Parameter Estimation of Weibull Growth Models in Forestry

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***Abstract*—The main aim of this paper is to discuss some certain properties of three Weibull growth models in forestry viewpoint. The parameters of these models are estimated using Newton-Raphson iteration method for the mean diameter at breast height data and top height growth data originated from the Bowmont Norway spruce thinning experiment, sample plot 3661. The average height of 12 weeping Higan Cherry trees planted in Washington, D.C. is also used. The initial value specifications of the parameters for use of any iterative methods of estimation are also provided.**

***Key words*— Initial value, Parameter estimation, Weibull model, Newton-Raphson method, Growth model, derivative, Mathematical Properties.**

I. Introduction

The Weibull model was first introduced by Ernst Hjalmar Waloddi Weibull in 1951. Initially it was described as a statistical distribution. It has many applications in population growth, agricultural growth, height growth and is also used to describe survival in cases of injury or disease or in population dynamic studies [7]. In 1997, Lianjun Zhang [13] used this model to describe tree height-diameter data of ten conifer species. In the paper by Fekedulegn et al. [6] used this model for the top height data of Norway spruce from the Bowmont Norway spruce Thinning Experiment. Colbert et al. [1] have tried to define some character developments such as forest trees height growth and diameter development by using the model. The paper by Karadavut et al. [8] used the model to evaluate the relative growth rate of silage corn. Ozel and ertekin in 2011 [10] studied the Weibull growth model and applied it to the oriental beach Juvenilities growth. Weibull model is also used to study the height growth of Pinus radiate by Colff and Kimberley in 2013 [2]. Lumbres et al. [9] used this model to describe the diameter at breast height of Pinus Kesiya.

The Weibull growth model can be derived from the one parameter Weibull distribution function, which is given by

$$F(x) = 1 - exp\left(-x^{\delta}\right) ; x > 0, \tag{1}$$

where $\delta > 0$ is a shape parameter. The distribution function has the point of inflection at $x = [(\delta - 1)/\delta]^{\frac{1}{\delta}}$ and $F = 1 - exp(-(1 - \delta^{-1}))$. Then the eq. (2) can be used to get a sigmoidal growth curve for empirical use [12].

$$f(x) = \beta + (\alpha - \beta)F(kx; \theta) \tag{2}$$

For the Weibull distribution,

$$f(x) = \alpha - (\alpha - \beta)exp\{-(kx)^{\delta}\}. \tag{3}$$

The form of the model (3) is the model that David Ratkowsky [11] calls the Weibull model. The equation (3) can also be written as,

$$w(t) = a - b\, exp(-ct^{m}). \tag{4}$$

Where $a = \alpha$, $b = \alpha - \beta$, $c = k^{\delta}$ and $m = \delta$. For this study, the model (4) will be considered as the Weibull model with four parameters. Philippe Grosjean [7] used a three parameter Weibull model with considering $\beta = 0$ in eq. (3), which is given in eq. (5). This eq. gives the Weibull model with three parameters.

$$w(t) = a(1 - exp(-ct^{m})) \tag{5}$$

In 2000, Alistair Duncan Macgregor Dove [3] used a two parameters Weibull model to investigate parasite richness of nine species of fishes. He used

$$w(t) = a(1 - exp(-ct)). \tag{6}$$

This model can be derived from the Weibull model with three parameters by considering $m = 1$ to the eq. (5). Dove also defines the parameters of the model (6) for parasite richness of fishes as a is the maximum regional fauna richness and c is an index of the mean infracommunity richness.

The properties and the derivations of the Weibull models play a crucial rule for estimating the parameters. Proper understanding of the mathematics of these models avoids problems encountered in the method of parameter estimations of the models. This paper provided the most fundamental properties and careful observations of the first and second derivatives of the models. This paper also provides some useful definition of the parameters for initial estimates of the Weibull models, which is also a major requirement for estimating the parameters using any iteration method. Three well-known Forestry data sets are considered to estimate the parameters of these models.

