$(\in, \in \vee q)$-Fuzzy ideals of d-algebra

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***Abstract*— Here we defined $(\in, \in \vee q)$-fuzzy ideal of d-algebra and investigated some of its properties.**

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I. Introduction

In 1991, Xi [1] applied the concept of fuzzy sets to BCK-algebras which were introduced by Imai and Iseki [2]. In 1996, J. Neggers and H. S. Kim [3] introduced the class of d-algebras which is a generalisation of BCK-algebras and investigated relation between d-algebras and BCK-algebras. M. Akram and K. H. Dar [4] introduced the concepts fuzzy d-algebra, fuzzy subalgebra and fuzzy d-ideals of d- algebras. S. K. Bhakat and P. Das [5] used the relation of "belongs to" and "quasi coincident with" between fuzzy point and fuzzy set to introduce the concept of $(\in, \in \vee q)$-fuzzy subgroup and $(\in, \in \vee q)$-fuzzy subring. D. K. Basnet and L. B. Singh [6] introduced $(\in, \in \vee q)$-fuzzy ideals of BG-algebra in 2010. In this paper, we define $(\in, \in \vee q)$-fuzzy ideals of d-algebra and got some interesting results.

II. PRELIMINARIES

Definition 2.1. A d-algebra is a non empty set X with a binary operation * satisfying following axioms.

(i) $x * x = 0$
(ii) $0 * x = 0$
(iii) $x * y = 0$ and $y * x = 0 \Rightarrow x = y \;\; \forall \;\; x, y \in X$

Definition 2.2. Let S be a non empty subset of a d-algebra X. Then S is called a subalgebra of X if $x * y \in S \;\forall\, x, y \in X$

Definition 2.3. A non empty subset I of a d-algebra X is called d-ideal of X if it satisfies the following conditions.

(i) $0 \in I$
(ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$
(iii) $x \in I$ and $y \in X \Rightarrow x * y \in I$
(iv)

Definition 2.4. A fuzzy subset μ of a d-algebra X is called fuzzy d-ideal of X if it satisfies following conditions.

(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$
(iii) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

Example 2.5. Consider a d-algebra X = {0, a, b, c} with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Define $\mu: X \to [0, 1]$ by $\mu(0) = 0.9$, $\mu(a) = 0.6$, $\mu(b) = 0.5$, $\mu(c) = 0.3$

Then it is easy to verify that μ is a fuzzy d-ideal X.

III. $(\in, \in \vee q)$-FUZZY IDEALS OF d-ALGEBRA

Definition 3.1.A fuzzy set μ of the form

$$\mu(y) = \begin{cases} t & \text{if } y = x,\ t \in (0,1] \\ 0 & \text{if } y \neq x \end{cases}$$

Is called a fuzzy point with support x and value t and it is denoted by x_t.

Definition 3.2. A fuzzy point x_t is said to belong to (respectively be quasi coincident with) a fuzzy set μ written as $x_t \in \mu$ (respectively $x_t\, q\, \mu$) if $\mu(x) \geq t$ (respectively $\mu(x) + t > 1$) if $x_t \in \mu$ or $x_t\, q\, \mu$ then we write $x_t \in\vee q\ \mu$

(Note $\overline{\in \vee q}$ means $\in \vee q$ does not hold)

Definition 3.3.A fuzzy subset μ of a d-algebra X is said to be $(\in, \in \vee q)$-fuzzy ideal of X if

(i) $(x * y)_t,\ y_s \in \mu \Rightarrow x_{m(t,s)} \in \vee q\ \mu$

(ii) $x_t,\ y_s \in \mu \Rightarrow (x * y)_{m(t,s)} \in \vee q\ \mu$ for all $x, y \in X$ where $m(t, s) = \min\{t, s\}$

Definition 3.4. A fuzzy subset μ of a d-algebra X is said to be (α, β)-fuzzy ideal of X if

(i) $(x * y)_t,\ y_s\ \alpha\ \mu \Rightarrow x_{m(t, s)}\ \beta\ \mu$

(ii) $x_t,\ y_s\ \alpha\ \mu \Rightarrow (x * y)_{m(t, s)}\ \beta\ \mu$ for all $x, y \in X$ where $m(t, s) = \min\{t, s\}$

and $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq\ \in \wedge q$

Theorem 3.5. A fuzzy subset μ of a d-algebra X is a fuzzy ideal iff μ is a $(\in, \in)$-fuzzy ideal of X.

Proof- Let μ be a fuzzy ideal of X. To prove that μ is $(\in, \in)$-fuzzy ideal.
Let $x, y \in X$, such that $(x * y)_t,\ y_s \in \mu$ where $t, s \in (0, 1)$, then $\mu(x * y) \geq t$, $\mu(y) \geq s$
Now $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \min\{t, s\} = m(t, s)$
$\Rightarrow x_{m(t,s)} \in \mu$
Again let $x_t \in \mu$, $y_s \in \mu$ then $\mu(x) \geq t$, $\mu(y) \geq s$
Now $\mu(x * y) \geq \min\{\mu(x), \mu(y)$
$\geq \min\{t, s\} = m(t, s)$
$\Rightarrow (x * y)_{m(t,s)} \in \mu$

$\therefore$ μ is $(\in, \in)$-fuzzy ideal of X.
Conversely,
let μ be a $(\in, \in)$-fuzzy ideal of X. To prove that μ is a fuzzy ideal of X.
Let $x, y \in X$ and $t = \mu(x * y)$, $s = \mu(y)$ then $\mu(x * y) \geq t$, $\mu(y) \geq s$
i.e. $(x * y)_t \in \mu$, $y_s \in \mu \Rightarrow x_{m(t,s)} \in \mu$ by given condition
i.e. $\mu(x) \geq m(t, s) \geq \min\{\mu(x * y), \mu(y)\}$
Again let $t = \mu(x)$, $s = \mu(y)$ then $\mu(x) \geq t$, $\mu(y) \geq s$
$\Rightarrow \quad x_t \in \mu$, $y_s \in \mu$
$\Rightarrow \quad (x * y)_{m(t,s)} \in \mu$ by given condition
$\Rightarrow \quad \mu(x * y) \geq m(t, s) = m\{\mu(x), \mu(y)\}$
Again taking $y = x$, we get
$\mu(x * x) \geq m\{\mu(x), \mu(x)\} = \mu(x)$
$\Rightarrow \mu(0) \geq \mu(x)$
Hence μ is a fuzzy ideal.

