A Common Fixed Point Theorem on Lohani and Bhadshah

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Abstract **--The aim of this paper is to present a common fixed point theorem in a metric space which extends the results of P.C.Lohani V.H.Bhadshah using the weaker conditions such as Weakly compatible and Associated sequence.**

***Keywords*-- Fixed point, Self maps, compatible mappings, weakly compatible, associated sequence.**

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1. INTRODUCTION:

G.Jungck gave a common fixed point theorem for commuting mapping maps, which generalizes the Banach's fixed point theorem. This result was further generalized and extended in various ways by many authors. S.Sessa[5] defined weak commutativity and proved common fixed point theorems for weakly commuting maps. Further G.Jungck [1] initiated the concept of compatible maps which is weaker than weakly commuting maps. After wards Jungck and Rhoades[4] defined weaker class of maps known as weakly compatible maps.

The purpose of this paper is to prove a common fixed point theorem for four self maps using weakly compatible mappings.

Definitions and Preliminaries:

Definition 1.1 If S and T are mappings from a metric space (X,d) into itself, are called weakly commuting mappings on X, if $d(STx,TSx) \le d(Sx,Tx)$ for all x in X.

Definition 1.2 Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n,TSx_n)=0$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n= t$ for some $t\in X$.
Clearly commuting mappings are weakly commuting, but the converse is not necessarily true.

Definition 1.3. Two self maps S and T of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence point. i.e if Su=Tu for some $u\in X$ then STu=TSu.

It is clear that every compatible pair is weakly compatible but its converse need not be true.

P.C.Lohani and V.H.Badshah [6] proved the following theorem.

Theorem (A): Let P,Q,S and T be self mappings from a complete metric space (X,d) into itself satisfying the following conditions

$S(X)\subset Q(X)$ and $T(X)\subset P(X)$ …….(1.5)

$$d(Sx,Ty) \le \alpha \frac{d(Qy,Ty)[1+d(Px,Sx)]}{[1+d(Px,Qy)]} + \beta\, d(Px,Qy)$$ ………(1.6)

for all x,y in X where $\alpha,\beta\ge 0$, $\alpha+\beta<1$.
One of P,Q,S and T is continuous ….(1.7)
Pairs S,P and T,Q are compatible on X …….(1.8)

Then P,Q,S and T have a unique common fixed point in X.

Associated Sequence: Suppose P,Q,S and T are self maps of a metric space (X,d) satisfying the condition (1.5). Then for an arbitrary $x_0 \in X$ such that $Sx_0 = Qx_1$ and for this point x_1, there exist a point x_2 in X such that $Tx_1 = Px_2$ and so on. Proceeding in the similar manner, we can define a sequence $<y_n>$ in X such that $y_{2n}=Sx_{2n}= Qx_{2n+1}$ and $y_{2n+1}=Px_{2n+2} = Tx_{2n+1}$ for $n \geq 0$. We shall call this sequence as an "*Associated sequence of x_0* "relative to the four self maps P,Q,S and T.

Lemma: Let P,Q,S and T be self mappings from a complete metric space (X,d) into itself satisfying the conditions (1.5) and (1.6). Then the associated sequence $\{y_n\}$ relative to four self maps is a Cauchy sequence in X.

Proof: From (1.6), we have

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\leq \alpha \frac{d(Qx_{2n+1}, Tx_{2n+1})[1 + d(Px_{2n}, Sx_{2n})}{[1 + d(Px_{2n}, Qy_{2n+1})]}$$

$$+ \beta\, d(Px_{2n}, Qy_{2n+1})$$

$$= \alpha \frac{d(y_{2n}, y_{2n+1})[1 + d(y_{2n-1}, y_{2n})}{[1 + d(y_{2n-1}, y_{2n})]} + \beta\ d(y_{2n-1}, y_{2n})$$

$$= \alpha\ d(y_{2n}, y_{2n+1}) + \beta\ d(y_{2n-1}, y_{2n})$$

$$(1-\alpha)\ d(y_{2n}, y_{2n+1}) \leq \beta\ d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{\beta}{(1-\alpha)}\ d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq h\ d(y_{2n-1}, y_{2n})\ \ where\ \ h = \frac{\beta}{(1-\alpha)}$$

Now

$$d(y_n, y_{n+1}) \leq h\ d(y_{n-1}, y_n)$$

$$\leq h^2\, d(y_{n-2}, y_{n-1}) \leq ...h^n\ d(y_0, y_1)$$

For every integer $p > 0$, we get

$$d(y_n, y_{n+p}) \leq\ d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + .. + d(y_{n+p-1}, y_{n+p})$$

$$\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + ... + h^{n+p-1} d(y_0, y_1)$$

$$\leq h^n \left(1 + h + h^2 + ... + h^{p-1}\right) d(y_0, y_1)$$

Since $h<1$, $h^n \to 0$ as $n \to \infty$, so that $d(y_n,y_{n+p}) \to 0$. This shows that the sequence $\{y_n\}$ is a cauchy sequence in X and since X is a complete metric space; it converges to a limit, say $z \in X$.

