Degree of Approximation of a Function Belonging to $W(L_r, \xi(t))$ $(r > 1)$-Class by $(E, 1)(C, 2)$ Product Summabilty Transform

Kusum Sharma[1] and S. S. Malik[2]

[1]Department of Mathematics, Faculty of Arts, Science & Commerce,
Mody Institute of Technology and Science (Deemed University),
Laxmangarh-332311, Sikar (Rajasthan), INDIA.
[2]Department of Mathematics, Bhartiya Institute of Engineering & Technology,
Sikar- Rajasthan, INDIA.
[1]e-mail: kusum31sharma@rediffmail.com,
[2]e-mail: ssmalik@live.in

Abstract. The field of approximation theory is so vast that it plays an increasingly important role in applications in pure and applied mathematics. The present study deals with a theorem concerning the degree of approximation of a function f belonging to $W(L_r, \xi(t))$ $(r > 1)$-class by using $(E, 1)(C, 2)$ of its Fourier series.

Key Words and Phrases : Degree of approximation, $W(L_r, \xi(t))(r > 1)$-class of function, $(E, 1)$ summability, $(C, 2)$ summability, $(E, 1)(C, 2)$ product summability, Fourier series, Lebesgue integral.

1 Introduction

The degree of approximation of a function belonging to the various classes $Lip\alpha$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ using different summability methods have been determined by several investigators like Alexits [2], Sahney and Goel [13], Quershi and Neha [11], Qureshi [9, 10], Chandra [1], Khan [4], Liendler [5], Mishra et al. [7] and Rhoades [12]. Recently Nigam [8] has obtained the degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by $(E, 1)(C, 2)$ summability method. In the present paper, a theorem on degree of approximation of a function f belonging to $W(L_r, \xi(t))(r \geq 1)$-class by $(E, 1)(C, 2)$ product summability transform of Fourier series has been obtained which in turn generalizes the result of Nigam [8].

2 Preliminaries

Let $f(x)$ be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series associated with f at a point x is defined as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

with n^{th} partial sums $s_n(f;x)$.

L_∞ - norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = sup\{|f(x)| : x \in R\} \tag{2.2}$$

L_r-norm of a function is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r \, dx \right)^{\frac{1}{r}}, \ r \geq 1. \tag{2.3}$$

The degree of approximation of a signal $f : R \to R$ by a trigonometric polynomial t_n of degree n under sup norm $\| \ \|_\infty$ is defined as

$$\|t_n - f\|_\infty = sup\{|t_n(x) - f(x)| : x \in R\} \text{ (Zygmund [14])} \tag{2.4}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = min \|t_n - f\|_r \tag{2.5}$$

This method of approximation is called trigonometric Fourier approximation (TFA).

A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O(|t|^\alpha) \ for \ 0 < \alpha \leq 1. \tag{2.6}$$

$f \in Lip(\alpha, r)$ for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx \right)^{\frac{1}{r}} = O(|t|^\alpha), 0 < \alpha \leq 1, \ r \geq 1. \tag{2.7}$$

(definition 5.38 of Mc Fadden [6], 1942).

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \, dx\right)^{\frac{1}{r}} = O(\xi(t)) \tag{2.8}$$

If $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class reduces to the $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the $Lip\alpha$ class.
and that $f \in W(L_r, \xi(t))$ if

$$\left(\int_0^{2\pi} \left|\{f(x+t) - f(x)\} \sin^\beta x \ \right|^r dx\right)^{\frac{1}{r}} = O(\xi(t) \ ), \beta \geq 0. \tag{2.9}$$

where $\xi(t)$ is a positive increasing function of t.