II. Material and methods

The growth models considered for this study are Weibull model with two parameters (6), Weibull model with three parameters (5) and Weibull model with four parameters (4). For these three models consider, w is the dependent growth variable, t is the independent variable, a, b, c and m are parameters to be estimated, $\log$ is the natural logarithms and $\exp(e)$ is the base of the natural logarithms. The properties of the parameters of this model are discussed in this paper. The parameters are estimated using the Newton –Raphson method of nonlinear regression relating the mean diameter at breast height data and top height growth data originated from the Bowmont Norway spruce thinning experiment, sample plot 3661 [5, 6]. The average height of 12 Weeping Higan cherry trees planted in Washington, D.C. are also used to testing the validity of the models for forestry viewpoint. At the time of planting, the trees were one year old and were all 6 feet in height [14].

Table 1. Mean diameter at breast height data from the Bowmont Norway spruce thinning experiment, sample plot 3661.

Age (years)	20	25	30	35	40	45	50	55	60	64
Mean DBH	8.4	10.4	12.35	14.74	17.13	19.5	21.49	23.82	25.55	26.5

The Weibull growth models can be written in the form as

$$w_i = f(t_i, \mathbf{B}) + \varepsilon_i, \tag{7}$$

$i = 1,2, \cdots, n$, where $w_i's$ are the response variables, $t_i's$ are the independent variable, $\mathbf{B}$ is the vector of parameters b_j $(b_1, b_2, \cdots, b_m)$ to be estimated. Where m is the number of parameter, n is the number of observations and $\varepsilon_i's$ are random errors in the models has mean zero and constant variance σ^2. Root mean square error (RMSE) and coefficient of determination test (R^2) are used to check the validity of these models. A FORTRAN programming has been developed for the Newton-Raphson algorithm. The confidence intervals of the parameters are also provided.

A. Method of Estimation

The parameters of these models are estimated by minimizing the sum of square residue $S(\boldsymbol{B})$ under the assumption that the $\varepsilon_i's$ are independent $N(0, \sigma^2)$ random variable.

$$S(\boldsymbol{B}) = \sum_{i=1}^{n} [w_i - f(t_i, \mathbf{B})]^2 \tag{8}$$

Since w_i and t_i are fixed observations, the $S(\boldsymbol{B})$ is a function of $\boldsymbol{B}$. Now the eq. (8) is differentiated with respect to $\boldsymbol{B}$ and setting the result to zero, we get

$$f_j = \sum_{i=1}^{n} [w_i - f(t_i, \mathbf{B})] \left[\frac{\partial f(t_i, \boldsymbol{B})}{\partial b_j}\right] = 0 \tag{9}$$

for $j = 1,2, \cdots, m$. This provides a system of m nonlinear equation with m unknown parameters and that must be solved for $\boldsymbol{B}$ using any iteration method. In this literature, the Newton-Raphson iteration method is used to solve the equations. The general Newton-Raphson method for system of nonlinear equation is given by

$$\mathbf{B}^{(k+1)} = \mathbf{B}^{(k)} - {J_k}^{-1} F(\mathbf{B}^{(k)}); \; k = 0, 1, 2, 3 \cdots \tag{10}$$

Where

$$J_k = \begin{bmatrix} \frac{\partial f_1}{\partial b_1} & \frac{\partial f_1}{\partial b_2} \cdots & \frac{\partial f_1}{\partial b_m} \\ \frac{\partial f_2}{\partial b_1} & \frac{\partial f_2}{\partial b_2} \cdots & \frac{\partial f_2}{\partial b_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial b_1} & \frac{\partial f_m}{\partial b_2} \cdots & \frac{\partial f_m}{\partial b_m} \end{bmatrix}_{[\mathbf{B}^{(k)}]}, \mathbf{B}^{(k)} = \left[{b_1}^{(k)}, {b_2}^{(k)}, \cdots, {b_m}^{(k)}\right]^T$$

and $F(\mathbf{B}^{(k)}) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}_{[\mathbf{B}^{(k)}]}$.

