On Isomorphism in C^*-Ternary Algebras for a Cauchy-Jensen Functional Equations

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Abstract: In this paper, we investigate isomorphisms between $C*$-ternary algebras by proving the Hyers-Ulam-Rassias stability of homomorphisms in C^*-ternary algebras and of derivations on $C*$-ternary algebras for the following Cauchy-Jensen additive mapping:

$$f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)=f(y)$$

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z)$$

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z)$$

Key words: Cauchy-Jensen functional equation, C^* -ternary algebra isomorphism, Hyers- Ulam-Rassias stability, C^*-ternary derivation.

1.INTRODUCTION AND PRELIMINARIES

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z)\to[x,y,z]$ of A^3 to A, which is C-linear in the outer variables, conjugate C-linear in the middle variable. Also it is associative in the sense that

$$[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$$

and satisfies $P[x,y,z]P\leq P_xP.P_yP.P_zP$ and $P[x,x,x]P=PxP^3$. If a C^*-ternary algebra (A ,[.,.,.]) has an identity, i.e., an element $e\in A$ such that $x=[x,e,e]=[e,e,x]$ for all $x\in A$, then it is routine to verify that A, endowed with $x\circ y=[x,e,y]$ and $x^*=[e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a C^*- algebra, then $[x,y,z]=x\circ y^*\circ z$ makes A into C^*-ternary algebra.

A C-linear mapping $H:A\to B$ is called a C^*-ternary algebra homomorphisms if $H([x,y,z])=[H(x),H(y),H(z)]$ for all $x,y,z\in A$. If, in addition, the mapping H is bijective, then the mapping $H: A \to B$ is called a C^*-ternary algebra isomorphisms. A C-linear mapping $\delta: A \to A$ is called a C^*-ternary derivation if

$\delta([x,y,z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$ for all $x, y, z \in A$.

The study of stability problems for functional equations is related to a question of Ulam [13] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [5]. It was further generalized and excellent results obtained by number of authors [3,4,10]. During the past two decades, a number of papers and research monographs have been published on various generalization and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, K-additive mappings, invariant means, multiplicative mappings, bounded n-th differences, convex functions, generalized orthogonality mappings, Euler-lagrange functional equations, different equations, and Navier-Stokes equation. Also, the stability problem of ternary homomorphisms and ternary derivations was es- tablished by Park [9] and J.M.Rassias, Kim [12].

2. STABILITY OF HOMOMORPHISMS IN C^*-TERNARY ALGEBRAS

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|.\|_A$ and that B is a C^*-ternary algebra with norm $\|.\|_B$. For a given mapping $f: A \to B$, we define

$$P_\mu f(x,y,z) = f\left(\frac{\mu x + \mu y}{2} + \mu z\right) - \mu f\left(\frac{x-y}{2} + 2\right) - \mu f(y)$$

$$Q_\mu f(x,y,z) = f\left(\frac{\mu x + \mu y}{2} + \mu z\right) + \mu f\left(\frac{x-y}{2} + 2\right) - \mu f(x) - 2\mu f(z)$$

$$R_\mu f(x,y,z) = 2f\left(\frac{\mu x + \mu y}{2} + \mu z\right) - \mu f(x) - \mu f(y) - 2\mu f(z)$$

For all $\mu \in T^1 = \{\lambda \in C : |\lambda| = 1\}$ and all $x, y, z \in A$.

We prove the Hyers-Ulam-Rassias stability of homomorphism in C^*-ternary algebras for the functional equation $P_\mu f(x,y,z) = 0$.

Theorem 1 . Let $r > 1$ and θ be positive real numbers, and let $f: A \to B$ be a mapping such that

$$\|P_\mu f(x,y,z)\|_B \le \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \quad \textbf{(1)}$$

$$\|f([x,y,z]) - [f(x), f(y), f(z)]\|_B \le \theta(\|x\|_A^r + \|y_A^r\| + \|z\|_A^r) \quad \textbf{(2)}$$

For all $\mu \in T^1$ and all $x, y, z \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H: A \to B$ such that

$$\|f(x)-H(x)\|_B \le \frac{3\theta}{2^r-2}\|x\|_A^r, \text{ for all } \quad x\in A. \qquad \textbf{(3)}$$

Proof: Let $\mu=1$ and $y=z=x$ in (1), we get

$$\|f(2x)-2f(x)\|_B \le 3\theta\|x\|_A^r \qquad \textbf{(4)}$$

For all $x\in A$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|_B \le \frac{3\theta}{2^r}\|x\|_A^r$$

For all $x\in A$. Hence

$$\left\|2^k f\left(\frac{x}{2^k}\right)-2^l f\left(\frac{x}{2^l}\right)\right\|_B \le \sum_{j=k}^{l-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B$$

$$\le \frac{3\theta}{2^r}\sum_{j=k}^{l-1}\frac{2^j}{2^{jr}}\|x\|_A^r \qquad \textbf{(5)}$$

for all non-negative integers l and k with $l>k$ and all $x\in A$. It follows from (5) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all x ∈ A. Since B is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $H: A\to B$ by

$$H(x)=\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

For all $x\in A$. Moreover, letting $k=0$ and passing the limit $l\to\infty$ in (5), we get (3).

