1}(V)) ⊂ Y-V, this implies f(F) =Y-V. Since V is θgs-open set in Y and so f(F) is θgs-closed set in Y. Hence f is θgs-closed function.

Theorem 3.7: A function f: X→Y is strongly θgs-closed, if and only if θgsCl(f(A)) ⊂ f(θgsCl(A)) for every subset A of X.

Proof: Let f be strongly θgs-closed function and A ⊂ X. Then f(θgsCl(A)) is θgs-closed in Y. Since f(A) ⊂ f(θgsCl(A)), implies θgsCl(f(A)) ⊂ θgsCl(f(θgsCl(A))) = f(θgsCl(A)). Therefore, θgsCl(f(A)) ⊂ f(θgsCl(A)).

Conversely, A is any θgs-closed set in X. Then θgsCl(A) =A, implies, f(A) = f(θgsCl(A)). By hypothesis, θgsCl(f(A)) ⊂ f(θgsCl(A)) = f(A). Hence θgsCl(f(A)) ⊂ f(A). But f(A) ⊂ θgsCl(f(A)) is always true. This shows, f(A) = θgsCl(f(A)) Therefore, f(A) is θgs-closed set in Y. Hence f is strongly θgs-closed function.

Remark 3.8: Composition of two strongly θgs-closed functions is strongly θgs-closed.

Theorem 3.9: Let f: X→Y and g: Y→Z be two functions, such that g∘f: X→Z is strongly θgs-closed function. Then

(i) f is θgs-irresolute and surjective implies g is strongly θgs-closed

(ii) g is θgs-irresolute and injective implies f is strongly θgs-closed

Proof: (i) Let A be θgs-closed set of Y. Since f is θgs-irresolute and surjective, $f^{-1}(A)$ is θgs-closed set in X. Also since g∘f is strongly θgs-closed function, implies $(g∘f)(f^{-1}(A)) = g(A)$ is θgs-closed in Z. Therefore g is strongly θgs-closed.
(ii) Let A be θgs-closed set of X. Since (g∘f) is strongly θgs-closed function (g∘f)(A) is θgs-closed in Z. Also since g is θgs-irresolute and injective, $g^{-1}(g∘f)(A) = f(A)$ is θgs-closed set in Y. Therefore f is strongly θgs-closed.

Theorem 3.10: For any bijection, f: X→Y the following statements are equivalent.

(i) Inverse of f is θgs-irresolute

(ii) f is a strongly θgs-open function.

(iii) f is a strongly θgs-closed function.

Proof: (i) → (ii) Suppose U is an θgs-open set in X, then by (i), $(f^{-1})^{-1}(U)=f(U)$ is θgs-open set in Y. Therefore f is strongly θgs-open function.
(ii) → (iii) Suppose F is a θgs-closed set in X, and then X-F is θgs-open set in X. By (ii), f(X-F) =Y-f(F) is θgs-open set in Y. This implies f(F) is θgs-closed set in Y. Therefore f is a strongly θgs-closed function.
(iii) → (i) Suppose F is a θgs-closed set in X, then by (iii) $f(F)=(f^{-1})^{-1}(F)$ is θgs-closed set in Y. Therefore f^{-1} is θgs--irresolute function

Theorem 3.11: Let f: X→Y be a function from a space X to a $T_{\theta gs}$-space Y. Then following are equivalent

(i) f is strongly θgs-closed function.

(ii) f is quasi θgs-closed function.

Proof: (i) → (ii) Suppose (i) holds. Let F be a θgs-closed set in X. Then f(F) is θgs-closed in Y. Since Y is $T_{\theta gs}$-space, f(F) is closed in Y. Therefore f is quasi θgs-closed function.
(ii) → (i) Suppose (ii) holds. Let F be a θgs-closed set in X. Then f(F) is closed and hence θgs-closed in Y. Therefore f is strongly θgs-closed function.

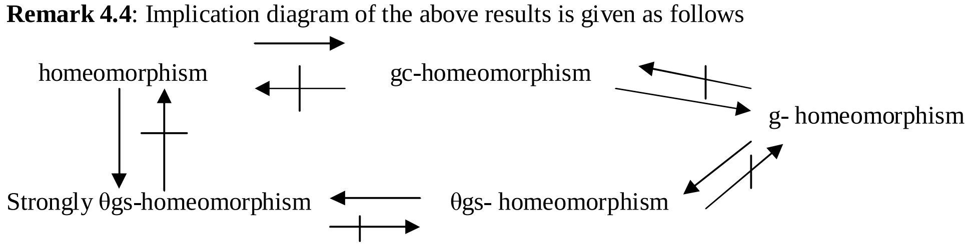
IV. Strongly θgs-Homeomorphism

Definition 4.1: A bijective function f: X→Y is called strongly θgs-homeomorphism if f is both θgs-irresolute and strongly θgs-open. The family of all strongly θgs-homeomorphism of space X on to itself is denoted by SθGS-H(X)

Remark 4.2: Every strongly θgs-homeomorphism is θgs-homeomorphism. But converse need not be true in general.