The eq. (10) is the $(k+1)th$ iteration of the parameters in Newton-Raphson method. Since this is an iteration method, so the process may be repeated using a pre-defined stopping criterion. This method requires specifications of the starting values of the parameters to be estimated. The initial value specifications of the parameters are given bellow.

III. Results

A. Properties of Weibull growth model

There is a clear relationship between the properties of various mathematical models and the estimation of their respective parameters. If the properties of nonlinear mathematical models are to be known then it may quite helpful to estimate the parameters to be estimated. Even some cases, due to lack of knowledge of these properties, it may seem to face different problems to use in various biological growth. Therefore, in this paper, an attempt has been made to discuss the different fundamental properties of the Weibull growth models.

1. Properties of Four Parameter Weibull Model

In the form of the model (4), a, b, c and m are the parameters and are defined correspondingly as: a is the asymptote or the limiting value of the response variable, b is the biological constant, c is the parameter governing the rate at which the response variable approaches its potential maximum and m is the allometric constant [6] and t is the independent variable w is the response variable of t.

The Weibull growth model rises from a point β, at the starting growth setting $t = 0; w = a - b = \alpha - \alpha + \beta = \beta$; to the limiting value of α, which is the maximum possible value of $w(t)$; that is when $t \to \infty \Rightarrow w \to \alpha$. Examining the model at the starting of the growth, which is most preferably when the independent variable (t) is zero, the only way to understand and make clear the meaning and possible range of the parameter b that is defined as a biological constant. In other word from the expression $w(0) = a - b$, it is logical to define the parameter b as a constant that should make the expression $a - b$ reasonably small enough to at least consider it as a possible value of the model parameter estimate at the starting growth. Based on the model assumption and evaluation of the model at the start of growth, it was evident that:

- $a > 0$, since a is the limiting value.
- The parameter b is always positive $(b > 0)$ and its size depends on the size of the parameter a. Since if $b = 0$ then $w = a$ at the starting of the growth and if $b < a$ then $w > a$ at the start of the growth and both

cases violate the model assumption concerning the parameter a which states that when $t \to \infty \Rightarrow w \to \alpha$.

- For biological growth analysis, the parameter c and m must be positive.
- The Weibull model is sigmoidal when $m > 1$, otherwise it has no inflexion point [7].

The Weibull model does not pass through the origin (when $t = 0; w \nrightarrow 0$) and that is the main limitation of the model for some biological growth. Now,

$$\frac{dw}{dt} = bcmt^{m-1}e^{-ct^m} \tag{11}$$

The first derivative of the model described the slope of the curve or the rate of change of the dependent variable with respect to the independent variable and is positive. That indicates that the model is an increasing function of the independent variable.
Now to investigate some important properties of the model, the second derivative of the model was derived and may be expressed as

$$\frac{d^2w}{dt^2} = bcm(m-1)t^{m-2}e^{-ct^m} - bc^2m^2t^{2m-2}e^{-ct^m}$$

$$\frac{d^2w}{dt^2} = cmt^{m-2}(a-w)(m-1-cmt^m) \tag{12}$$

It is seen that the second derivative of the model is positive $\left(\frac{d^2w}{dt^2} > 0\right)$ for $w < a - be^{\frac{1-m}{m}}$; zero $\left(\frac{d^2w}{dt^2} = 0\right)$ for $w = a - be^{\frac{1-m}{m}}$ and negative $\left(\frac{d^2w}{dt^2} < 0\right)$ for $w > a - be^{\frac{1-m}{m}}$. In terms of the predictor variable; the second derivative is positive for $0 \le t < \left(\frac{m-1}{cm}\right)^{\frac{1}{m}}$; zero for $t = \left(\frac{m-1}{cm}\right)^{\frac{1}{m}}$ and negative for $\left(\frac{m-1}{cm}\right)^{\frac{1}{m}} < t < \infty$.
So the model approaches the asymptote at an increasing rate for $w < a - be^{\frac{1-m}{m}}$ and at a decreasing rate for $w < a - be^{\frac{1-m}{m}}$. The point where the model makes a transition from an increasing to a decreasing slope, the second derivative of the model is zero $\left(\frac{d^2w}{dt^2} = 0\right)$ and at the same point the growth function has a constant slope. The point is known as the point of inflection and it occurs at $w = a - be^{\frac{1-m}{m}}$. This is the most important property of the model.