It follows from (1) that

$$\left\|H\left(\frac{x+y}{2}+z\right)-H\left(\frac{x-y}{2}+z\right)-H(y)\right\|_B = \lim_{n\to\infty} 2^n\left\|f\left(\frac{x+y}{2^{n+1}}+\frac{z}{2^n}\right)-f\left(\frac{x-y}{2^{n+1}}+\frac{z}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right\|_B$$

$$\le \lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right)=0$$

For all $x, y, z\in A$. So

$$H\left(\frac{x+y}{2}+z\right)-H\left(\frac{x-y}{2}+z\right)=H(y)$$

For all $x, y, z \in A$. By the same way of [1, Lemma 2.1], the mapping $H : A \to B$ is Cauchy additive.

Letting $y = z = x$ in (1), we get

$$\left\| f(2\mu x) - 2\mu f(x) \right\|_B \le 3\theta \|x\|_A^r$$

For all $\mu \in T^1$ and all $x \in A$. So

$$H(\mu x) = \lim_{n\to\infty} 2^n f\left(\frac{\mu x}{2^n}\right) = \lim_{n\to\infty} \mu 2^n f\left(\frac{x}{2^n}\right) = \mu H(x)$$

For all $\mu \in T^1$ and all $x \in A$. So the mapping $H : A \to B$ is C-linear by the same reasoning as in the proof of [8, Theorem 2.1]. It follows from (2) that

$$\|H([x,y,z]) - [H(x), H(y), H(z)]\|_B = \lim_{n\to\infty} 8^n \left\| f\left(\left[\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right]\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right]\right\|_B$$

$$\le \lim_{n\to\infty} \frac{8^n \theta}{2^{nr}} \left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r \right) = 0$$

For all $x, y, z \in A$. So

$$H([x,y,z]) = [H(x), H(y), H(z)]$$

For all $x, y, z \in A$. Now let $S : A \to B$ be another Cauchy-Jensen additive mapping satisfying (3). Then we have

$$\|H(x) - S(x)\|_B = 2^n \left\| H\left(\frac{x}{2^n}\right) - S\left(\frac{x}{2^n}\right) \right\|_B$$

$$\le 2^n \left(\left\| H\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_B + \left\| S\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\|_B \right)$$

$$\le \frac{6.2^n.\theta}{2^{nr}\left(2^r - 2\right)} \|x\|_A^r$$

Which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = S(x)$ for all $x \in A$.

This proves the uniqueness of H.

Thus the mapping $H : A \to B$ is a unique C^*-ternary algebra homomorphism satisfying (3).

Theorem 2. Let $r<1$ and θ be positive real numbers, and let $f:A\to B$ be a mapping satisfying (1) and (2). Then there exists a unique C^*-ternary algebra homomorphism $H:A\to B$ such that

$$\left\|f(x)-H(x)\right\|_B \le \frac{3\theta}{2-2^r}\|x\|_A^r \quad \text{for all } x\in A. \qquad \textbf{(6)}$$

Proof: It follows from (4) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|_B \le \frac{3\theta}{2}\|x\|_A^r$$

For all $x\in A$. Hence

$$\left\|\frac{1}{2^k}f\left(2^k x\right)-\frac{1}{2^l}f\left(2^l x\right)\right\|_B \le \sum_{j=k}^{l-1}\left\|\frac{1}{2^j}f\left(2^j x\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\|_B$$

$$\le \frac{3\theta}{2}\sum_{j=k}^{l-1}\frac{2^{jr}}{2^r}\|x\|_A^r$$

for all non-negative integers l and k with $l>k$ and all $x\in A$. It follows from (7) that the sequence $\left\{\frac{1}{2^n}f\left(2^n x\right)\right\}$ is a Cauchy sequence for all x ∈ A. Since B is complete, the sequence $\left\{\frac{1}{2^n}f\left(2^n x\right)\right\}$ converges. So one can define the mapping $H:A\to B$ by

$$H(x)=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^n x\right)$$

For all $x\in A$. Moreover, letting $k=0$ and passing the limit $l\to\infty$ in (7), we get (6).