Theorem 3.6. If μ is a (q ,q)-fuzzy ideal of a d-algebra X, then it is also a $(\in, \in)$-fuzzy ideal of X.
Proof: Let μ be a (q ,q)-fuzzy ideal of a d-algebra X. Let $x, y \in X$ such that $(x * y)_t$, $y_s \in \mu$
$\Rightarrow \quad \mu(x * y) \geq t$ and $\mu(y) \geq s$
$\Rightarrow \quad \mu(x * y) + \delta > t$ and $\mu(y) + \delta > s$ for any $\delta > 0$.
$\Rightarrow \quad \mu(x * y) + \delta - t + 1 > 1$ and $\mu(y) + \delta - s + 1 > 1$
$\Rightarrow (x * y)_{\delta-t+1}\, q\, \mu$ and $(y)_{\delta-s+1}\, q\, \mu$ [since μ is a (q q)-fuzzy ideal X].
$\therefore$ We have $x_{m(\delta-t+1, \delta-s+1)}\, q\, \mu$
$\Rightarrow \quad \mu(x) + m(\delta - t + 1, \delta - s + 1) > 1$
$\Rightarrow \quad \mu(x) + \delta + 1 - M(t, s) > 1$, where $M(t, s) = \max\{t, s\}$
$\Rightarrow \quad \mu(x) > M(t, s) - \delta$
$\Rightarrow \quad \mu(x) \geq M(t, s)$, since δ is arbitrary
$\Rightarrow \quad \mu(x) \geq M(t, s) \geq m(t, s)$
$\Rightarrow \quad x_{m(t, s)} \in \mu$
Again let x_t, $y_s \in \mu$
$\Rightarrow \mu(x) \geq t$ and $\mu(y) \geq s$
$\Rightarrow \mu(x) + \delta > t$ and $\mu(y) + \delta > s$, where $\delta > 0$ is arbitrary.
$\Rightarrow \mu(x) + 1 + \delta - t > 1$ and $\mu(y) + 1 + \delta - s > 1$
$\Rightarrow x_{(1+\delta-t)}\, q\, \mu$ and $(y)_{(1+\delta-s)}\, q\, \mu$
$\Rightarrow (x * y)_{m(1+\delta-t, 1+\delta-s)} q\mu$ [Since μ is (q, q)-fuzzy ideal].
$\Rightarrow \quad \mu(x * y) + m(1 + \delta - t, 1 + \delta - s) > 1$
$\Rightarrow \quad \mu(x * y) + 1 + \delta - M(t, s) > 1$
$\Rightarrow \quad \mu(x * y) + \delta - M(t, s) > 0$
$\Rightarrow \quad \mu(x * y) > M(t, s) - \delta \geq M(t, s) \geq m(t, s)$, since $\delta > 0$ is arbitrary.
$\therefore (x * y)_{m(t,s)} \in \mu$
Hence μ is $(\in, \in)$-fuzzy ideal of X.

Remark 3.7. Converse of above is not true i.e. every $(\in, \in)$-fuzzy ideal is not a (q, q)-fuzzy ideal as shown in the following example.
Example 3.8. Consider d-algebra $X = \{0, a, b, c\}$ with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	c	0

Define $\mu: X \to [0, 1]$ by $\mu(0) = 0.7$, $\mu(a) = 0.6$, $\mu(b) = 0.4$, $\mu(c) = 0.3$ then it is east to verify that μ is a $(\in, \in)$-fuzzy d-ideal X. But it is not a (q, q)-fuzzy ideal of X because if $x = b$, $y = c$, $t = 0.35$, $s = 0.75$ then $(x * y)_t \, q \, \mu$, $y_s \, q \, \mu$ but $\mu(x) + m(t, s) = \mu(b) + m(0.35, 0.75) = 0.4 + 0.35 = 0.75 < 1$

Theorem 3.9. A fuzzy subset μ of a d-algebra X is a $(\in, \in \vee q)$-fuzzy ideal of X iff

$$\mu(x) \geq m(\mu(x * y), \mu(y), 0.5)$$
$$\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) \qquad \forall\, x, y \in X$$

Proof. First let μ be a $(\in, \in \vee q)$-fuzzy ideal of X

Case I. Let $m(\mu(x * y), \mu(y)) < 0.5 \quad \forall x, y \in X$ then

$m(\mu(x * y), \mu(y), 0.5) = m(\mu(x * y), \mu(y))$

If possible $\mu(x) < m(\mu(x * y), \mu(y))$. Choose a real number t such that

$\mu(x) < t < m(\mu(x * y), \mu(y))$, then $(x * y)_t, (y)_t \in \mu$

but $\mu(x) < t$ i.e. $x_t \notin \mu$.

Also $\mu(x) + t < 2t$

i.e. $\mu(x) + t < 2m(\mu(x * y), \mu(y)) < 2 \times 0.5 = 1$

$\Rightarrow \mu(x) + t < 1$

Which contradicts the fact that μ is a $(\in, \in \vee q)$-fuzzy ideal of X

$\mu(x) \geq m(\mu(x * y), \mu(y)) = m(\mu(x * y), \mu(y), 0.5)$

Case II. Let $m(\mu(x * y), \mu(y)) \geq 0.5$

Then $m(\mu(x * y), \mu(y), 0.5) = 0.5$. If possible $\mu(x) < m(\mu(x * y), \mu(y), 0.5) = 0.5$

Then $\mu(x * y) \geq 0.5$ and $\mu(y) \geq 0.5$

$\therefore$ $(x * y)_{0.5}, y_{0.5} \in \mu$ but $\mu(x) < 0.5$

$\therefore$ $x_{0.5} \notin \mu$ and $\mu(x) + 0.5 < 0.5 + 0.5 = 1$

Which is again a contradiction to the fact that μ is a $(\in, \in \vee q)$-fuzzy ideal of X.