2. Properties of Three Parameters Weibull Model

From the equation (5), the Weibull model with three parameters was given by:

$$w(t) = a(1 - exp(-ct^m))$$

Here a, c and m are the parameters and they have the similar properties and definitions with the Weibull model with four parameters, t is the independent variable and w is the response variable.
The Weibull model with three parameters passes through the origin that is when $t = 0; w = 0$. for the case of some biological growth, this is an advantage of this model. Now the derivatives of the model with respect to the independent variable are given bellow:

$$\frac{dw}{dt} = acmt^{m-1}e^{-ct^m} \tag{13}$$

$$\frac{d^2w}{dt^2} = cmt^{m-2}(a-w)(m-1-cmt^m) \tag{14}$$

For the Weibull model with three parameters it is seen that the second derivative of the model is positive $\left(\frac{d^2w}{dt^2} > 0\right)$ for $w < a - ae^{\frac{1-m}{m}}$; zero $\left(\frac{d^2w}{dt^2} = 0\right)$ for $w = a - ae^{\frac{1-m}{m}}$ and negative $\left(\frac{d^2w}{dt^2} < 0\right)$ for $w > a - ae^{\frac{1-m}{m}}$. In terms of the predictor variable; the second derivative is positive for $0 \le t < \left(\frac{m-1}{cm}\right)^{\frac{1}{m}}$ zero for $t = \left(\frac{m-1}{cm}\right)^{\frac{1}{m}}$ and negative for $\left(\frac{m-1}{cm}\right)^{\frac{1}{m}} < t < \infty$. Again from (13), it is seen that the first derivative of the Weibull model $\left(\frac{dw}{dt}\right)$ is positive. This means that the yield model is an increasing function of the independent variable.
From the above discussion, it is achieved that the three parameter Weibull model approaches the asymptote at an increasing rate for values of the dependent variable less than $a - ae^{\frac{1-m}{m}}$. The differential form of the growth model is a decreasing function of the independent variable for $w > a - ae^{\frac{1-m}{m}}$. This implies that the yield function of the three parameter Weibull model is approaching the asymptote at a decreasing rate for values of the dependent variable greater than $a - ae^{\frac{1-m}{m}}$. The point where the growth function of the Weibull model makes a transition from an increasing to a decreasing slop, the growth function has a constant slop and the point is termed as the point of inflection of the model which occurs at $= a - ae^{\frac{1-m}{m}}$.

3. Properties of Two Parameters Weibull Model

The two parameters Weibull growth model is given by the eq. (6) and which can be written as

$$w(t) = a(1 - exp(-ct)).$$

The parameters of this model a and c also have the similar properties and definitions with the Weibull model with four parameters.
The Weibull model is started from 0 (as when $t = 0; w = 0$) to the limiting value of a (as $t \to \infty; w = a$), which is also known as the upper asymptote of this model. Also since this model passes through the origin, so for some biological growth it may be an advantage of this model. The first and second derivatives of the model with respect to the independent variable may be expressed as follows:

$$\frac{dw}{dt} = ace^{-ct} \tag{15}$$

$$\frac{d^2w}{dt^2} = -ac^2e^{-ct} \tag{16}$$

This describes that the first derivative of the model is always positive, which indicates that the model parameter estimates increase monotonically as the independent variable increases. Also a careful examination of the second derivative of the model revealed that it is always negative that is the growth curve is a decreasing function of t and this implies that increment of a stand parameter, decreases over the entire range of the independent variable. The slop of the yield curve decreases as the independent variable increases. Since the second derivative does not show a change in sign so the model has no point of inflection and this is the major drawback of the model for forestry growth and yield modeling. The **Table 2** illustrate the fundamental mathematical properties of the Weibull growth model to proper understand the mathematics of the models and avoids problems encountered in the method of parameter estimation of the nonlinear Weibull growth models.