If $\beta = 0$ then $W(L_r, \xi(t))$ reduces to the class $Lip(\xi(t), r)$ and if $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class coincides with the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the class $Lip\alpha$.
We observe that

$$Lip\alpha \subseteq Lip(\alpha, r) \subseteq Lip(\xi(t), r) \subseteq W(L_r, \xi(t)) \text{ for } 0 < \alpha \leq 1, \ r \geq 1.$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its n^{th} partial sum $\{s_n\}$.
The $(E, 1)$ transform is defined as the n^{th} partial sum of $(E, 1)$ summability and is given by

$$(E, 1) = E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} s_k \to s \text{ as } n \to \infty \tag{2.10}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be $(E, 1)$ summable to a definite number s (Hardy [3]).
The $(C, 2)$ transform is defined as the nth partial sum of $(C, 2)$ summability and is given by

$$t_n = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1) s_k \to s \ as \ n \to \infty \tag{2.11}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is $(C, 2)$ summable to a definite number s.

The $(E, 1)$ transform of the $(C, 2)$ transform defines $(E, 1)(C, 2)$ transform and we denote it by $E_n^1 C_n^2$.

Thus if

$$E_n^1 C_n^2 = \frac{1}{2^n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) C_k^2 \to s \ as \ n \to \infty. \tag{2.12}$$

then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E,1)(C,2)$ summability transform to a definite number s.

We use the following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$M_n(t) = \frac{1}{2^n \pi} \sum_{k=0}^{n} \left\{ \left(\begin{array}{c} n \\ k \end{array} \right) \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k-\nu+1) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}$$

3 Main Theorem

If f is a 2π-periodic function, Lebesgue integrable on $(0, 2\pi)$, belonging to $W(L_r, \xi(t))$ class then its degree of approximation by $(E,1)(C,2)$ summability transform of its Fourier series is given by

$$\left\| E_n^1 C_n^2 - f \right\|_r = O\left[(n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1} \right) \right] \tag{3.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t \ |\phi(t)|}{\xi(t)} \right)^r \sin^{\beta r} \ dt \right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1} \right) \tag{3.2}$$

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta} |\phi(t)|}{\xi(t)} \right)^r dt \right\}^{\frac{1}{r}} = O\left\{ (n+1)^{\delta} \right\} \tag{3.3}$$

and

$$\left\{ \frac{\xi(t)}{t} \right\} \text{ is non-increasing in t,} \tag{3.4}$$

where δ is an arbitrary number such that $s(1-\delta) - 1 > 0$, $\frac{1}{r} + \frac{1}{s} = 1$, $1 \le r \le \infty$, conditions (3.2) and (3.3) hold uniformly in x and $E_n^1 C_n^2$ is $(E,1)(C,2)$ means of the series (2.1).

Note 3.1 For $\beta = 0$, our theorem reduces to the theorem of Nigam [8] and thus generalizes it.

4 Lemma

In order to prove our theorem, we need following lemmas:

Lemma 4.1. *For* $0 \leq t \leq \frac{1}{n+1}$,

$$|M_n(t)| = O\,(n+1) \tag{4.1}$$

Proof. For $0 \leq t \leq \frac{1}{n+1}$, $\sin nt \leq n\ \sin t$,

$$\begin{aligned}
|M_n\,(t)| &\leq \frac{1}{2^n\pi}\left|\sum_{k=0}^{n}\left\{\binom{n}{k}\frac{1}{(k+1)\,(k+2)}\sum_{\nu=0}^{k}(k-\nu+1\ )\quad\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2^n\pi}\left|\sum_{k=0}^{n}\left\{\binom{n}{k}\frac{1}{(k+1)\,(k+2)}\sum_{\nu=0}^{k}(k-\nu+1\ )\quad\frac{(2\nu+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2^n\pi}\left|\sum_{k=0}^{n}\left\{\binom{n}{k}\frac{1}{(k+1)\,(k+2)}\quad(2k+1)\sum_{\nu=0}^{k}(k-\nu+1\ )\right\}\right| \\
&\leq \frac{1}{2^n\pi}\left|\sum_{k=0}^{n}\left\{\binom{n}{k}\frac{1}{(k+1)\,(k+2)}\quad(2k+1)\,\frac{(k+1)\,(k+2)}{2}\right\}\right| \\
&= \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\left\{\binom{n}{k}(2k+1)\right\} \\
&= \frac{1}{2^{n+1}\pi}\left\{2^n\,(n+1)\right\} \\
&= O(n+1)
\end{aligned}$$