The converse of the Lemma is not true, that is P,Q,S and T are self maps of a metric space (X,d) satisfying (1.5) and (1.6), even if for $x_0 \in X$ and for associated sequence of x_0 converges, the metric space (X,d) need not be complete. The following example establishes this.

Example: Let X= (-1,1) with $d(x,y)=|x-y|$

$$Sx = Tx = \begin{cases} \frac{1}{5} & if\ -1 < x < \frac{1}{6} \\ \frac{1}{6} & if\ \frac{1}{6} \leq x < 1 \end{cases}$$

$$Px = \begin{cases} \frac{1}{5} & if\ -1 < x < \frac{1}{6} \\ \frac{6x+5}{36} & if\ \frac{1}{6} \leq x < 1 \end{cases} \quad Qx = \begin{cases} \frac{1}{5} & if\ -1 < x < \frac{1}{6} \\ \frac{1}{3} - x & if\ \frac{1}{6} \leq x < 1 \end{cases}$$

Then $S(X)=T(X)=\left\{\frac{1}{5}, \frac{1}{6}\right\}$, while $P(X) = \left\{\frac{1}{5} \cup \left[\frac{1}{6}, \frac{11}{36}\right)\right\}$, $Q(X) = \left\{\frac{1}{5} \cup \left[\frac{1}{6}, \frac{-2}{3}\right)\right\}$ so that $S(X) \subset Q(X)$ and $T(X) \subset P(X)$ proving the condition (1.5). Clearly (X,d) is not a complete metric space. It is easy to prove that the associated sequence $Sx_0,Tx_1,Sx_2,Tx_3,..,Sx_{2n},Tx_{2n+1}..$, converges to $\frac{1}{5}$.

If $-1 < x < \frac{1}{6}$ or $\frac{1}{6} \leq x < 1$, the associated sequence is converges to $\frac{1}{6}$. Now we prove our theorem.

Theorem (B): Let P,Q, S and T are self maps of a metric space (X,d) satisfying (1.5), (1.6) and the conditions

The pairs (S, P) and (T,Q) are weakly compatible ..(1.7)

Also

The associated sequence relative to four self maps P,Q, S and T such that the sequence $Sx_0,Tx_1, Sx_2,Tx_3, \ldots\ldots$, $Sx_{2n},Tx_{2n+1},\ldots$ converges to $z\in X$ as $n\to\infty$. …....(1.8)
Then P,Q,S and T have a unique common fixed point z in X.

Proof: From the condition (1.8), $Sx_0,Tx_1,Sx_2,Tx_3,\ldots\ldots, Sx_{2n},Tx_{2n+1},\ldots$converges to $z\in X$ as $n\to\infty$.
Since $S(X)\subset Q(X)$ then there exists $u\in X$ such that z=Qu we prove that Qu=Tu=z

we consider $d(Tu,z)=d(z,Tu)=d(Sx_{2n},Tu)$

$$\le \lim_{n\to\infty}\left\{\alpha\frac{d(Qu,Tu)[1+d(Px_{2n},Sx_{2n})]}{[1+d(Px_{2n},Qu)]}+\beta\, d(Px_{2n},Qu)\right\}$$

$= d(z,Tu)+\beta\, d(z,z)$

$d(z,Tu)\le d(z,Tu)\Rightarrow(1-\propto)\ d(z,Tu)=0\Rightarrow z=Tu.$

$\therefore Qu=Tu=z$

Since (Q,T) is weakly compatible $QTu=TQu\Rightarrow Qz=Tz$ and $T(X)\subset P(X)$ there exists $v\in X$ such that z=Pv.

We solve Sv=Pv
Consider $d(Sv,Tx_{2n+1})$

$$\le\left\{\alpha\frac{d(Qx_{2n+1},Tx_{2n+1})[1+d(Pv,Sv)]}{[1+d(Pv,Qx_{2n+1})]}+\beta\, d(Pv,Qx_{2n+1})\right\}$$

$$\Rightarrow d(Sv,z)\le 0\Rightarrow d(Sv,z)=0\Rightarrow Sv=z$$

Since Sv=Pv=z and (S,P) is weakly compatible $SPv=PSv\Rightarrow Sz=Pz$.

