On Some Generalized Well Known Results of Fixed Point Theorems of T- Contraction Mappings in Cone Metric Spaces.

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***Abstract*: In this paper, we obtain sufficient conditions for the existence of a common fixed point of T- Contraction mapping in the setting on complete cone metric spaces. Our results generalized well known recent result of Garg and Ansari [2].**

Key word: Common fixed point, cone metric space, complete cone metric spaces, convergent sequence.

I. INTRODUCTION

In 2007, Huang and Zhang [1] generalized the concept of metric space, replacing the set of real numbers by an ordered Banach space, and defined a cone metric space. These authors also described the convergence of sequences in the cone metric spaces and introduced the completeness. Also, they have proved following theorem:

Theorem 1.1 Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq Kd(x, y)$ for all $x, y \in X$,where $k \in [0,1]$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to a fixed point.

Theorem 1.2 Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$,and $x \neq y$. Then T has a unique fixed point in X.

Theorem 1.3 Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T: X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq K(d(Tx, y) + d(Ty, x))$ for all $x, y \in X$,where $k \in [0,1]$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to a fixed point.

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The above result of [1] were generalized by Sh. Rezapour and Hamlbarani in [3] omitting the assumption of normality of the cone. Subsequently, many authors have generalized the results of [1] and have studied fixed point theorems for normal and non-normal cone. Recently, A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh [4] introduced a new class of contractive mappings: T – Contraction and T – Contractive functions, extending the Banach Contraction Principle and the Edelstein's fixed point theorem [5], respectively. S. Moradi [6] introduced the notion a T – Kannan contractive mapping which extend the well known Kannan's fixed point theorem [7] and Morales and Rojes in (2009) gave the following theorems:

Theorems 1.3 Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T, S : X \to X$ be one to one continuous function and TK_1 –contraction (T-Kannan Contraction). Then for all $x, y \in X$.

[i] For every $x_0 \in X$

$$\lim_{n\to\infty} d(TS^n x_{0,} TS^{n+1} x_0 = 0$$

[ii] There is $v \in X$ $\quad \lim_{n\to\infty} d(TS^n x_{0,} TS^{n+1} x_0 = v$

[iii] If T is sub sequentially convergent, then $(S^n x_0)$ has a convergent subsequence.

[iv]There is a unique $u \in X$ Such that $su = u$.

[v] If T is sequentially convergent, then for each $x_0 \in X$, iterates sequence $\{S^n x_0\}$ convergent to u.

Theorems 1.3 Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose the mapping $T, S : X \to X$ be one to one continuous function and TK_2 –contraction (T-Chatterjee Contraction). Then for all $x, y \in X$.

[i] For every $x_0 \in X$

$$\lim_{n\to\infty} d(TS^n x_{0,} TS^{n+1} x_0 = 0$$

[ii] There is $v \in X$ $\lim_{n\to\infty} d(TS^n x_{0,} TS^{n+1} x_0 = v$

[iii] If T is sub sequentially convergent, then $(S^n x_0)$ has a convergent subsequence.

[iv]There is a unique $u \in X$ Such that $su = u$.

[v] If T is sequentially convergent, then for each $x_0 \in X$, iterates sequence $\{S^n x_0\}$ convergent to u.

View of these facts, thereby the purpose of this paper is to study the existence on some generalized known results of fixed point theorem **of** T – contractive mapping defined complete cone metric space (X, d) of results Garg and Ansari [2].

II. PRELUMINARIES

First, we recall some standard notations and definitions in cone metric spaces with some of their properties [1].

Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:

(i) P is closed, non – empty and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and non – negative real number a, b,

(iii) $x \in P$ and $- x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll$ y if $y - x \in$ into P, int P denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all, $y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq \| y \|$.

The least positive number satisfying the above is called the normal constant P.

Definition 2.1 [1] Let X be a non – empty set. Suppose the mapping $d: X \times X \to E$ satisfies

(d_1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x) = 0$ iff $x = y$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all, $x, y \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space [1]. It is obvious that cone metric spaces generalize metric space.

Example 2.2 [1] Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $d : X \times X \rightarrow E$ defined by $d(x, y) = (|x - y|, \alpha |x, y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.3 [1] Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X. Then,

(i) $\{x_n\}_{\geq 1}$ converges to x whenever for every $c \in E$ with $o \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \rightarrow x, (n \rightarrow \infty)$

(ii) $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $o \ll c$, there is a

natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(ii) (X, d) is called a complete cone metric space if every Cauchy sequence in X is

converge.