Hence we must have $\mu(x) \geq m(\mu(x * y), \mu(y), 0.5)$

Case III. Let $m(\mu(x), \mu(y)) < 0.5 \ \forall x, y, \in X$, therefore $m(\mu(x), \mu(y), 0.5) = m(\mu(x), \mu(y))$

If possible $\mu(x * y) < m(\mu(x), \mu(y))$ choose a real number t such that

$\mu(x * y) < t < m(\mu(x), \mu(y))$ then $x_t \in \mu$, $y_t \in \mu$

But $\mu(x * y) < t \Rightarrow (x * y)_t \notin \mu$

Again $\mu(x * y) + t < 2t < 2m(\mu(x), \mu(y)) < 2 \times 0.5 = 1$

$\Rightarrow \mu(x * y) + t < 1$

Which contradict the fact that μ is $(\in, \in \vee q)$-fuzzy ideal

Case IV. Let $m(\mu(x), \mu(y)) \geq 0.5$ then $m(\mu(x), \mu(y), 0.5) = 0.5$

If possible $\mu(x * y) < m(\mu(x), \mu(y), 0.5) = 0.5$, then $\mu(x) \geq 0.5$, $\mu(y) \geq 0.5$

i.e. $x_{0.5} \in \mu$, $y_{0.5} \in \mu$. But $\mu(x * y) < 0.5$

i.e. $(x * y)_{0.5} \notin \mu$ and $\mu(x * y) + 0.5 < 0.5 + 0.5 = 1$, which is again a contradiction.

$\therefore \mu$ is $(\in, \in \vee q)$-fuzzy ideal.

Conversely,

let $\mu(x) \geq m(\mu(x * y), \mu(y), 0.5)$ (3.1)

Let $x, y \in X$, such that $\mu(x * y) = t$ and $\mu(y)$. Then $(x * y)_t, y_s \in \mu$

Hence $\mu(x) \geq m(t, s, 0.5)$ [By (3.1)]

Now if $m(t, s) \leq 0.5$ then $m(t, s, 0.5) = m(t, s)$ therefore, $\mu(x) \geq m(t, s)$ i.e. $x_{m(t,s)} \in \mu$ (3.2)

Again if $m(t, s) > 0.5$ then $m(t, s, 0.5) = 0.5$ $\therefore \mu(x) \geq m(t, s, 0.5) = 0.5$

i.e. $\mu(x) + m(t, s) > 0.5 + 0.5 = 1$

$\Rightarrow x_{m(t,s)} q\, \mu$ (3.3)

(3.2) and (3.3) $\Rightarrow x_{m(t,s)} \in \vee q\, \mu$. Hence μ is a $(\in, \in \vee q)$-fuzzy ideal.

Again Let $\mu(x * y) \geq m(\mu(x), \mu(y), 0.5)$ (3.4)

Let $x, y \in X$ so that $\mu(x) = t$ and $\mu(y)$. Then $x_t, y_s \in \mu$

Therefore $m(\mu(x), \mu(y)) \geq m(t, s)$

By (3.4), $\mu(x * y) \geq m(t, s, 0.5)$

Now, if $m(t, s) \leq 0.5$ then $\mu(x * y) \geq m(t, s, 0.5) \geq m(t, s)$

$\Rightarrow (x * y)_{m(t,s)} \in \mu$ (3.5)

Again if $m(t, s) \geq 0.5$ then $\mu(x * y) \geq m(t, s, 0.5) = 0.5$

$\Rightarrow \quad \mu(x * y) + m(t, s) > 0.5 + 0.5 = 1$

$\Rightarrow \quad (x * y)_{m(t,s)}\, q\mu$ (3.6)

From (3.5) and (3.6)

$(x * y)_{m(t,s)} \in \vee q\, \mu$. Therefore μ is $(\in, \in \vee q)$-fuzzy ideal of X.

Remark 3.10. A $(\in, \in)$-fuzzy ideal is always a $(\in, \in \vee q)$-fuzzy ideal of X but not conversely which can be seen from the following example.

Example 3.11. Consider d-algebra X = {0, a, b, c} with the following cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Define $\mu: X \rightarrow [0, 1]$ by $\mu(0) = 0.58$, $\mu(a) = \mu(b) = 0.75$, $\mu(c) = 0.55$

Then it is easy to verify that μ is $(\in, \in \vee q)$-fuzzy ideal by above theorem.

But $a_{0.7} = (c * b)_{0.7}$, $b_{0.7} \in \mu$ but $c_{0.7} \notin \mu$ also $a_{0.7}, b_{0.7} \in \mu$ but $(a * b)_{0.7} = 0_{0.7} \notin \mu$

Remark 3.12. Every $(\in, q)$-fuzzy ideal of d-algebra X is always a $(\in, \in \vee q)$-fuzzy ideal of X

Theorem 3.13. If a fuzzy subset μ of a d-algebra X is a $(\in, \in \vee q)$-fuzzy ideal of X and $\mu(x) < 0.5$ $\forall x \in X$, then μ is also a $(\in, \in)$-fuzzy ideal of X.

