

Fixed Point Theorems for Cyclic Weak Φ – Contraction in Menger Space

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Abstract - Cyclic weak ϕ -contraction mapping is introduced in Menger space and fixed point theorem for such mappings are studied in Menger space.

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I. INTRODUCTION

In 1969, Boyd and Wong introduced the notion of ϕ – contraction in metric space. Alber and Guerre-Delabriere, gave the definition of weak ϕ –contraction for Hilbert space and proved the existence of fixed points in Hilbert space in 1997.

The existence application potential of fixed point theory in various fields resulted in several generalizations of the metric spaces. One such generalization is Menger space, initiated by Menger. It was observed by many authors that contraction condition in metric space could be extended to Menger space. V. M. Sehgal and A.T. Bharucha-Reid first introduced the contraction mapping principle in probabilistic metric space which is a milestone for the development of fixed point theory in Menger space. Sunny Chauhan proved common fixed point theorems for weakly compatible mappings in Menger space satisfying ϕ – contractive conditions. Recently D. Gopal *et al*, established a common fixed point theorem in fuzzy metric space using cyclic weak ϕ – contraction.

The main objective of this paper is to study cyclic weak ϕ – contraction. We introduced the mapping with cyclic weak ϕ – contraction and obtained unique fixed point theorem for such mapping.

II. PRELIMINARIES

Before giving our main results, we recall some of the basic concepts and results in Menger space.

Definition 2.1. A mapping $F : R \rightarrow R^+$ is called a distribution if it is non-decreasing left continuous with $\inf \{F(t):t \in R\} = 0$ and $\sup \{F(t): t \in R\} = 1$. We shall denote by the L the set of all distribution functions while H will always denote the specific distribution function defined by $H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases}$

Definition 2.2. A probabilistic metric space (PM-space) is an ordered pair (X, F) where X is an arbitrary set of elements and $F : X \times X \rightarrow L$ is defined by $(p,q) \rightarrow F_{p,q}$, where L is the set of all distribution functions, that is, $L = \{ F_{p,q}; p,q \in X\}$, where the functions $F_{p,q}$ satisfy:

- i) $F_{p,q}(x) = 1$ for all $x > 0$, $F_{p,q}(x) = 0$ if and only if $p = q$;
- ii) $F_{p,q}(0) = 0$;
- iii) $F_{p,q} = F_{q,p}$;
- iv) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$.

Definition 2.3. A mapping $t : [0,1] \times [0,1] \rightarrow [0,1]$ is called a t -norm if

- i) $t(a, 1) = a, t(0, 0) = 0$;
- ii) $t(a, b) = t(b, a)$;
- iii) $t(c, d) \geq t(a, b)$ for $c \geq a, d \geq b$;
- iv) $t(t(a, b), c) = t(a, t(b, c))$.

Definition 2.4. A Menger space is a triplet (X, F, t) where (X, F) is a PM- space and t is a t - norm such that for all p, q, r in X and for all $x, y \geq 0$, $F_{p,r}(x+y) \geq t(F_{p,q}(x), F_{q,r}(y))$.

Example 2.1. If (X, d) is a metric space then the metric d induces a mapping F from $X \times X$ to L , defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$ and $x \in R$. If the t -norm $t : [0,1] \times [0,1] \rightarrow [0,1]$ is defined by $t(a, b) = \min\{a, b\}$, then (X, F, t) so obtained is called the induced Menger space.

Example 2.2. Let (X, d) be a metric space. We defined the t -norm $*$ by $a * b = ab$ for all $a, b \in [0,1]$ and $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, F, *)$ is a Menger space.

Definition 2.5. Let $(X, F,*)$ be a Menger space. Then

- i) a sequence $\{x_n\}$ in X is said to converge to a point x in X , if and only if $\lim_{n \rightarrow \infty} F_{x_n,x}(t) = 1$ for all $t > 0$, that is, for each $\lambda \in (0, 1)$ and $t > 0$, there is an integer $n_0 \in \mathbb{N}$ such that $F_{x_n,x}(t) > 1 - \lambda$ for all $n \geq n_0$ and we denote $x_n \rightarrow x$.