The rest of the proof is similar to the proof of Theorem (1).

Similarly we can obtain the results for the functional equations $Q_\mu f(x,y,z)=0$ and $R_\mu f(x,y,z)=0$.

3. ISOMORPHISMS BETWEEN C^*-TERNARY ALGEBRAS

Throughtout this section, assume that A is a C^*-ternary algebra with norm $\mathrm{P.P}_{\mathrm{A}}$ and unit e, and that B is a C^*-ternary algebra with norm $\mathrm{P.P}_{\mathrm{B}}$ and unit e'.

We investigate isomorphism between C^*-ternary algebras, associated to the functional equation $Q_\mu f(x,y,z)=0$.

Theorem 3. Let $r>1$ and θ be positive real numbers, and let $f:A\to B$ be a mapping such that

$$\left\|Q_\mu f(x,y,z)\right\|_B \le \theta\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right), \tag{8}$$

$$f([x,y,z]) = [f(x), f(y), f(z)] \tag{9}$$

For all $\mu\in T^1$ and all $x,y,z\in A$. If $\lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right)=e'$, then the mapping $f:A\to B$ is a C^*-ternary algebra isomorphism.

Proof: Let $\mu=1$ and $y=z=x$ in (8), we get

$$\|f(2x)-2f(x)\|_B \le 3\theta\|x\|_A^r \tag{10}$$

For all $x\in A$. So

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|_B \le \frac{3\theta}{2^r}\|x\|_A^r$$

For all $x\in A$. Hence

$$\left\|2^k f\left(\frac{x}{2^k}\right)-2^l f\left(\frac{x}{2^l}\right)\right\|_B \le \sum_{j=k}^{l-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B$$

$$\le \frac{3\theta}{2^r}\sum_{j=k}^{l-1}\frac{2^j}{2^{jr}}\|x\|_A^r \tag{11}$$

for all non-negative integers l and k with $l>k$ and all $x\in A$. It follows from (11) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all x ∈ A. Since B is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $H:A\to B$ by

$$H(x)=\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

For all $x\in A$. Moreover, letting $k=0$ and passing the limit $l\to\infty$ in (11), we get

$$\|f(x)-H(x)\|_B \le \frac{3\theta}{2^r-2}\|x\|_A^r, \text{ for all } x\in A.$$

It follows from (8) that

$$\left\| H\left(\frac{x+y}{2}+z\right)+H\left(\frac{x-y}{2}+z\right)-H(x)-2H(z)\right\|_B$$

$$=\lim_{n\to\infty}2^n\left\| f\left(\frac{x+y}{2^{n+1}}+\frac{z}{2^n}\right)+f\left(\frac{x-y}{2^{n+1}}+\frac{z}{2^n}\right)-f\left(\frac{x}{2^n}\right)-2f\left(\frac{z}{2^n}\right)\right\|_B$$

$$\le\lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right)=0$$

For all $x, y, z \in A$. So

$$H\left(\frac{x+y}{2}+z\right)-H\left(\frac{x-y}{2}+z\right)=H(x)+2H(z)$$

For all $x, y, z \in A$. By the same way of [1, Lemma 2.1], the mapping $H : A \to B$ is Cauchy additive. Letting $y = z = x$ in (8), we get

$$\| f(2\mu x)-2\mu f(x)\|_B \le 3\theta\|x\|_A^r$$

For all $\mu \in T^1$ and all $x \in A$. So

$$H(\mu x)=\lim_{n\to\infty}2^n f\left(\frac{\mu x}{2^n}\right)=\lim_{n\to\infty}\mu 2^n f\left(\frac{x}{2^n}\right)=\mu H(x)$$

For all $\mu \in T^1$ and all $x \in A$. So the mapping $H : A \to B$ is C-linear by the same reasoning as in the proof of [8, Theorem 2.1].

Since $f([x, y, z]) = [f(x), f(y), f(z)]$ for all $x, y, z \in A$.

$$H([x,y,z])=\lim_{n\to\infty}8^n f\left(\left[\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right]\right)$$

$$=\lim_{n\to\infty}\left[2^n f\left(\frac{x}{2^n}\right),2^n f\left(\frac{y}{2^n}\right),2^n f\left(\frac{z}{2^n}\right)\right]$$

$$=\left[H(x),H(y),H(z)\right]$$

For all $x, y, z \in A$. So the mapping $H : A \to B$ is a C^*-ternary homomorphism. It follows from (9) that

$$H(x)=H([e,e,x])$$

$$= \frac{2^n}{2^n} \lim_{n\to\infty} 2^n f\left(\frac{1}{2^n}[e,e,x]\right)$$

$$= \lim_{n\to\infty} 4^n f\left(\left[\frac{1}{2^n}e, \frac{1}{2^n}e, x\right]\right)$$

$$= \lim_{n\to\infty}\left(\left[2^n f\left(\frac{1}{2^n}e\right), 2^n f\left(\frac{1}{2^n}e\right), f(x)\right]\right)$$

$$= \left[e', e', f(x)\right] = f(x) \text{ for all } x \in A.$$

Hence the bijective mapping $f: A \to B$ is a C^*-ternary algebra isomorphism.