Proof. Let μ is $(\in, \in \vee q)$-fuzzy ideal of X and $\mu(x) < 0.5$ and $\mu(x * y) < 0.5$ $\forall x, y \in X$

Let $(x * y)_t \in \mu$, $y_s \in \mu$

Therefore $t \leq \mu(x * y) < 0.5$ and $s \leq \mu(x) < 0.5$

$\therefore m(t, s) < 0.5$ also $\mu(x) + m(t, s) < 0.5 + 0.5 = 1$

Since, μ is $(\in, \in \vee q)$-fuzzy ideal of X i.e. $\mu(x) \geq m(t, s)$ or $\mu(x) + m(t, s) > 1$

So, we must have $x_{m(t,s)} \in \mu$ (3.7)
Again, let $x_t \in \mu$, $y_s \in \mu$
$t \leq \mu(x) < 0.5$ and $s \leq \mu(y) < 0.5$
$\therefore m(t, s) < 0.5$ also $\mu(x*y) + m(t, s) < 0.5 + 0.5 = 1$
Since μ is $(\in, \in \vee q)$-fuzzy ideal of X so $\mu(x*y) \geq m(t, s)$ or $\mu(x*y) + m(t, s) > 1$
Therefore we must have $\mu(x*y) \geq m(t, s)$ i.e. $(x*y)_{m(t,s)} \in \mu$ (3.8)
(3.7) and (3.8) $\Rightarrow \mu$ is $(\in, \in)$-fuzzy ideal.

Theorem 3.14. A fuzzy set μ in a d-algebra X is an $(\in, \in \vee q)$-fuzzy ideals of X iff the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is an ideal of X for all $t \in (0, 0.5)$.

Proof: Assume μ is an $(\in, \in \vee q)$-fuzzy ideal of X and $t \in (0, 0.5]$.

Let $x, y \in X$ such that $x*y, y \in \mu_t$
Therefore $\mu(x*y) \geq t$, $\mu(y) \geq t$,
Now by theorem 3.9
$\mu(x) \geq m(\mu(x * y), \mu(y), 0.5) \geq m(t, t, 0.5) = t$
$\Rightarrow \mu(x) \geq t \Rightarrow x \in \mu_t$
Therefore, $x*y, y \in \mu_t \Rightarrow x \in \mu_t$ (3.9)
Again, let $x, y \in \mu_t$
Therefore $\mu(x) \geq t$, $\mu(y) \geq t$
Again by theorem 3.9
$\mu(x*y) \geq m(\mu(x), \mu(y), 0.5) \geq m(t, t, 0.5) = t$
$\Rightarrow \mu(x*y) \geq t \Rightarrow x*y \in \mu_t$
Therefore $x, y \in \mu_t \Rightarrow x*y \in \mu_t$ (3.10)
From (3.9) and (3.10) μ_t is an ideal of X.
Conversely,
Let μ be a fuzzy set in X and $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ be an ideal of X, to prove μ is $(\in, \in \vee q)$-fuzzy ideal X. Suppose μ is not an $(\in, \in \vee q)$-fuzzy ideal of X, then there exist $a, b \in X$ such that at least one of $\mu(a) < m(\mu(a * b), \mu(b), 0.5)$ and $\mu(a * b) < m(\mu(a), \mu(b), 0.5)$ hold.
Suppose $\mu(a) < m(\mu(a * b), \mu(b), 0.5)$ holds. Let $t = [\mu(a) + m(\mu(a * b), \mu(b), 0.5)]/2$, then $t \in (0, 0.5)$
and $\mu(a) < t < m(\mu(a * b), \mu(b), 0.5)$ (3.11)
i.e. $\mu(a * b) > t$, $\mu(b) > t$
$\Rightarrow a * b \in \mu_t$, $b \in \mu_t$
$\Rightarrow a \in \mu_t$ [since μ_t is ideal]
Therefore $\mu(a) > t$, which contradict (3.11)
Hence we must have $\mu(x) \geq m(\mu(x * y), \mu(y), 0.5)$
Next let $\mu(a * b) < m(\mu(a), \mu(b), 0.5)$ holds. Let $t = [\mu(a*b) + m(\mu(a), \mu(b), 0.5)]/2$, then $t \in (0, 0.5)$
And $\mu(a*b) < t < m(\mu(a), \mu(b), 0.5)$ (3.12)
i.e. $\mu(a) > t$, $\mu(b) > t$
$\Rightarrow a \in \mu_t$, $b \in \mu_t \Rightarrow a*b \in \mu_t$ [since μ_t is ideal]
Therefore $\mu(a*b) > t$, which contradict (3.12)
Therefore we must have $\mu(x*y) \geq m(\mu(x), \mu(y), 0.5)$
Hence μ is an $(\in, \in \vee q)$-fuzzy ideal of X.

Theorem 3.15. Let S be a subset of a d-algebra X. Consider the fuzzy set μ_s in X defined by

$$\mu_s = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise} \end{cases}$$

Then S is an ideal of X iff μ_s is an $(\in, \in \vee q)$-fuzzy ideal X.

Proof. Let S be an ideal of X. Now $(\mu_s)_t = \{ x \in X \mid \mu_s(x) \geq t\} = S$, which is an ideal.
Hence by theorem 3.14, μ_s is an $(\in, \in \vee q)$-fuzzy ideal X.

Conversely, assume that μ_s is an $(\in, \in \vee q)$-fuzzy ideal X, to prove S is an ideal of X.

Let $x*y, y \in S$. Then $\mu_s(x) \geq m(\mu_s(x * y), \mu_s(y), 0.5) = m(1, 1, 0.5)=0.5 \Rightarrow \mu_s(x) \geq 0.5 \Rightarrow \mu_s(x) =1 \Rightarrow x \in S$.
Next let $x, y \in S$. Then $\mu_s(x*y) \geq m(\mu_s(x), \mu_s(y), 0.5) = m(1, 1, 0.5) = 0.5 \Rightarrow \mu_s(x*y) \geq 0.5$
$\Rightarrow \mu_s(x*y) =1 \Rightarrow x*y \in S$. Hence S is an ideal of X.

Theorem 3.16. Let S be an ideal of X, then for every $t \in (0, 0.5]$ there exists $(\in, \in \vee q)$-fuzzy ideal μ of X such that $\mu_t = S$.