Now consider d(Sz,z)=d(Sz,Tu)

$$d(Sz,z)=\lim_{n\to\infty}d(Sz,Tu)$$

$$\le \lim_{n\to\infty}\left\{\alpha\frac{d(Qu,Tu)[1+d(Pz,Sz)]}{[1+d(Pz,Qu)]}+\beta\, d(Pz,Qu)\right\}$$

$$=\beta\, d(Sz,z)$$

$$\Rightarrow d(Sz,z)\le\beta\, d(Sz,z)$$

Since $\alpha+\beta\le 1$, $d(Sz,z)=0$. This shows that Sz=z. This implies Sz=Pz therefore z is common fixed point of S and P

Again we consider

$$d(z,Tz)=d(Sz,Tz)$$

$$\le\left\{\alpha\frac{d(Qz,Tz)[1+d(Pz,Sz)]}{[1+d(Pz,Qz)]}+\beta\, d(Pz,Qz)\right\}$$

$$=\beta\, d(z,Tz)$$

This gives $d(z,Tz)\le\beta\, d(z,Tz)$, Since $\beta\ge 0,\alpha+\beta<1$ giving that d(z,Tz)=0. Thus Tz=z.

Therefore z=Qz=Tz then z is common fixed point of T and Q, this gives $d(Sz,z)\le\beta\, d(Sz,z)$, Since $\beta\ge 0,\alpha+\beta<1$ giving that d(Sz,z)=0. Thus Sz=z.

Hence Sz=Pz=z=Qu. This shows that 'z' is a common fixed point of P and S.

Therefore Pz=Qz=Sz=Tz=z, showing that 'z' is a common fixed point of P,Q,S and T.

The uniqueness of the fixed point can be easily proved.

Remark 1: From the example given above, clearly the pairs (S,P) and (T,Q) are weakly compatible as they commute at coincident points $\frac{1}{8}$ and $\frac{1}{10}$. But the pairs (S,P) and (T,Q) are not compatible For this, take a sequence $x_n=\left(\frac{1}{10}+\frac{1}{10^n}\right)$ for $n\ge 1$, then $\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Px_n=\frac{1}{10}$ and $\lim_{n\to\infty}SPx_n=\frac{1}{8}$ also $\lim_{n\to\infty}PSx_n=\frac{1}{10}$. So that $\lim_{n\to\infty}d(SPx_n,PSx_n)\ne 0$. Also note that none of the mappings are continuous and the rational inequality holds for the values of $\alpha,\beta\ge 0\ and\ \alpha+\beta\le 1$. Clearly $\frac{1}{6}$ is the unique common fixed point of P,Q,S and T.

Remark 2: Theorem (B) is a generalization of Theorem (A) by virtue of the weaker conditions such as weakly compatibility of the pairs (S,P) and (T,Q) in place of compatibility; and associated sequence relative to four self maps P,Q,S and T in place of the complete metric space.

REFERENCES

[1]. G.Jungck, *Compatible mappings and common fixed points*, Internat.J.Math.&Math.Sci.9,771- 778,1986.

[2]. R.P.Pant, *A Common fixed point theorem under a new condition*, Indian J. of Pure and App. Math., 30(2), 147-152, 1999.

[3]. G.Jungck, *Compatible mappings and common fixed points (2)*, Internat.J.Math.&Math.Sci.11, 285-288, 1988.

[4]. G.Jungck and B.E. Rhoades, *Fixed point for set valued functions without continuity*, Indian J. Pure.Appl.Math., 29 (3), 227-238, 1998.

[5]. S.Sessa, *On weak commutativity condition of mappings in fixed point considerations*, Publ. Inst.Math.32 (46),149-153,1980.

[6]. P.C.Lohani and V.H.Badshah, *Compatible mappings and common fixed point for four mappings*, Bull.Cal.Math. Soc., 90,301-308, 1998.

[7]. B.Fisher, *Common fixed points of four mappings*, Bull.Inst.Math.Acad.Sinica, 11, 103, 1983.

[8]. V.Srinivas and R.Umamaheshwar Rao, *A fixed point theorem for four self maps under weakly compatible maps*, Proceeding of world congress on engineering,vol.II,WCE 2008,London,U.K.

[9]. Umamaheshwar Rao.R and V.Srinivas, *A generalization of Djoudi's common fixed point theorem***,** International J. of Math. Sci.& Engg. Appls., Vol.1, (No.2), 229-238, 2007.

[10]. Umamaheshwar Rao.R ,V.Srinivas and P.Srinkanth Rao**,** *A fixed point theorem on reciprocally continuous self maps*" Indian Journal of Mathematics and Mathematical Sciences, Vol.3, No.2, 207-215, 2007.