Theorem 4. Let $r < 1$ and θ be positive real numbers, and let $f: A \to B$ be a mapping satisfying (8) and (9). If $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f: A \to B$ is a C^*-ternary algebra isomorphisms.

Proof: It follows from (10) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_B \le \frac{3\theta}{2} \|x\|_A^r$$

For all $x \in A$. Hence

$$\left\| \frac{1}{2^k} f(2^k x) - \frac{1}{2^l} f(2^l x) \right\|_B \le \sum_{j=k}^{l-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|_B$$

$$\le \frac{3\theta}{2} \sum_{j=k}^{l-1} \frac{2^{jr}}{2^r} \|x\|_A^r \tag{12}$$

for all non-negative integers l and k with $l > k$ and all $x \in A$. It follows from (12) that the sequence $\left\{ \frac{1}{2^n} f(2^n x) \right\}$ is a Cauchy sequence for all x ∈ A. Since B is complete, the sequence $\left\{ \frac{1}{2^n} f(2^n x) \right\}$ converges. So one can define the mapping $H: A \to B$ by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

For all $x \in A$. Moreover, letting $k = 0$ and passing the limit $l \to \infty$ in (12), we get

$$\left\| f(x) - H(x) \right\|_B \le \frac{3\theta}{2-2^r} \|x\|_A^r \qquad \text{for all } x \in A.$$

The rest of the proof is similar to the proof of Theorem (3).

Similarly we can obtain the results for the functional equations $P_\mu f(x,y,z) = 0$ and $R_\mu f(x,y,z) = 0.$

4. STABILITY OF DERIVATIONS ON C^*-TERNARY ALGEBRAS

Throughtout this section, assume that A is a C^*-ternary algebra with norm $\|.\|_A$.

We prove the Hyers-Ulam-Rassias stability of derivations in C^*-ternary algebras for the functional equation $R_\mu f(x,y,z) = 0.$

Theorem 5. Let $r > 1$ and θ be positive real numbers, and let $f : A \to A$ be a mapping such that

$$\left\| R_\mu f(x,y,z) \right\|_A \le \theta\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r \right), \tag{13}$$

$$\left\| f([x,y,z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] \right\|_A \le \theta\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r \right) \tag{14}$$

For all $\mu \in T^1$ and all $x, y, z \in A.$ Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that

$$\left\| f(x) - \delta(x) \right\|_A \le \frac{3\theta}{2(2^r - 2)} \|x\|_A^r \qquad \text{for all } x \in A. \tag{15}$$

Proof: Let $\mu = 1$ and $y = z = x$ in (13), we get

$$\left\| 2f(2x) - 4f(x) \right\|_A \le 3\theta \|x\|_A^r \tag{16}$$

For all $x \in A$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_A \le \frac{3\theta}{2.2^r} \|x\|_A^r$$

For all $x \in A$. Hence

$$\left\| 2^k f\left(\frac{x}{2^k}\right) - 2^l f\left(\frac{x}{2^l}\right) \right\|_A \le \sum_{j=k}^{l-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|_A$$

$$\le \frac{3\theta}{2.2^r} \sum_{j=k}^{l-1} \frac{2^j}{2^{jr}} \|x\|_A^r \tag{17}$$

for all non-negative integers l and k with $l > k$ and all $x \in A$. It follows from (11) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all x ∈ A. Since B is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $H : A \to B$ by

$$\delta(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

For all $x \in A$. Moreover, letting $k = 0$ and passing the limit $l \to \infty$ in (17), we get (15).

It follows from (13) that

$$\left\|2\delta\left(\frac{x+y}{2}+z\right)-\delta(x)-\delta(y)-2\delta(z)\right\|_A$$

$$= \lim_{n\to\infty} 2^n \left\|2f\left(\frac{x+y}{2^{n+1}}+\frac{z}{2^n}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)-2f\left(\frac{z}{2^n}\right)\right\|_A$$

$$\leq \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}}\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right)=0.$$

For all $x, y, z \in A$. So

$$2\delta\left(\frac{x+y}{2}+z\right)=\delta(x)+\delta(y)+2\delta(z)$$

For all $x, y, z \in A$. By the same way of [1, Lemma 2.1], the mapping $\delta : A \to A$ is Cauchy additive.