Lemma 2.4 [1] Let (X, d) be a cone metric space, $P \subset E$ a normal cone with normal constant K. Let $\{x_n\}$, $\{y_n\}$ be a sequence in X and $x, y \in X$. Then,

(i) $\{x_n\}$ **converges** to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$;

(ii) **If** $\{x_n\}$ converges to $x$and $\{x_n\}$ converges to y then $x = y$. That is the limit of $\{x_n\}$ is unique;

(iii) If $\{x_n\}$ converges to x, then $\{x_n\}$ is Cauchy sequence.

(iv) $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n, m\to\infty} d(x_n, x_m) = 0$;

(v) If $x_n \rightarrow x$ and $\{y_n\}$ is another sequence in X such that $y_n \rightarrow y$ $(n \rightarrow \infty)$ then $d(x_n, y_n) \rightarrow d(x, y)$

Definition 2.5 Let (X, d) be a cone metric space. If for any sequence $\{x_n\}$ in X, there is a sub sequence $\{x_{ni}\}$ of $\{x_n\}$ such that $\{x_n\}$ is convergent in X then X is called a sequentially compact cone metric space.

Definition 2.6[2] Let (X, d) be a cone metric space, P be a normal cone with normal constant K and $T : X \rightarrow X$ then

(i) T is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$ for all $\{x_n\}$ in X;

(ii) T is said to be subsequentialy convergent, if for every sequence $\{y_n\}$ that $\{Ty_n\}$ is convergent, implies $\{y_n\}$ has a convergent subsequence.

(iii) T is said to be sequentially convergent if for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent, then $\{y_n\}$ is also convergent.

III. Main Result

The results which we will give are generalization of theorem6.1 of [2].

In 2003, V. Berinde (see [9] and [10]) introduced a new class of contraction mappings on metric spaces, which are called weak contractions. In sequel, J.R. Moral and E. Rojas [8] extended these kind of mappings by introducing a new function T and are called T-weak contraction. In 2014, Arun, G. and Z. K. Ansari [2] introduce some new definition on cone metric spaces based on the idea of Moradi [6] and are called T-Hardy- Rogers contraction(T-HR_1contraction).

Definition 3.1(T-Hardy-Rogers contraction [2]) Let (X, d) be a cone metric spaces and $T, S: X \to X$ two functions. Then S is called a T- Hardy and Rogers contraction (T-HR_1contraction**)** **,** if there exist a constant $b \in [0,1]$ such that

$$d(TSx, TSy) \leq a_1\, d(Tx, Ty) + a_2 d(Tx, TSx) + a_3 d(Tt, TSy) + a_4 d(Tx, TSy) + a_5 d(Ty, TS),$$

for all $x, y \in X$.

Theorems 3.2 Let (X, d) be a cone metric space, P is a normal cone with normal constant K. in addition $T : X \to X$ be one to one continuous function and the mapping $R, S : X \to X$ be pair of T- Hardy-Rogers contraction (T-HR_1contraction**)**,by definition[3.1], if there is a constant b = $a_1 + a_2 + a_3 + a_4 + a_5 \in [0,1]$ such that

$$d(TRx, TSy) \leq a_1\, d(Tx, Ty) + a_2 d(Tx, TRx) + a_3 d(Ty, TSy) + a_4 d(Tx, TSy) + a_5 d(Ty, TRx),$$

for all $x, y \in X$.Then

[i] For every $x_0 \in X$

$$\lim_{n\to\infty} d(TR^{n+1}x_{0,} TR^{n+2}x_0 = 0 \text{ and}$$

$$\lim_{n\to\infty} d(TS^{n+2}x_{0,} TS^{n+3}x_0 = 0;$$

[ii] There is $v \in X$ such that $\lim_{n\to\infty} TR^{n+1}x_{0,} = v = \lim_{n\to\infty} TS^{n+2}x_{0,}$;

[iii] If T is sub sequentially convergent, then $(R^{n+1}x_0)$ and$(S^{n+2}x_0)$ have a convergent

subsequence.

[iv] There is a unique $u \in X$ Such that $Ru = u = Su$.

[v] If T is sequentially convergent, then for each $x_0 \in X$, iterates sequence $(R^{n+1}x_0)$ and $(S^{n+2}x_0)$ convergent to u.