Proof. Let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} 1 \ if \ \ x \in S, \\ s \ \ Otherwise \end{cases}$$

Where $s < t \in (0, 0.5]$. Therefore $\mu_t = \{x \in X \mid \mu(x) \geq t\} = S$ and hence μ_t is an ideal. Now if μ is not an $(\in, \in \vee q)$-fuzzy ideal of X then there exist $a, b \in X$ such that at least one of $\mu(a) < m(\mu(a * b), \mu(b), 0.5)$ and $\mu(a * b) < m(\mu(a), \mu(b), 0.5)$ hold.

Suppose $\mu(a) < m(\mu(a * b), \mu(b), 0.5)$ holds. Then choose a real number $t \in (0, 1)$ such that

$$\mu(a) < t < m(\mu(a * b), \mu(b), 0.5) \tag{3.13}$$

i.e. $\mu(a * b) > t, \mu(b) > t$
$\Rightarrow a*b \in \mu_t, b \in \mu_t$
$\Rightarrow a \in \mu_t = S$ [since μ_t is ideal]
Therefore $\mu(a) =1 > t$, which contradicts (3.13)

Hence we must have $\mu(x) \geq m(\mu(x * y), \mu(y), 0.5)$

Again if $\mu(a * b) < m(\mu(a), \mu(b), 0.5)$ holds then choose a real number $t \in (0, 1)$

and $\mu(a*b) < t < m(\mu(a), \mu(b), 0.5)$ (3.14)

i.e. $\mu(a) > t, \mu(b) > t$
$\Rightarrow a \in \mu_t, b \in \mu_t \Rightarrow a*b \in \mu_t = S$ [since μ_t is ideal]
Therefore $\mu(a*b) =1 > t$, which contradicts (3.14)

Hence we must have $\mu(x*y) \geq m(\mu(x), \mu(y), 0.5)$

Thus, μ is an $(\in, \in \vee q)$-fuzzy ideal X.

Definition 3.17 Let μ be a fuzzy set in d-algebra X and $t \in (0,1]$, then let
$\mu_t = \{ x \in X \mid x_t \in \mu \} = \{ x \in X \mid \mu(x) \geq t\}$
$<\mu>_t = \{x \in X \mid x_t \, q \, \mu \} = \{ x \in X \mid \mu(x) + t > 1\}$
$[\mu]_t = \{ x \in X \mid x_t \in \vee q \, \mu \} = \{ x \in X \mid \mu(x) \geq t \text{ or } \mu(x) + t > 1 \}$
Where μ_t is called t level set of μ, $<\mu>_t$ is called q level set of μ and $[\mu]_t$ is called $\in \vee q$ level set of μ
Clearly, $[\mu]_t = <\mu>_t \cup \mu_t$

Theorem 3.18. Let μ be a fuzzy set in d-algebra X, then μ is an $(\in, \in \vee q)$-fuzzy ideal X iff $[\mu]_t$ is an ideal of X for all $t \in (0,1]$. We call $[\mu]_t$ as $\in \vee q$ level ideal of μ.

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy ideal X, to prove $[\mu]_t$ is an ideal of X.

Let $x, y \in [\mu]_t$ for $t \in (0,1]$
Then $x_t \in \vee q \, \mu$ and $y_t \in \vee q \, \mu$
i.e. $\mu(x) \geq t$ or $\mu(x) + t > 1$ and $\mu(y) \geq t$ or $\mu(y) + t > 1$
Since, μ is an $(\in, \in \vee q)$-fuzzy ideal X

$\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) \quad \forall x, y \in X$

Now we have the following cases.

Case I : $\mu(x) \geq t$, $\mu(y) \geq t$, let $t > 0.5$

Then $\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) \geq m(t, t, 0.5) = 0.5$

$\Rightarrow \quad \mu(x * y) \geq 0.5 \Rightarrow \mu(x * y) + t > 0.5+0.5=1 \Rightarrow (x * y)_t\, q\, \mu$

Again if $t \leq 0.5$, then $\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) \geq m(t, t, 0.5) = t$

$\Rightarrow \quad \mu(x * y) \geq t \Rightarrow (x * y)_t \in \mu$

Hence $(x * y)_t \in \vee q\, \mu$

$\Rightarrow \; x * y \in [\mu]_t$

Case II : $\mu(x) \geq t$, $\mu(y) + t > 1$, let $t > 0.5$

Then $\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) > m(t, 1-t, 0.5) = 1-t$

$\Rightarrow \quad \mu(x * y) > 1-t \Rightarrow \mu(x * y) + t > 1 \Rightarrow (x * y)_t\, q\, \mu$

Again if $t \leq 0.5$, then $\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) \geq m(t, 1-t, 0.5) = t$

$\Rightarrow \quad \mu(x * y) \geq t \Rightarrow (x * y)_t \in \mu$

Hence $(x * y)_t \in \vee q\, \mu$ i.e. $x * y \in [\mu]_t$

Case III : $\mu(x) + t > 1$, $\mu(y) \geq t$

This is similar to case II

Case IV : $\mu(x) + t > 1$, $\mu(y) + t > 1$, let $t > 0.5$

Then $\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) > m(1-t, 1-t, 0.5) = 1-t$

$\Rightarrow \quad \mu(x * y) > 1-t \Rightarrow \mu(x * y) + t > 1 \Rightarrow (x * y)_t\, q\, \mu$

Again if $t \leq 0.5$, then $\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) \geq m(1-t, 1-t, 0.5) = 0.5 \geq t$

$\Rightarrow \quad \mu(x * y) \geq t \Rightarrow (x * y)_t \in \mu$

Hence $(x * y)_t \in \vee q\, \mu \Rightarrow \; x * y \in [\mu]_t$

Hence from above four cases $x, y \in [\mu]_t \Rightarrow \; x * y \in [\mu]_t$

Again let $x*y, y \in [\mu]_t$ for $t \in (0,1]$,

Then $(x * y)_t \in \vee q\, \mu$ and $y_t \in \vee q\, \mu$

i.e $\mu(x *y) \geq t$ or $\mu(x *y) + t > 1$ and $\mu(y) \geq t$ or $\mu(y) + t > 1$