Letting $y = z = x$ in (13), we get

$$\left\|2f(2\mu x)-4\mu f(x)\right\|_A \leq 3\theta\|x\|_A^r$$

For all $\mu \in T^1$ and all $x \in A$. So

$$\delta(\mu x)=\lim_{n\to\infty} 2^n f\left(\frac{\mu x}{2^n}\right)=\lim_{n\to\infty} \mu.2^n f\left(\frac{x}{2^n}\right)=\mu\delta(x)$$

For all $\mu \in T^1$ and all $x \in A$. So the mapping $\delta : A \to A$ is C-linear by the same reasoning as in the proof of [8, Theorem 2.1]. It follows from (14) that

$$\left\|\delta([x,y,z])-[\delta(x),y,z]-[x,\delta(y),z]-[x,y,\delta(z)]\right\|_A$$

$$=\lim_{n\to\infty}8^n\left\|f\left(\frac{[x,y,z]}{8^n}\right)-\left[f\left(\frac{x}{2^n}\right),\frac{y}{2^n},\frac{z}{2^n}\right]-\left[\frac{x}{2^n},f\left(\frac{y}{2^n}\right),\frac{z}{2^n}\right]-\left[\frac{x}{2^n},\frac{y}{2^n},f\left(\frac{z}{2^n}\right)\right]\right\|_A$$

$$\le\lim_{n\to\infty}\frac{8^n\theta}{2^{nr}}\left(\|x\|_A^r+\|y\|_A^r+\|z\|_A^r\right)=0.$$

For all $x,y,z\in A$. So

$$\delta([x,y,z])=[\delta(x),y,z]+[x,\delta(y),z]+[x,y,\delta(z)] \quad \text{For all } x,y,z\in A.$$

Now we let $S:A\to A$ be another Cauchy-Jensen additive mapping satisfying (15). Then we have

$$\|\delta(x)-S(x)\|_A=2^n\left\|\delta\left(\frac{x}{2^n}\right)-S\left(\frac{x}{2^n}\right)\right\|_A$$

$$\le 2^n\left(\left\|\delta\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|_A+\left\|S\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|_A\right)$$

$$\le\frac{3.2^n.\theta}{2^{nr}\left(2^r-2\right)}\|x\|_A^r$$

Which tends to zero as $n\to\infty$ for all $x\in A$. So we can conclude that $\delta(x)=S(x)$ for all $x\in A$. This proves the uniqueness of δ. Thus the mapping $\delta:A\to A$ is a C^*-ternary derivation satisfying (15).

Theorem 6. Let $r<1$ and θ be positive real numbers, and let $f:A\to A$ be a mapping satisfying (13) and (14). Then there exists a unique C^*-ternary derivation $\delta:A\to A$ such that

$$\|f(x)-\delta(x)\|_A\le\frac{3\theta}{2\left(2-2^r\right)}\|x\|_A^r \quad \text{for all } x\in A. \qquad \textbf{(18)}$$

Proof: It follows from (16) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\|_A\le\frac{3\theta}{4}\|x\|_A^r$$

For all $x\in A$. Hence

$$\left\|\frac{1}{2^k}f\left(2^kx\right)-\frac{1}{2^l}f\left(2^lx\right)\right\|_A\le\sum_{j=k}^{l-1}\left\|\frac{1}{2^j}f\left(2^jx\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\|_A$$

$$\leq \frac{3\theta}{4}\sum_{j=k}^{l-1}\frac{2^{jr}}{2^r}\|x\|_A^r \tag{19}$$

for all non-negative integers l and k with $l > k$ and all $x \in A$. It follows from (19) that the sequence $\left\{\frac{1}{2^n} f\left(2^n x\right)\right\}$ is a Cauchy sequence for all x ∈ A. Since B is complete, the sequence $\left\{\frac{1}{2^n} f\left(2^n x\right)\right\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta\left(x\right) = \lim_{n\to\infty}\frac{1}{2^n} f\left(2^n x\right)$$

For all $x \in A$. Moreover, letting $k = 0$ and passing the limit $l \to \infty$ in (19), we get (18).

The rest of the proof is similar to the proof of Theorem (5).

Similarly we can obtain the results for the functional equations $P_\mu f\left(x, y, z\right) = 0$ and $Q_\mu f\left(x, y, z\right) = 0.$

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