Proof: Suppose $x_0 \in X$ is an arbitrary point and the Picard iteration associated to R,(x_{n+1}) given by

$$x_{n+2} = Rx_{n+1} = R^{n+1}x_0,\ n = 0, 1, 2 \text{-----------------}$$

Similarly, associated to S, (x_{n+3}) given by

$$x_{n+3} = Rx_{n+2} = R^{n+2}x_0,\ n = 0, 1, 2 \text{-----------------}$$

Since R and S are pair of $T - HR_1$ contractions, we have

$$\begin{aligned} d(Tx_{n+1,}Tx_{n+2}) &= d(TRx_{n,}\, TRx_{n+1}) \\ &\leq a_1 d(Tx_{n,}\, Tx_{n+1}) + a_2 d(Tx_n, TRx_n) \\ &+ a_3 d(Tx_{n+1}, TRx_{n+1}) + a_4 d(Tx_n, TRx_{n+1}) + a_5 d(Tx_{n+1}, TRx_n) \\ &\leq a_1 d(Tx_{n,}\, Tx_{n+1}) + a_2 d(Tx_n, Tx_{n+1}) \\ &+ a_3 d(Tx_{n+1}, Tx_{n+2}) + a_4 d(Tx_n, Tx_{n+2}) + a_5 d(Tx_{n+1}, Tx_{n+1}) \\ &\leq (a_1 + a_2 + a_4) d(Tx_{n,}\, Tx_{n+1}) + (a_3 + a_4)\, d(Tx_{n+1}, Tx_{n+2}) \end{aligned}$$

$$1 - (a_3 + a_4)\, d(Tx_{n+1}, Tx_{n+2}) \leq (a_1 + a_2 + a_4) d(Tx_{n,}\, Tx_{n+1})$$

Similarly

$$\begin{aligned} d(Tx_{n+2,}Tx_{n+3}) &= d(TSx_{n+1,}\, TSx_{n+2}) \\ &\leq a'_1 d(Tx_{n+1,}\, Tx_{n+2}) + a'_2 d(Tx_{n+1}, TSx_{n+1}) \\ &+ a'_3 d(Tx_{n+2}, TSx_{n+2}) + a'_4 d(Tx_{n+1}, TSx_{n+2}) \\ &+ a'_5 d(Tx_{n+2}, TSx_{n+1}) \end{aligned}$$

$$\leq a'_1 d(Tx_{n+1,} Tx_{n+2}) + a'_2 d(Tx_{n+1}, Tx_{n+2})$$

$$+ a'_3 d(Tx_{n+1}, Tx_{n+3}) + a'_4 d(Tx_{n+1}, Tx_{n+2})$$

$$+ a'_5 d(Tx_{n+2}, Tx_{n+2})$$

$$\leq (a'_1 + a'_2 + a'_4) d(Tx_{n+1,} Tx_{n+2}) + (a'_3 + a'_4) d(Tx_{n+2}, Tx_{n+3})$$

$1 - (a'_3 + a'_4)\, d(Tx_{n+2}, Tx_{n+3}) \leq (a'_1 + a'_2 + a'_4) d(Tx_{n+1,} Tx_{n+2})$

So,

$$d(Tx_{n+1}, Tx_{n+2}) \leq h\, d(Tx_{n,} Tx_{n+1})$$

Where $h = \frac{(a_1 + a_2 + a_4)}{1-(a_3 + a_4)} < 1 \Rightarrow a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and

$$d\left(Tx_{n+2,} Tx_{n+3}\right) \leq h'\, d(Tx_{n+1,} Tx_{n+2}$$

Where $h' = \frac{(a'_1 + a'_2 + a'_4)}{1-(a'_3 + a'_4)} < 1 \Rightarrow a'_1 + a'_2 + a_{'3} + a'_4 + a'_5 < 1$,

We can conclude, by repeating the same argument, that

$$d(TR^{n+1}x_{0,} TR^{n+2}x_0) \leq h^{n+1} d(Tx_0, TRx_0) \text{ ------------------------- (3.2.1)}$$

$$d(TS^{n+2}x_{0,} TS^{n+3}x_0) \leq h^{n+2} d(Tx_0, TSx_0) \text{ ------------------------- (3.2.2)}$$

From (3.2.1) we have,

$\|d(TR^{n+1}x_0, TR^{n+2}x_0\| \leq h^{n+1} K \|d(Tx_0, TRx_0)\|$, Where K is the normal constant of E. By inequality above we get

$$\lim_{n\to\infty} \|d(TR^{n+1}x_0, TR^{n+2}x_0\| = 0$$

Hence

$$\lim_{n\to\infty} d(TR^{n+1}x_{0,} TR^{n+2}x_0) = 0 \text{ ----------------------- (3.2.3)}$$

Similarly, from (3.2.2) we have

$$\lim_{n\to\infty} d(TS^{n+2}x_{0,} TS^{n+3}x_0 = 0 \text{--------------------- (3.2.4)}$$