- ii) a sequence $\{x_n\}$ in X is said to be a G -Cauchy sequence if and only if $\lim_{n \rightarrow \infty} F_{x_{n+p}, x_n}(t) = 1$ for any $p > 0$ and $t > 0$.
- iii) the Menger space $(X, F, *)$ is called G -complete if every G -Cauchy sequence is convergent.

Definition 2.6. Let X be a non-empty set, m be a positive integer and $f : X \rightarrow X$ an operator. $X = \cup_{i=1}^m X_i$ is a cyclic representation of X with respect to f if

- i) $X_i, i = 1, 2, \dots, m$ are non-empty sets.
- ii) $f(X_1) \subset X_2, f(X_2) \subset X_3, \dots, f(X_{m-1}) \subset X_m$ and $f(X_m) \subset X_1$.

Example 2.3. Let $X = \mathbb{R}$. Assume $A_1 = A_3 = [-1, 0]$ and $A_2 = A_4 = [0, 1]$ so that $Y = \cup_{i=1}^4 A_i = [-1, 1]$. Define $f : Y \rightarrow Y$ such that $f(y) = \frac{-y}{2}$ for all $y \in Y$. It is clear that $Y = \cup_{i=1}^4 A_i$ is a cyclic representation of X with respect to f .

III. MAIN RESULTS

Definition 3.1. Let $(X, F, *)$ be a Menger space, A_1, A_2, \dots, A_m be closed subsets of X and $Y = \cup_{i=1}^m A_i$. An operator $f : X \rightarrow X$ is called a cyclic weak ϕ -contraction if the following conditions hold:

- i) $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f .
- ii) There exists continuous non-decreasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(r) > 0$ for $r > 0$ and $\phi(0) = 0$ such that

$$\left(\frac{1}{F_{fx, fy}(t)} - 1\right) \leq \left(\frac{1}{F_{x, y}(t)} - 1\right) - \phi\left(\frac{1}{F_{x, y}(t)} - 1\right) \quad (3.1)$$

for any $x \in A_i, y \in A_{i+1}$ ($i = 1, 2, \dots, m$ and $A_{m+1} = A_1$) and each $t > 0$.

Theorem 3.1. Let $(X, F, *)$ be a Menger space, A_1, A_2, \dots, A_m be closed subsets of X and $Y = \cup_{i=1}^m A_i$ be G -complete. Suppose that $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous non-decreasing function with $\phi(r) > 0$ for each $r \in (0, \infty)$ and $\phi(0) = 0$. If $f : Y \rightarrow Y$ is a cyclic weak ϕ -contraction then f has a unique fixed point $y \in \cap_{i=1}^m A_i$.

Proof: Let $x_0 \in Y = \cap_{i=1}^m A_i$ and set $x_n = f x_{n-1}, n \geq 1$.

Then $F_{x_n, x_{n+1}}(t) = F_{fx_{n-1}, fx_n}(t)$ for any $t > 0$. For any $n \geq 0$, there exists $i_n \in \{1, 2, \dots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$.

For $t > 0$,

$$\begin{aligned} \left(\frac{1}{F_{x_n, x_{n+1}}(t)} - 1\right) &= \left(\frac{1}{F_{fx_{n-1}, fx_n}(t)} - 1\right) \\ &\leq \left(\frac{1}{F_{x_{n-1}, x_n}(t)} - 1\right) - \phi\left(\frac{1}{F_{x_{n-1}, x_n}(t)} - 1\right) \quad (3.2) \\ &\leq \left(\frac{1}{F_{x_{n-1}, x_n}(t)} - 1\right) \end{aligned}$$

which implies that $F_{x_n, x_{n+1}}(t) \geq F_{x_{n-1}, x_n}(t)$ for all $n \geq 0$ and so $\{F_{x_{n-1}, x_n}(t)\}$ is a non-decreasing sequence of positive real numbers in $(0, 1]$.

Let $S(t) = \lim_{n \rightarrow \infty} F_{x_{n-1}, x_n}(t)$.