Since μ is an $(\in, \in \vee q)$-fuzzy ideal X

$\mu(x) \geq m(\mu(x), \mu(y), 0.5) \qquad \forall x, y \in X$

Case I : $\mu(x *y) \geq t$, $\mu(y) \geq t$, let $t > 0.5$

Then $\mu(x) \geq m(\mu(x *y), \mu(y), 0.5) \geq m(t, t, 0.5) = 0.5$

$\Rightarrow \quad \mu(x) \geq 0.5 \Rightarrow \mu(x) + t > 0.5+0.5=1 \Rightarrow x_t\, q\, \mu$

Again if $t \leq 0.5$, then $\mu(x) \geq m(\mu(x *y), \mu(y), 0.5) \geq m(t, t, 0.5) = t$

$\Rightarrow \quad \mu(x) \geq t \Rightarrow x_t \in \mu$

Hence $x_t \in \vee q\, \mu \Rightarrow x \in [\mu]_t$

Case II : $\mu(x *y) \geq t$, $\mu(y) + t > 1$, let $t > 0.5$

Then $\mu(x) \geq m(\mu(x *y), \mu(y), 0.5) > m(t, 1-t, 0.5) = 1-t$

$\Rightarrow \quad \mu(x) > 1-t \Rightarrow \mu(x) + t > 1 \Rightarrow x_t\, q\, \mu$

Again if $t \leq 0.5$, then $\mu(x) \geq m(\mu(x *y), \mu(y), 0.5) \geq m(t, 1-t, 0.5) = t$

$\Rightarrow \quad \mu(x) \geq t \Rightarrow x_t \in \mu$

Hence $x_t \in \vee q\, \mu \Rightarrow x \in [\mu]_t$

Case III : $\mu(x) + t > 1$, $\mu(y) \geq t$

This is similar to case II

Case IV : $\mu(x *y) + t > 1$, $\mu(y) + t > 1$, let $t > 0.5$

Then $\mu(x) \geq m(\mu(x *y), \mu(y), 0.5) > m(1-t, 1-t, 0.5) = 1-t$

$\Rightarrow \quad \mu(x) > 1-t \Rightarrow \mu(x) + t > 1 \Rightarrow x_t\, q\, \mu$

Again if $t \leq 0.5$, then $\mu(x) \geq m(\mu(x *y), \mu(y), 0.5) \geq m(1-t, 1-t, 0.5) = 0.5 \geq t$

$\Rightarrow \quad \mu(x) \geq t \Rightarrow x_t \in \mu$

Hence $x_t \in \vee q\ \mu \Rightarrow x \in [\mu]_t$

Hence from above four cases $x * y,\ y \in [\mu]_t \Rightarrow x \in [\mu]_t$

Hence $[\mu]_t$ is an ideal of X.

Conversely

Let μ be a fuzzy set in X and $t \in (0, 1]$ be such that $[\mu]_t$ is an ideal of X, to prove μ is an $(\in, \in \vee q)$-fuzzy ideal X. If μ is not an $(\in, \in \vee q)$-fuzzy ideal X, then there exist a, b $\in$ X such that at least one of $\mu(a) < m(\mu(a * b), \mu(b), 0.5)$ and $\mu(a * b) < m(\mu(a), \mu(b), 0.5)$ must hold.

Suppose $\mu(a) < m(\mu(a * b), \mu(b), 0.5)$ is true, then choose $t \in (0, 1]$ such that

$\mu(a) < t < m(\mu(a * b), \mu(b), 0.5)$ (3.15)

Then $\mu(a * b) \geq t$, $\mu(b) \geq t \Rightarrow a * b, b \in \mu_t \subset [\mu]_t$ which is an ideal

Therefore $a \in [\mu]_t \Rightarrow \mu(a) \geq t$ or $\mu(a) + t > 1$ which contradict (3.15)

Again if $\mu(a * b) < m(\mu(a), \mu(b), 0.5)$ is true, then choose $t \in (0, 1]$ such that

$\mu(a*b) < t < m(\mu(a), \mu(b), 0.5)$ (3.16)

Then $\mu(a) \geq t$, $\mu(b) \geq t \Rightarrow a, b \in \mu_t \subset [\mu]_t$ which is an ideal

therefore $a*b \in [\mu]_t \Rightarrow \mu(a * b) \geq t$ or $\mu(a*b) + t > 1$ which contradict (3.16)

hence we must have

$\mu(x) \geq m(\mu(x * y), \mu(y), 0.5)$

$\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) \quad \forall x, y \in X$

Hence μ is an $(\in, \in \vee q)$-fuzzy ideal X

Theorem 3.19. Every $(\in \vee q, \in \vee q)$-fuzzy ideal is an $(\in, \in \vee q)$-fuzzy ideal.

Proof. It follows from definition.

Theorem 3.20. Let μ_1 and μ_2 be two $(\in, \in \vee q)$-fuzzy ideals of a d-algebra X. Then $\mu_1 \cap \mu_2$ is also a $(\in, \in \vee q)$-fuzzy ideal of X.

Proof. Let x, y $\in$ X. Now

We have $(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\} = m\{\mu_1(x), \mu_2(x)\}$

Again $(\mu_1 \cap \mu_2)(x) = m\{\mu_1(x), \mu_2(x)\}$

$\geq m\{m\{\mu_1(x * y), \mu_1(y), 0.5\}, m\{\mu_2(x * y), \mu_2(y), 0.5\}\}$ [μ_1 is $(\in, \in \vee q)$-fuzzy ideal]

$= m\{m\{\mu_1(x * y), \mu_2(x * y)\}, m\{\mu_1(y), \mu_2(y)\}, 0.5\}$

$= m\{(\mu_1 \cap \mu_2)(x * y), (\mu_1 \cap \mu_2)(y), 0.5\}$ (3.17)

And $(\mu_1 \cap \mu_2)(x*y) = m\{\mu_1(x*y), \mu_2(x*y)\}$

$\geq m\{m\{\mu_1(x), \mu_1(y), 0.5\}, m\{\mu_2(x), \mu_2(y), 0.5\}\}$ [μ_2 is $(\in, \in \vee q)$-fuzzy ideal]

$= m\{m\{\mu_1(x), \mu_2(x)\}, m\{\mu_1(y), \mu_2(y)\}, 0.5\}$

$= m\{(\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y), 0.5\}$ (3.18)

(3.17) and (3.18) $\Rightarrow \mu_1 \cap \mu_2$ is $(\in, \in \vee q)$-fuzzy ideal of X.