By inequality (3.2.1), for every $m, n \in N$ with $m > n$, we have

$$d(Tx_{n+1,}Tx_{m+1}) \leq d\,(Tx_{n+1,}Tx_{n+2}) + \text{---------------------} + d(Tx_{m,}Tx_{m+1})$$

$$\leq (h^{n+1} + h^{n+2} + -------- + h^{m})\, d(Tx_0, TRx_0)$$

$$\leq \frac{h^{n+1}}{1-h}\, d(Tx_0, TRx_0)$$

$$d(TR^{n+1}x_{0,}TR^{m+1}x_0) \leq \frac{h^{n+1}}{1-h} K\, \|d(Tx_0, TRx_0)\| \text{ ----------------------------- } (3.2.5)$$

From (3.2.4) we have,

$$\|d(TR^{n+1}x_{0,}TR^{m+1}x_0)\| \leq \frac{h^{n+1}}{1-h} K\, \|d(Tx_0, TRx_0)\|$$

Where K is the normal constant of E. Taking limit and by $h < 1$, we obtain

$$\lim_{n,m\to\infty}\|d(TR^{n+1}x_{0,}TR^{m+1}x_0)\| = 0.$$

In this way, we have

$\lim_{n\to\infty} d(TR^{n+1}\,x_{0,}TR^{m+1}x_0) = 0$, which implies that $(TR^{n+1}x_0)$ is Cauchy sequence in X. Since X is a complete cone metric space, then there is $v \in X$ such that

$$\lim_{n\to\infty} TR^{n+1}\,x_{0,} = v -------- \quad (3.2.6)$$

Now, if T is subsequently convergent, $(R^{n+1}x_0)$ has a convergent subsequence. So there exist $u \in X$ and $x_{(n+1)i}$ such that

$$\lim_{i\to\infty} R^{(n+1)i}\,x_{0,} = u ----------- \quad (3.2.7)$$

Since T is continuous and by (3.2.7) we obtain

$$\lim_{i\to\infty} TR^{(n+1)i}\,x_{0,} = Tu ---------- \quad (3.2.8)$$

By (3.2.6) and (3.2.8) we conclude that

$$Tu = v \quad ------------- \quad (3.2.9)$$

On the other hand,

$$d(TRu, Tu) \leq d(TRu, TR^{(n+1)i}x_0 + d(TR^{(n+1)i}x_{0,}\, TR^{(n+1)i+1}x_0)$$

$+ d(TR^{(n+1)i+1}x_{0,}Tu)$

$\leq a_1 d(Tu, \mathrm{TR}x_{(n+1)i-1}) + a_2 d(Tu, TRu \ \ + a_3 d(TRx_{(n+1)i-1,} TRx_{(n+1)i})$

$+ a_4 d(Tu, TRx_{(n+1)i} + a_5 d\big(TRx_{(n+1)i-1,} TRu\big)$

$+ d\big(TRx_{(n+1)i} + TRx_{(n+1)i,} TRx_{(n+1)i+1}\big) + d(TRu, Tu)$

$\leq \frac{a_1}{1-a_2}\ d(Tu, TRx_{(n+1)-i}) + \frac{a_3}{1-a_2}\ d(TRx_{(n+1)i-1,} TRx_{(n+1)i}$

$+ \ \frac{a_1}{1-a_2} d(Tu, TRx_{(n+1)i}) + \frac{h^{n+1}}{1-a_2}\ \ d(Tx_0, TRx_0)$

Thus,

$\|d(TRu, Tu)\| \leq \frac{a_1}{1-a_2} K \left\|d(Tu, TRx_{(n+1)-i})\right\| + \frac{a_3}{1-a_2} K \left\| d(TRx_{(n+1)i-1,} TRx_{(n+1)i} \right\|$

$+ \ \frac{a_1}{1-a_2} K \left\|d(Tu, TRx_{(n+1)i})\right\| + \frac{h^{n+1}}{1-a_2}\ K\ \|d(Tx_0, TRx_0)\| \to 0 (i \to \infty),$

Where K is the normal constant of X. The convergences above give us that

$d(TRu, Tu)$ = 0.which implies that $TRu = Tu$. Since T is one to one, then $Ru = u$, consequently R has a fixed point.

If v is another fixed point of R, then fixed point is unique. Finally, if T is sequentially convergent, by replacing $(n+1)$ for $(n+1)i$ we conclude that

$\lim_{i\to\infty} R^{(n+1)}\, x_{0,} = u$. This shows that $(R^{n+1}x_0)$ converges to the fixed point of R.

Similarly, we can prove that $(S^{n+2}x_0)$ converges to the fixed point of S.

$i.e.$ $\quad \lim_{n\to\infty} TR^{n+1}\, x_{0,} = u = \lim_{n\to\infty} TS^{n+2}\, x_{0,}.$

This completes the proof of the theorem.

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