□

Lemma 4.2. *For* $\frac{1}{n+1} \leq t \leq \pi$,

$$|M_n(t)| = O\left(\frac{1}{t}\right) \tag{4.2}$$

Proof. For $\frac{1}{n+1} \leq t \leq \pi$, $\sin \frac{t}{2} \geq \frac{t}{\pi}$ and $\sin nt \leq 1$

$$\begin{aligned}
|M_n(t)| &\leq \frac{1}{2^n \pi} \left| \sum_{k=0}^{n} \left\{ \begin{pmatrix} n \\ k \end{pmatrix} \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k-\nu+1\ ) \quad \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\} \right| \\
&\leq \frac{1}{2^n \pi} \left| \sum_{k=0}^{n} \left\{ \begin{pmatrix} n \\ k \end{pmatrix} \frac{1}{(k+1)(k+2)} \sum_{\nu=0}^{k} (k-\nu+1\ ) \quad \left(\frac{1}{\frac{t}{\pi}}\right) \right\} \right| \\
&\leq \frac{1}{2^n\ t} \left| \sum_{k=0}^{n} \left\{ \begin{pmatrix} n \\ k \end{pmatrix} \frac{1}{(k+1)(k+2)} \ \sum_{\nu=0}^{k} (k-\nu+1\ ) \right\} \right| \\
&\leq \frac{1}{2^n\ t} \left| \sum_{k=0}^{n} \left\{ \begin{pmatrix} n \\ k \end{pmatrix} \frac{1}{(k+1)(k+2)} \ \frac{(k+1)(k+2)}{2} \right\} \right| \\
&= \frac{1}{2^{n+1} t} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \\
&= \frac{1}{2^{n+1}\ t} \{2^n\} \\
&= O\left(\frac{1}{t}\right)
\end{aligned}$$

□

5 Proof of Main Theorem

Let $s_n(x)$ denote the partial sum of the series (2.1), then we have

$$s_n(x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt$$

Therefore, the $E_n^1 C_n^2$ transform of $s_n(f;x)$ is given by

$$\begin{aligned}
E_n^1 C_n^2 - f(x) = \frac{1}{\pi\ 2^n} \Bigg[\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{1}{(k+1)(k+2)} \\
\int_0^{\pi} \frac{\phi(t)}{\sin\frac{t}{2}} \left\{ \sum_{\nu=0}^{k} (n-\nu+1) \sin\left(\nu+\frac{1}{2}\right) t \right\} dt \Bigg] \\
= \int_0^{\pi} \phi(t)\ \ M_n(t)\, dt
\end{aligned}$$

$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right] \phi\left(t\right) M_n\left(t\right))dt$$

$$= I_1 + I_2 \text{ (say)} \tag{5.1}$$

We consider,

$$\left|I_1\right| \leq \int_0^{\frac{1}{n+1}} \left|\phi\left(t\right)\right| \left|M_n\left(t\right)\right| dt$$

Applying Hölder's inequality and the fact that $\phi\left(t\right) \in W\left(L_r, \xi\left(t\right)\right)$ due to the fact that $f \in W\left(L_r, \xi\left(t\right)\right)$, condition (3.2) and Lemma 4.1, we have

$$\begin{aligned} \left|I_1\right| &\leq \left[\int_0^{\frac{1}{n+1}} \left\{\frac{t\ \left|\phi\left(t\right)\right| \sin^{\beta} t}{\xi\left(t\right)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi\left(t\right) \left|M_n\left(t\right)\right|}{t\ \sin^{\beta} t}\right\}^s dt\right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{n+1}\right) \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi\left(t\right) \left|M_n\left(t\right)\right|}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{n+1}\right) \left[\int_{0^+}^{\frac{1}{n+1}} \left\{\frac{\xi\left(t\right)\left(n+1\right)}{t^{1+\beta}}\right\}^s dt\right]^{\frac{1}{s}} \end{aligned}$$