The above theorem can be generalised as

Theorem 3.21. Let $\{\mu_i \mid i = 1, 2, 3, \ldots\}$ be a family of $(\in, \in \vee q)$-fuzzy ideals of a d-algebra X, then $\bigcap^n_{i=1} \mu_i$ is also a $(\in, \in \vee q)$-fuzzy ideal of X, where $\bigcap \mu_i = \min\{\mu_i(x) : i = 1, 2, \ldots\}$.

IV. CARTESIAN PRODUCT OF d-ALGEBRAS AND THEIR IDEALS

Theorem 4.1. Let X, Y be two d-algebras, then their Cartesian product $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is also a d-algebra under the binary operation * defined in $X \times Y$ by $(x, y)*(p, q) = (x*p, y*q)$ for all $(x, y), (p, q) \in X \times Y$.

Proof. Clearly $0 \in X$, $0 \in Y$ therefore $(0, 0) \in X \times Y$

Let $(x, y), (p, q) \in X \times Y$

Now

(i) $(x, y)*(x, y)=(x*x, y*y)=(0, 0) \in X \times Y$

(ii) $(0, 0) *(x, y)=(0*x,0*y)=(0, 0) \in X \times Y$

(iii) $(x, y)*(p, q)=(0, 0)$ and $(p, q)*(x, y)=(0, 0)$

$\Rightarrow (x* p, y*q)=(0, 0)$ and $(p*x, q*y)=(0, 0)$

$\Rightarrow x*p =0\ y*q=0$ and $p*x =0\ q*y=0$

$\Rightarrow x*p =0$ and $p*x =0$, $y*q=0$ and $q*y=0$

$\Rightarrow x = p , y = q$

$(x, y) = (p, q)$

Which shows that $(X \times Y, (0, 0),*)$ is a d-algebra.

Definition 4.2. Let μ_1 and μ_2 be two $(\in, \in \vee q)$-fuzzy ideals of a d-algebra X. Then their Cartesian product $\mu_1 \times \mu_2$ is defined by $(\mu_1 \times \mu_2)(x, y) = \min\{\mu_1(x), \mu_2(y)\}$ where $(\mu_1 \times \mu_2): X \times X \rightarrow [0\ 1]$ $\forall x, y \in X$.

Theorem 4.3. Let μ_1 and μ_2 be two $(\in, \in \vee q)$-fuzzy ideals of d-algebra X. Then $\mu_1 \times \mu_2$ is also a $(\in, \in \vee q)$-fuzzy ideal of $X \times X$.

Proof. Similar to theorem 3.14.

V. HOMOMORPHISM OF d-ALGEBRAS AND FUZZY IDEALS

Definition 5.1. Let X and $X^{/}$ be two d-algebras, then a mapping $f: X \rightarrow X^{/}$ is said to be homomorphism if $f(x * y) = f(x) * f(y)$ $\forall x, y \in X$.

Theorem 5.2. Let X and $X^{/}$ be two d-algebras and $f : X \rightarrow X^{/}$ be homomorphism. If μ be a $(\in, \in \vee q)$-fuzzy ideal of $X^{/}$, then $f^{-1}(\mu)$ is $(\in, \in \vee q)$-fuzzy ideal of X.

Proof. $f^{-1}(\mu)$ is defined as $f^{-1}(\mu)(x) = \mu(f(x))\ \forall x \in X$. Let μ be a $(\in, \in \vee q)$-fuzzy ideal of $X^{/}$

Let $x, y \in X$ such that $(x * y)_t, y_s \in f^{-1}(\mu)$ then $f^{-1}(\mu)(x * y) \geq t$ and $f^{-1}(\mu)(y) \geq s$

$\mu(f(x * y)) \geq t$ and $\mu(f(y)) \geq s$

$\Rightarrow$ $(f(x * y))_t \in \mu$ and $f(y)_s \in \mu$

$\Rightarrow$ $(f(x) * f(y))_t \in \mu$ and $f(y)_s \in \mu$ [Since f is a homomorphism]

$\Rightarrow$ $((f(x))_{m(t,s)} \in \vee q\ \mu$ [Since μ is a $(\in, \in \vee q)$- fuzzy ideal of $X^{/}$]

$\Rightarrow$ $((f(x))_{m(t,s)} \in \mu$ or $\mu(f(x)) + m(t, s) > 1$

$\Rightarrow$ $\mu(f(x)) \geq m(t, s)$ or $\mu(f(x)) + m(t, s) > 1$

$\Rightarrow$ $f^{-1}(\mu)(x) \geq m(t, s)$ or $f^{-1}(\mu)(x) + m(t, s) > 1$

$\Rightarrow$ $x_{m(t,s)} \in f^{-1}(\mu)$ or $x_{m(t,s)}\ q f^{-1}(\mu)$

$\Rightarrow$ $x_{m(t,s)} \in \vee q\ f^{-1}(\mu)$ (5.1)

Again, let $x, y \in X$ such that $x_t, y_s \in f^{-1}(\mu)$ then $f^{-1}(\mu)(x) \geq t$, $f^{-1}(\mu)(y) \geq s$