Table 2. Summary of the properties of the Weibull growth models

	Weibull 4 parameter	**Weibull 3 parameter**	**Weibull 2 parameter**
Integral form of the growth function $(w(t))$	$a - b\ exp(-ct^m)$	$a(1 - exp(-ct^m))$	$a(1 - \ exp\ (-ct))$
Lower asymptote	$-\infty$	$-\infty$	$-\infty$
Upper asymptote	a	a	a
Starting point of the growth function	$a - b$	0	0
Growth rate $\left(\frac{dw}{dt}\right)$	$bcmt^{m-1}e^{-ct^m}$	$acmt^{m-1}e^{-ct^m}$	$ac\ e^{-ct}$
Maximum growth rate	$bmc^{\frac{1}{m}}e^{\frac{1-m}{m^2}}$	$amc^{\frac{1}{m}}e^{\frac{1-m}{m^2}}$	$Undefined$
Relative growth rate as function of time	$bcmt^{m-1}\left(ae^{ct^m} - b\right)^{-1}$	$a^2cmt^{m-1}\left(e^{ct^m} - 1\right)^{-1}$	$ce^{-ct}(1 - e^{-ct})^{-1}$
Relative growth rate as function of biomass	$cm\left(\frac{1}{c}\ln\frac{b}{a-w}\right)^{\frac{m-1}{m}}\frac{a-w}{w}$	$cm\left(\frac{1}{c}\ln\frac{a}{a-w}\right)^{\frac{m-1}{m}}\frac{a-w}{w}$	$c(a-w)/_{w}$
Second derivative of the growth function $\left(\frac{d^2w}{dt^2}\right)$	$cmt^{m-2}(a-w)(m-1-cmt^m)$	$cmt^{m-2}(a-w)(m-1-cmt^m)$	$-ac^2e^{-ct}$
Point of inflection $w(t) =$	$a - be^{\frac{1-m}{m}}$	$a - ae^{\frac{1-m}{m}}$	$No\ point\ of\ inflection$
Domain of the independent variable	$(0, \infty)$	$(0, \infty)$	$(0, \infty)$
Domain of the dependent variable	$(a - b, a)$	$(0, a)$	$(0, a)$

B. *Starting Value Specification*

All iteration procedures require initial values of the parameters to be estimates and the better these initial estimates are, the faster will be the convergence to the fitted value. Initial value specification is one of the most difficult problems encountered in estimating parameters of nonlinear model [4]. If the initial estimates are poor, convergence to the wrong final values can easily occur. There is no any general method for obtaining initial estimates. One uses whatever information is available. For Weibull models,

Starting value of $\boldsymbol{a}$: From the earlier discussion, it is noticed that for Weibull growth model, the parameter a is defined as the limiting value of the dependent variable. Therefore for the biological growth, the parameter a was specified as the maximum value of the dependent variable in the data.

Starting value of $\boldsymbol{b}$: The starting value for the biological constant, b, was specified by evaluating the model at the start of the growth when the predictor variable is zero. To specify the starting value of the parameter b for the Weibull model is given bellow:

$$w_0 = a_0 - b_0$$

$$b_0 = a_0 - w_0 \qquad (17)$$

Where a_0 is the starting value of the parameter a, w_0 is the value of the response variable at time $t = 0$.