Next we show that $S(t) = 1$ for all $t > 0$.

If not there exists some $t > 0$ such that $S(t) < 1$.

Then, on making $n \rightarrow \infty$ in (3.2) we obtain

$$\frac{1}{S(t)} - 1 \leq \left(\frac{1}{S(t)} - 1\right) - \phi\left(\frac{1}{S(t)} - 1\right),$$

which is a contradiction.

Therefore $S(t) = 1$ for all $t > 0$. That is, $F_{x_{n-1}, x_n}(t) \rightarrow 1$

as $n \rightarrow \infty$.

Now for each positive integer p , we have

$$F_{x_n, x_{n+p}}(t) \geq F_{x_n, x_{n+1}}(t/p) * \dots * F_{x_{n+p-1}, x_{n+p}}(t/p)$$

It follows that $\lim_{n \rightarrow \infty} F_{x_n, x_{n+p}}(t) \geq 1 * 1 * \dots * 1 = 1$. This implies that $\{x_n\}_{n \geq 0}$ is a G -Cauchy sequence. As Y is complete, there exists $y \in Y$ such that $\lim_{n \rightarrow \infty} x_n = y$. On the other hand by (3.1), it follows that the iterative sequence, sequence $\{x_n\}$ has an infinite number of terms in A_i , for each $i = 1, 2, \dots, m$. From each $A_i, i = 1, 2, \dots, m$, one can extract a subsequence of $\{x_n\}$ that converges to y . By virtue of the fact that each $A_i, i = 1, 2, \dots, m$ is closed, we conclude that $y \in \cap_{i=1}^m A_i$. Now fix $i \in \{1, 2, \dots, m\}$ such that $y \in A_i$ and $f y \in A_{i+1}$. We take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$.

Now,

$$\begin{aligned} F_{y, fy}(t) &\geq F_{y, x_{n_k+1}}(t/2) * F_{x_{n_k+1}, fy}(t/2) \\ &= F_{y, x_{n_k+1}}(t/2) * F_{fx_{n_k}, fy}(t/2) \end{aligned}$$

As $k \rightarrow \infty, F_{y, fy}(t) \geq 1 * 1 = 1$ and so $fy = y$. Thus $y \in \cap_{i=1}^m A_i$ is a fixed point of f .

In order to prove the uniqueness of fixed point, let z be another fixed point of f .

By (3.1)

$$\left(\frac{1}{F_{fx_n,z}(t)} - 1\right) \leq \left(\frac{1}{F_{x_{n-1},z}(t)} - 1\right) - \phi\left(\frac{1}{F_{x_{n-1},z}(t)} - 1\right)$$

Letting $n \rightarrow \infty$ we get

$$\left(\frac{1}{F_{y,z}(t)} - 1\right) \leq \left(\frac{1}{F_{y,z}(t)} - 1\right) - \phi\left(\frac{1}{F_{y,z}(t)} - 1\right)$$

which is contradiction if $F_{y,z}(t) < 1$, and so, we conclude that $y = z$. Hence y is the unique fixed point of f .

The following example illustrate the above theorem

Example 3.1. Let $X = \mathbb{R}$, $F_{x,y}(t) = \frac{t}{t + |x-y|}$, for all $x, y \in X$, $t > 0$ and the t -norm $*$ is defined as $a * b = ab$. Then $(X, F, *)$ is a Menger space. Assume $A_1 = A_2 = \dots = A_m = [0, 1]$ so that $Y = \bigcup_{i=1}^m A_i = [0, 1]$ is G -complete. Define $f: Y \rightarrow Y$ such that $fy = \frac{y^2}{4}$. Furthermore if $\phi: [0, \infty) \rightarrow [0, \infty)$ is defined by $\phi(s) = \frac{s}{2}$, we have

$$\begin{aligned} \frac{1}{F_{fx,fy}(t)} - 1 &= \frac{1}{\frac{t}{t + |\frac{x^2}{4} - \frac{y^2}{4}|}} - 1 = \frac{|x^2 - y^2|}{4t} \\ &\leq \frac{|x-y|}{2t} \end{aligned} \tag{i}$$