$\Rightarrow$ $\mu(f(x)) \geq t$ and $\mu(f(y)) \geq s$

$\Rightarrow$ $f(x)_t \in \mu$ and $f(y)_s \in \mu$

$\Rightarrow$ $[f(x) * f(y)]_{m(t,s)} \in \mu$ or $[f(x) * f(y)]_{m(t,s)}\ q\ \mu$ [Since μ be a $(\in, \in \vee q)$- fuzzy ideal of $X^{/}$]

i.e. $\mu[f(x) * f(y)] \geq m(t, s)$ or $\mu[f(x) * f(y)] + m(t, s) > 1$

$\Rightarrow$ $\mu[f(x * y)] \geq m(t, s)$ or $\mu[f(x * y)] + m(t, s) > 1$

$\Rightarrow$ $f^{-1}(\mu)(x * y) \geq m(t, s)$ or $f^{-1}(\mu)(x * y) + m(t, s) > 1$

$\Rightarrow$ $(x * y)_{m(t,s)} \in f^{-1}(\mu)$ or $(x * y)_{m(t,s)}\ q\ f^{-1}(\mu)$

$\Rightarrow$ $(x * y)_{m(t,s)} \in \vee q\ f^{-1}(\mu)$ (5.2)

(5.1) and (5.2) $\Rightarrow$ $f^{-1}(\mu)$ is a $(\in, \in \vee q)$-fuzzy ideal of X.

Theorem 5.3. Let X and $X^{/}$ be two d- algebras and $f : X \to X^{/}$ be an onto homomorphism. If μ be a fuzzy subset of $X^{/}$ such that $f^{-1}(\mu)$ is a $(\in, \in \vee q)$-fuzzy ideal of X then μ is also $(\in, \in \vee q)$-fuzzy ideal of $X^{/}$

Proof. Let $x^{/}, y^{/} \in X^{/}$ such that $(x^{/}*y^{/})_t, y^{/}_s \in \mu$ where $t, s \in [0, 1]$ then $\mu(x^{/}*y^{/}) \geq t$ and $\mu(y^{/}) \geq s$. Since f is on to so there exists $x, y \in X$ such that $f(x) = x^{/}$, $f(y) = y^{/}$ also f is homomorphism so

$f(x * y) = f(x) * f(y) = x^{/} * y^{/}$

So, $\mu(f(x * y)) \geq t$ and $\mu((f(y)) \geq s$

$\Rightarrow$ $f^{-1}(\mu)(x * y) \geq t$ and $f^{-1}(\mu)(y) \geq s$

$\Rightarrow$ $f^{-1}(\mu)(x) \geq m(t, s)$ or $f^{-1}(\mu)(x) + m(t, s) > 1$ [Since $f^{-1}(\mu)$ is a $(\in, \in \vee q)$ -fuzzy ideal of X]

$\Rightarrow$ $\mu(f(x)) \geq m(t, s)$ or $\mu(f(x)) + m(t, s) > 1$

$\Rightarrow$ $\mu(x^{/}) \geq m(t, s)$ or $\mu(x^{/}) + m(t, s) > 1$

$\Rightarrow$ $x^{/}_{m(t,s)} \in \mu$ or $x^{/}_{m(t,s)}\, q\, \mu$

$\Rightarrow$ $x^{/}_{m(t,s)} \in \vee q\, \mu$ (5.3)

Again, let $x^{/}, y^{/} \in X$ such that $x^{/}_t, y^{/}_s \in \mu$

$\therefore \mu(x^{/}) \geq t$ and $\mu(y^{/}) \geq s$

$\Rightarrow$ $\mu(f(x)) \geq t$ and $\mu(f(x)) \geq s$

$\Rightarrow$ $f^{-1}(\mu)(x) \geq t$ and $f^{-1}(\mu)(y) \geq s$

$\Rightarrow$ $x_t \in f^{-1}(\mu)$ and $y_s \in f^{-1}(\mu)$

$\Rightarrow$ $(x * y)_{m(t, s)} \in \vee q\, f^{-1}(\mu)$ [Since $f^{-1}(\mu)$ is a $(\in, \in \vee q)$ -fuzzy ideal of X]

$\Rightarrow$ $(x * y)_{m(t, s)} \in f^{-1}(\mu)$ or $(x * y)_{m(t, s)}\, q\, f^{-1}(\mu)$

$\Rightarrow$ $f^{-1}(\mu)(x * y) \geq m(t, s)$ or $f^{-1}(\mu)(x * y) + m(t, s) > 1$

$\Rightarrow$ $\mu(f(x * y)) \geq m(t, s)$ or $\mu(f(x * y)) + m(t, s) > 1$

$\Rightarrow$ $\mu(f(x) * f(y)) \geq m(t, s)$ or $\mu(f(x) * f(y)) + m(t, s) > 1$

$\Rightarrow$ $\mu(x^{/} * y^{/}) \geq m(t, s)$ or $\mu(x^{/} * y^{/}) + m(t, s) > 1$

$\Rightarrow$ $(x^{/} * y^{/})_{m(t,s)} \in \mu$ or $((x^{/} * y^{/})_{m(t,s)} \in q\mu$

$\Rightarrow$ $(x^{/} * y^{/})_{m(t,s)} \in \vee q\mu$ (5.4)

(5.3) and (5.4) $\Rightarrow$ μ is a $(\in, \in \vee q)$-fuzzy ideal of $X^{/}$

VI. CONCLUSIONS

In this paper, we have introduced the concept of $(\in, \in \vee q)$-fuzzy ideals of d- algebra and investigated some of their useful properties. In my opinion, these definitions and results can be extended to other algebraic systems also. In the notions of (α, β)-fuzzy ideals we can define twelve different types of ideals by three choices of α and four choices of β. In the present paper, we have mainly discussed $(\in, \in \vee q)$-type fuzzy ideal. In future, the following studies may be carried out : (1) $(\in, \in \vee q)$-intuitionistic fuzzy ideals of d-algebra (2) $(\in, \in \vee q)$-doubt fuzzy ideals of d-algebra.

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