Starting value of $\boldsymbol{c}$: The parameter c is defined as the rate constant at which the response variable approaches its maximum possible value a. On the basis of this definition one can write, $c = \frac{(w_n - w_1)}{a_0(t_n - t_1)}$; where w_1 and w_n are the value of the response variable corresponding to the first (t_1) and the last observations (t_n) respectively. a_0 is the starting value specified for the parameter a.

Starting value of $\boldsymbol{m}$: The equation (4) can be written as

$$m = \frac{1}{\log t}\left\{\log\left(\frac{1}{c}\left(\log\frac{b}{a-w}\right)\right)\right\}.$$

Now at time $t = {}^{h}/_{2}$, where h is the last value of the independent variable in the data set.

$$m = \frac{1}{log(h/2)}\left\{log\left(\frac{1}{c_0}\left(log\frac{b_0}{a_0-w_{h/2}}\right)\right)\right\} \qquad (18)$$

From the equation (18), we can estimate the starting value for the parameter m. Here a_0, b_0, c_0 are the starting values for the respective parameters and $w_{h/2}$ is the value of the response variable at time $t = \frac{h}{2}$.

For finding the initial values of the parameters of three parameters and two parameters Weibull model, one can proceed similarly as the Weibull model with four parameters.

C. Parameter Estimations

Three Weibull growth models have been fitted to mean diameter at breast height, top height and average cherry height growth data. The parameters of these models are estimated using Newton-Raphson method of estimation. Parameter estimates for the Weibull models with the corresponding observed and predicted to mean diameter at breast height are presented in Table 3. The RMSE and χ^2 values along with the value of coefficient of determination (R^2) are also presented. It is observed that the RMSE values for Weibull models for Mean diameter at breast height are given by 0.165m, 0.888m and 1.313m. The table values of χ^2 for 99.5% level of significance is found to be higher than our calculated χ^2 values for Weibull model with four parameters whereas it is 99% higher in the case of Weibull model with three parameters and 90% for Weibull models with two parameters at $(n-1-k)$ degree of freedom, where n is the number of observation and k is the number of parameters to be estimated. The R^2 values of the Weibull models are 99.929%, 97.926% and 95.349% for four, three and two parameters respectively.

Table 3: Fitting of Weibull growth models for mean diameter at breast height.

Age(Year)	MDBH	Weibull model with four parameters	Weibull model with three parameters	Weibull model with two parameters
20	8.40	8.4188	6.5375	4.9531
25	10.40	10.2364	10.3155	9.1343
30	12.35	12.4403	13.3088	12.6640
35	14.74	14.8111	15.8310	15.6436
40	17.13	17.1983	18.0218	18.1589
45	19.50	19.4970	19.9605	20.2823
50	21.49	21.6368	21.6980	22.0748
55	23.82	23.5748	23.2699	23.5879
60	25.55	25.2897	24.7023	24.8652
65	26.50	26.7768	26.0151	24.8652
a		33.15763	49.02721	24.8652
b		25.74324		
c		0.03980	0.14311	0.1694
m		1.54465	0.72304	
χ^2		0.013	0.783	2.771
RMSE(m)		0.165	0.888	1.313
R^2(%)		99.929	97.926	95.349

Estimated parameters and the observed and predicted values for top height growth data along with the RMSE, χ^2 and R^2 values are presented in the Table 4. The table values of χ^2 for 99.5% level of significance is found to be higher than our calculated χ^2 values for both models and the RMSE values are given by 0.110m and 0.486m for Weibull models with four parameters and three parameters respectively. The Determination coefficient values of the Weibull model with four and three parameters are found to be 99.93% and 98.612% respectively. The Weibull model with two parameters cannot fit the top height growth data with Newton-Raphson method of estimation due to the occurrence of a singular or badly scaled matrix in the iteration process.