Since $\xi(t)$ is positive increasing function and using second mean value theorem for integrals, we have

$$\begin{aligned} \left|I_1\right| = &\ O\left\{\xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{\in}^{\frac{1}{n+1}} \frac{dt}{t^{(1+\beta)s}}\right]^{\frac{1}{s}} \text{ for some } 0 \leq \in < \frac{1}{n+1} \\ &= O\left[\xi\left(\frac{1}{n+1}\right) \left\{\frac{t^{-(1+\beta)s+1}}{-\left(1+\beta\right)s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}} \\ &= O\left[\xi\left(\frac{1}{n+1}\right) \left(n+1\right)^{1+\beta-\frac{1}{s}}\right] \\ &= O\left\{\left(n+1\right)^{\beta+1-\frac{1}{s}} \xi\left(\frac{1}{n+1}\right)\right] \\ &= O\left\{\left(n+1\right)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r} + \frac{1}{s} = 1,\ 1 \leq r \leq \infty. \end{aligned} \tag{5.2}$$

Now we consider,

$$\left|I_2\right| \leq \int_{\frac{1}{n+1}}^{\pi} \left|\phi\left(t\right)\right| \left|M_n\left(t\right)\right| dt$$

Using Hölder's inequality, $|\sin t| < 1$ and $\sin t \geq \left(\frac{2t}{\pi}\right)$,

$$|I_2| \leq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}\,|\phi(t)\ |\sin^{\beta} t}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)\,|M_n(t)|}{t^{-\delta}\sin^{\beta} t}\right\}^s dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)\,|M_n(t)|}{t^{-\delta+\beta}}\right\}^s dt\right]^{\frac{1}{s}} \text{ by (3.3)}$$

$$= O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{1-\delta+\beta}}\right\}^s dt\right]^{\frac{1}{s}} \text{ by Lemma 4.2}$$

Now putting $t = \frac{1}{y}$,

$$I_2 = O\left\{(n+1)^{\delta}\right\} \left[\int_{\frac{1}{\pi}}^{n+1} \left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}}\right\}^s \frac{dy}{y^2}\right]^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function and $\frac{\xi\left(\frac{1}{y}\right)}{\frac{1}{y}}$ is also increasing function and using second mean value theorem for integrals,

$$I_2 = O\left\{(n+1)^{\delta}(n+1)\,\xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{\eta}^{n+1} \frac{dy}{y^{\delta s+2-\beta s}}\right]^{\frac{1}{s}}, \text{ for some } \frac{1}{\pi} \leq \eta \leq n+1$$

$$= O\left\{(n+1)^{\delta+1}\,\xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{1}^{n+1} \frac{dy}{y^{\delta s+2-\beta s}}\right]^{\frac{1}{s}}, \text{ for some } \frac{1}{\pi} \leq 1 \leq n+1$$

$$= O\left\{(n+1)^{\delta+1}\,\xi\left(\frac{1}{n+1}\right)\right\} \left[\left\{\frac{y^{-s\delta+\beta s-1}}{-s\delta+\beta s-1}\right\}_1^{n+1}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta+1}\,\xi\left(\frac{1}{n+1}\right)\right\} \left[(n+1)^{-\delta-\frac{1}{s}+\beta}\right]$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\} \left\{(n+1)^{1-\frac{1}{s}+\beta}\right\}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1 \tag{5.3}$$

Now combining (5.1), (5.2) and (5.3), we get

$$\left|E_n^1 C_n^2 - f\right| = O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

Now using L_r- norm, we get

$$
\begin{aligned}
\left\|E_n^1C_n^2 - f\right\|_r &= \left\{\int_0^{2\pi}\left|E_n^1C_n^2 - f\right|^r dx\right\}^{\frac{1}{r}}\\
&= \left[\left\{\int_0^{2\pi}\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}\right]\\
&= \left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right]\\
&= \left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}.
\end{aligned}
$$

This completes the proof of the main theorem .