Example 4.3: Let X=Y= {a, b, c}, τ={X, Ø, {b}, {c}, {a, b}, {a, c}}, σ ={Y, Ø, {b}, {c}, {b, c}} be topologies on X and Y respectively. We have θGSC(X) = {X, Ø, {a}, {c}, {a, b}, {a, c}} and θGSC(Y) = {Y, Ø, {a}, {b}, {c}, {a, b}, {a, c}}. Define a function by f(a) = a, f(b) = b and f(c) = c. Then f is θgs-homeomorphism .But not strongly θgs-homeomorphism, because f is not θgs-irresolute, i. e, for θgs-closed set {b} in Y, $f^{-1}(\{b\}) = \{b\}$ is not θgs-closed set in X.

Remark 4.4: Implication diagram of the above results is given as follows



Where A→B (resp. A $\nleftarrow$ B) represents A implies B but not conversely (resp. B does not implies A).

Theorem 4.5: If f: X→Y and g: Y→Z be two strongly θgs-homeomorphism functions, then g◦f: X→Z is also strongly θgs-homeomorphism.

Proof: Suppose U be a θgs-open set in Z, then $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$ where $g^{-1}(U)=V$. Since g is θgs-irresolute, implies V is θgs-open in Y and again f is θgs-irresolute, implies $f^{-1}(V)$ is θgs-open in X. Therefore, g◦f is θgs-irresolute. Also, for a θgs-open set G in X, (g◦f)(G)= g(f(G)) = g(W) where W= f(G), since f is strongly θgs-open, implies W= f(G) is θgs-open in Y and again g is strongly θgs-open, implies, g(W) = g(f(G)) is θgs-open in Z. This implies g◦f is strongly θgs-open. Therefore, g◦f is strongly θgs-homeomorphism.

Theorem 4.6: If f: X→Y is strongly θgs-homeomorphism, then $\theta gsCl(f^{-1}(A)) = f^{-1}(\theta gsCl(A))$, for every subset A of Y.

Proof: Suppose f: X→Y is a strongly θgs-homeomorphism, then f is both θgs-irresolute and strongly θgs-open. Since θgsCl(A) is a θgs-closed set in Y, implies $f^{-1}(\theta gsCl(A))$ is θgs-closed in X. Since $f^{-1}(A) \subset f^{-1}(\theta gsCl(A))$, implies $\theta gsCl(f^{-1}(A)) \subset \theta gsCl(f^{-1}(\theta gsCl(A))) = f^{-1}(\theta gsCl(A))$. This implies $\theta gsCl(f^{-1}(A)) \subset f^{-1}(\theta gs\, Cl(A))$. ---- (i). Again, since $\theta gsCl(f^{-1}(A))$ is a θgs-closed set in X and f is strongly θgs-open, implies $f(\theta gsCl(f^{-1}(A)))$ is θgs closed in Y. Since $f^{-1}(A) \subset \theta gsCl(f^{-1}(A))$, implies $A \subset f(\theta gsCl(f^{-1}(A)))$, therefore $\theta gsCl(A) \subset f(\theta gsCl(f^{-1}(A))$. This implies $f^{-1}(\theta gs\text{-}Cl(A)) \subset \theta gsCl(f^{-1}(A))$.----(ii). Thus, from (i) and (ii), $\theta gsCl(f^{-1}(A)) = f^{-1}(\theta gsCl(A))$, for every subset A of Y.

Corollary 4.7: If f: X→Y is strongly θgs-homeomorphism then θgsCl(f(A))=f(θgsCl(A)), for every subset A of X.

Proof: Since f: X→Y is a strongly θgs-homeomorphism, f^{-1}: Y→X is also strongly θgs-homeomorphism. By theorem 4.6, $\theta gsCl((f^{-1})^{-1}(A)) = (f^{-1})^{-1}(\theta gsCl(A))$, for every subset A of X. Hence θgsCl(f(A))=f(θgsCl(A)), for every subset A of X.

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