Average height growth data of Cherry trees are also used and the parameter estimates with the observed and predicted values are presented in table 5. The computed RMSE value for each models along with the χ^2 and R^2 values are also presented. It is observed that table values of χ^2 for 99.5% level of significance is found to be higher than our calculated χ^2 values for the Weibull models. The RMSE for both the models are also acceptable and are given by 0.133ft and 0.193ft for Weibull models with four parameters and three parameters respectively. The R^2 value of Weibull four and three parameters growth models fitted to average height growth of cherry trees are estimated as 99.911% and 99.812% respectively. For this data set also, the Weibull model of two parameters cannot fit with Newton-Raphson method of estimation due to the occurrence of singular matrix in the iteration process. The 95%

confidence intervals of each parameter of the Weibull models, a, b, c and m are calculated for the forestry data sets and presented in the Table 6.

Table 4: Fitting of Weibull growth models for top height growth data.

Age(Year)	Top height data	Weibull model with four parameters	Weibull model with three parameters
20	7.30	7.3574	6.3041
25	9.00	9.0215	9.1973
30	10.90	10.7636	11.3371
35	12.60	12.4661	13.0626
40	13.90	14.0689	14.5143
45	15.40	15.5408	15.7676
50	16.90	16.8680	16.8690
55	18.20	18.0474	17.8494
60	19.00	19.0831	18.7307
65	20.00	19.9833	19.5294
a		24.64759	33.57433
b		18.53032	
c		0.06927	0.20797
m		1.29917	0 .62228
χ^2		0.009	0.252
RMSE(m)		0.110	0.486
R^2		99.930	98.612

Table 5: Fitting of Weibull growth models for average height growth data of Cherry trees.

Age(Year)	Height(feet)	Weibull model with four parameters	Weibull model with three parameters
1	6.0	5.9429	5.5247
2	9.5	9.7502	9.7765
3	13.0	12.7259	12.8617
4	15.0	14.9420	15.0626
5	16.5	16.5465	16.6175
6	17.5	17.6853	17.7084
7	18.5	18.4812	18.4700
8	19.0	19.0305	18.9995
9	19.5	19.4057	19.3663
10	19.7	19.6597	19.6197
11	19.8	19.8302	19.7942
a		20.16013	20.17351
b		18.37222	
c		0.25638	0.32001
m		1.14778	1.05056
χ^2		0.016	0.055
RMSE(ft)		0.133	0.193
R^2		99.911	99.812

Table 6: 95% Confidence intervals of the parameters of Weibull growth models.

Data	Models	a		b		c		m	
		Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit
DBH	Wiebull_4	28.408	37.908	20.426	31.061	.029	.051	1.291	1.798
	Weibull_3	41.153	56.901			-.008	.294	.433	1.013
	Weibull_2	25.093	38.477			0.103	0.236		
Top Height	Wiebull_4	21.024	28.271	14.309	22.751	.052	.087	1.052	1.546
	Weibull_3	27.856	39.292			.027	.389	.423	.821
Average Height	Wiebull_4	19.723	20.597	16.381	20.364	.184	.329	.970	1.325
	Weibull_3	19.787	20.560			.296	.345	.998	1.103

IV. Discussion

The main Focus of this paper is to discuss some fundamental properties of the Weibull models and estimate the parameters using Newton-Raphson iteration method and construe some of the appropriate statistical outputs from forestry viewpoint. Good initial estimates are required to estimate the parameters from any iteration method. Also to specify good initial estimates of the parameters one should know the properties of the parameters. This paper developed some expressions to specify the initial values of the parameters based on the definitions and properties of the parameter of the Weibull models. These expressions will be very useful to specify the initial values of the parameters for Weibull models in the forestry data. From the result it is noticed that the Newton-Raphson algorithm is a very useful method in case of Weibull models. It is observed from the above results that the Weibull model with four parameter produce the best fit for all three forestry data sets. The Weibull model with three parameters also provides a satisfactory result for the data sets. But unfortunately the Weibull model with two parameters is failed to fit two data sets.

The Properties of the curve of the Weibull functions will help to select the Weibull models for appropriate field of forestry and use these models to predict and control a forestry system in a more mathematical manner.

V. Reference

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