6 Applications

Following corollaries can be derived from our main theorem:

6.1 Corollary

If $\xi(t) = t^\alpha, 0 < \alpha \leq 1$, then weighted class $W(L_r, \xi(t))$, $r \geq 1$, reduces to the class $Lip(\alpha, r)$ and the degree of approximation of 2π- periodic function f, belonging to the class $Lip(\alpha, r\ )$, $\frac{1}{r} < \alpha < 1$ is given by

$$
\left|E_n^1C_n^2 - f\right| = O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\} \tag{6.1}
$$

Proof. The result follows by setting $\beta = 0$ in (3.1). □

6.2 Corollary

If $\xi(t) = t^\alpha\ for\ 0 < \alpha < 1$ and $r = \infty$ in corollary 6.1, then $f \in Lip\alpha$. In this case, using (6.1), we have

$$
\left|E_n^1C_n^2 - f\right| = O\left\{\frac{1}{(n+1)^\alpha}\right\}
$$

Proof. For $r = \infty$, we get

$$\left\|E_n^1 C_n^2 - f\right\|_\infty = \sup_{0 \le x \le 2\pi} \left|E_n^1 C_n^2 - f\right| = O\left\{\frac{1}{(n+1)^\alpha}\right\}$$

that is,

$$\left|E_n^1 C_n^2 - f\right| = O\left\{\frac{1}{(n+1)^\alpha}\right\}$$

□

References

[1] P. Chandra, "Trigonometric approximation of functions in L_p norm", *J. Math. Anal. Appl.* Vol. 275 No. 1, 2002, pp. 13-26.

[2] G. Alexits, Convergence problems of orthogonal series, Pergamon Press, London, 1961.

[3] G. H. Hardy, *"Divergent series"*, first edition, Oxford University Press, 1949, 70.

[4] H. H. Khan, "On degree of approximation of functions belonging to the class Lip(α, p)", *Indian J. Pure Appl. Math,* Vol. 5, No. 2, 1974, pp. 132-136.

[5] L. Leindler, "Trigonometric approximation of functions in L_p norm", *J. Math. Anal. Appl.,* Vol. 302, No. 1, 2005, pp. 129-136.

[6] L. Mc Fadden, "Absolute Nörlund summability", *Duke Math. J.,* Vol. 9, 1942, pp. 168-207.

[7] V. N. Mishra, H. H. Khan, I. A. Khan, K. Khatri and L. N. Mishra, "Trigonometric Approximation of signals (functions) belonging to the $Lip(\xi(t), r), (r \ge 1)$- class by $(E, q)(q > 0)$-Means of the conjugate series of its Fourier Series", *Advances in Pure Mathematics*, Vol. 3, 2013, pp. 353-358.

[8] H. K. Nigam, "Degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by $(E, 1)(C, 2)$ summability means", *Asian Journal of Fuzzy and Applied Mathematics* 1 No. 3, 2013, pp. 61-68.

[9] K. Qureshi, "On the degree of approximation of a periodic function f by almost Nörlund means", *Tamkang J. Math.* 12 No. 1, 1981, pp. 35-38.

[10] K. Qureshi, "On the degree of approximation of a function belonging to the class Lipα", *Indian J. pure Appl. Math.* 13 No. 8, 1982, pp. 560-563.

[11] K. Qureshi, H. K. Neha, "A class of functions and their degree of approximation", *Ganita.* 41 No. 1, 1990, pp. 37-42.

[12] B. E. Rhaodes, "On the degree of approximation of functions belonging to Lipschitz class by Hausdorff means of its Fourier series", *Tamkang J. Math.* 34 no. 3, 2003, pp. 245-247.

[13] B. N. Sahney, D. S. Goel, "On the degree of continuous functions", *Ranchi University, Math. Jour.,* 4, 1973, pp. 50-53.

[14] A. Zygmund, *"Trigonometric series"*, 2^{nd} rev. ed., Vol. 1, Cambridge Univ. Press, Cambridge, 1959.