$$\begin{aligned} \left(\frac{1}{F_{x,y}(t)} - 1\right) - \phi\left(\frac{1}{F_{x,y}(t)} - 1\right) &= \left(\frac{t + |x-y|}{t} - 1\right) - \phi\left(\frac{t + |x-y|}{t} - 1\right) \\ &= \frac{|x-y|}{t} - \phi\left(\frac{|x-y|}{t}\right) = \frac{|x-y|}{2t} \end{aligned} \tag{ii}$$

From (i) and (ii),

$$\frac{1}{F_{fx,fy}(t)} - 1 \leq \left(\frac{1}{F_{x,y}(t)} - 1\right) - \phi\left(\frac{1}{F_{x,y}(t)} - 1\right).$$

Thus f is a cyclic weak ϕ -contraction. All the conditions of the theorem are satisfied and f has a unique fixed point $0 \in \bigcap_{i=1}^m A_i$.

Definition 3.2. Let $(X, F, *)$ be a Menger space. The distribution function F is triangular if it satisfies the condition $\left(\frac{1}{F_{x,y}(t)} - 1\right) \leq \left(\frac{1}{F_{x,z}(t)} - 1\right) + \left(\frac{1}{F_{y,z}(t)} - 1\right)$ for every $x, y, z \in X$ and every $t > 0$.

Example 3.2. Let $X = [0, \infty)$ with the metric d and for each $t > 0$ define $F_{x,y}(t) = \frac{t}{t + d(x,y)}$, for $x, y \in X$ and $*$ is defined as $a * b = ab$. Then $(X, F, *)$ is a Menger space.

For any $x, y \in X$ and $t > 0$

$$\begin{aligned} \frac{1}{F_{x,y}(t)} - 1 &= \frac{1}{\frac{t}{t + d(x,y)}} - 1 = \frac{d(x,y)}{t} \\ &\leq \frac{d(x,z)}{t} + \frac{d(z,y)}{t} \\ &= \left(\frac{1}{F_{x,z}(t)} - 1\right) + \left(\frac{1}{F_{y,z}(t)} - 1\right) \end{aligned}$$

Theorem 3.2. If we add to the hypotheses of theorem 3.1 the following condition: there exists a sequence $\{y_n\}$ in Y such that $F_{y_n, fy_n}(t) \rightarrow 1$ as $n \rightarrow \infty$ for any $t > 0$, then $y_n \rightarrow y$ as $n \rightarrow \infty$, provided that the distribution function F is triangular, and y is the unique fixed point of f in Y .

Proof: By theorem 3.1, $y \in \bigcap_{i=1}^m A_i$ is the unique fixed point of f . Now, from the triangularity of F and from (3.1) we have

$$\begin{aligned} \frac{1}{F_{y_n,y}(t)} - 1 &\leq \left(\frac{1}{F_{y_n, fy_n}(t)} - 1\right) + \left(\frac{1}{F_{fy_n,y}(t)} - 1\right) \\ &\leq \left(\frac{1}{F_{y_n, fy_n}(t)} - 1\right) + \left(\frac{1}{F_{y_n,y}(t)} - 1\right) - \phi\left(\frac{1}{F_{y_n,y}(t)} - 1\right) \end{aligned}$$

which is equivalent to $\phi\left(\frac{1}{F_{y_n,y}(t)} - 1\right) \leq \left(\frac{1}{F_{y_n, fy_n}(t)} - 1\right)$

Since $\lim_{n \rightarrow \infty} \left(\frac{1}{F_{y_n, fy_n}(t)} - 1\right) = 0$, from the above inequality

$$\lim_{n \rightarrow \infty} \phi\left(\frac{1}{F_{y_n,y}(t)} - 1\right) = 0$$

Then by the properties of ϕ , we conclude that $F_{y_n,y}(t) \rightarrow 1$, which is equivalent to say $y_n \rightarrow y$ as $n \rightarrow \infty$.

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