Generalized Hyers-Ulam Stability of a Sextic Functional Equation in Paranormed Spaces

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Abstract. In this paper, we obtain the general solution and prove the generalized Hyers-Ulam stability of a new sextic functional equation in paranormed spaces. We also present a counter-example for singular case.

1. Introduction

In 1940, S.M.Ulam [15] while he was giving a series of lectures in the University of Wisconsin; he raised a question concerning the stability of homomorphism.

"Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$. Then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?"

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The first partial solution to Ulam's question was provided by D.H. Hyers [3]. Indeed, he proved the following celebrated theorem.

Theorem 1.1. Assume that X and Y are Banach spaces. If a function f : X → Y satisfies the inequality

$$\|f(x+y)-f(x)-f(y)\|\leq \varepsilon$$

for some $\varepsilon \geq 0$ and for all $x \in X$, then the limit

$$a(x)=\lim_{n\to\infty} 2^{-n} f(2^n x)$$

exist for each x in X and a : $X \to Y$ is the unique additive function such that

$$\| f(x)-a(x) \| \leq \varepsilon$$

for any $x \in X$. Moreover, if $f(tx)$ is continuous in t for each led $x \in E$, then a is linear.

From the above case, we say that the additive functional equation $f(x + y) = f(x)+f(y)$ has the Hyers-Ulam stability on (X, Y). D.H. Hyers explicity constructed the additive function $a : X \to Y$ directly from the given function f. This method is called a direct method and it is a powerful tool for studying stability of functional equations.

Th.M.Rassias [14] proved the following substantial generalization of the result of Hyers:

Theorem 1.2. Let X and Y be Banach spaces, let $\theta \in [0; \infty)$, and let $P \in [0; 1)$. If a functional equation $f : X \to Y$ satisfies

$$\| f(x+y)-f(x)-f(y)\| \leq \theta(\|x\|^p+\|y\|^p)$$

for all $x, y \in X$, then there is a unique additive mapping $A : X \to Y$

$$\| f(x)-A(x)\| \leq \frac{2\theta}{2-2^p}\| f(x)\|^p$$

for all $x \in X$. If in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

Due to this fact, the Cauchy functional equation $f(x + y) = f(x) + f(y)$ is said to have the Hyers-Ulam-Rassias stability property on (X, Y).

Recently, Ch. Park and D.Y Shin [8] proved the Hyers-Ulam stability of the Cauchy additive functional equation

$$f(x + y) = f(x) + f(y); \tag{1.1}$$

the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \tag{1.2}$$

the cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.3}$$

and the quartic functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y) \tag{1.4}$$

in paranormed spaces.

C. Park [9] proved the Hyers-Ulam stability of an additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} f(x + 2y) + f(x - 2y) &= 4f(x + y) + 4f(x - y) - 6f(x) \\ &\quad + f(2y) + f(-2y) - 4f(y) - 4f(-y) \end{aligned} \tag{1.5}$$

in paranormed spaces using the fixed point method and direct method.

K. Ravi, J.M. Rassias and B.V. Senthil Kumar [11] investigated the generalized Hyers-Ulam stability of the reciprocal difference functional equation

$$f\left(\frac{x+y}{2}\right) - f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)} \tag{1.6}$$

and the reciprocal adjoint functional equation

$$f\left(\frac{x+y}{2}\right) + f(x+y) = \frac{3f(x)f(y)}{f(x)+f(y)} \tag{1.7}$$

in paranormed spaces using fixed point method and direct method.

In this paper, we obtain the general solution and prove the generalized Hyers-Ulam stability of a sextic functional equation

$$f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny)$$
$$= (n^4 + n^2)\,[f(x + y) + f(x - y)] + 2(n^6 - n^4 - n^2 + 1)\,[f(x) + f(y)] \qquad (1.8)$$

in paranormed spaces. For a given mapping f, we define

$$D_s f(x,y) = f(nx+y) + f(nx-y) + f(x+ny)\, f(x-ny)$$
$$-(n^4+n^2)[f(x+y) + f(x-y) + 2(n^6-n^2-n^2+1)[f(x)+f(y)].$$

Throughout this paper, let (X, P) be a Fréchet space and that $(Y, \|.\|)$ be a Banach space.

2. Generalized Hyers-Ulam stability of equation (1.8)

Theorem 2.1. Let $\varphi : Y \times Y \to [0, \infty)$ be a function satisfying

$$\sum_{i=0}^{\infty} \frac{1}{n^{6i}} \varphi(n^i x, n^i y) < \infty \qquad (2.1)$$

for all $x, y \in Y$. If an even function $f : Y \to X$ with the condition $f(0) = 0$ satisfies the functional inequality

$$P\,(D_s f(x, y)) \le \varphi\,(x, y) \qquad (2.2)$$

for all $x, y \in Y$, then there exists a unique sextic mapping $S : Y \to X$ which satisfies (1.8) and the inequality

$$P\big(f(x) - S(x)\big) \le \frac{1}{n^6} \sum_{i=0}^{\infty} \frac{1}{n^{6i}} \varphi(n^i x, 0) \qquad (2.3)$$

for all $x \in Y$.

Proof. Letting $y = 0$ in (2.2) and dividing by n^6, we get

$$P\left(\frac{1}{n^6} f(nx) - f(x)\right) \le \frac{1}{n^6} \varphi(x, 0) \qquad (2.4)$$

for all $x \in Y$. Now, replacing x by nx in (2.4), dividing by n^6 and adding the resulting inequality with (2.4), we obtain

$$\mathrm{P}\left(\frac{1}{n^{12}} f\left(n^2 x\right) - f\left(x\right)\right) \le \frac{1}{n^6} \sum_{i=0}^{1} \frac{1}{n^{6i}} \varphi(\mathrm{n}^i \mathrm{x}, 0)$$

for all $x \in Y$: Hence by mathematical induction, we arrive

$$\mathrm{P}\left(\frac{1}{n^{6m}} f\left(n^m x\right) - f\left(x\right)\right) \le \frac{1}{n^6} \sum_{i=0}^{n-1} \frac{1}{n^{6i}} \varphi(\mathrm{n}^i \mathrm{x}, 0) \tag{2.5}$$

for all $x \in Y$. For any non-negative integers l, m with $l > m$, we obtain by using the triangle inequality

$$\begin{aligned}
\mathrm{P}\left(\frac{1}{n^{6l}} f\left(n^l x\right) - \frac{1}{n^{6m}} f\left(n^m x\right)\right) &\le \mathrm{P}\left(\frac{1}{n^{6l}} f\left(n^l x\right) - f\left(x\right)\right) + \mathrm{P}\left(f\left(x\right) - \frac{1}{n^{6m}} f\left(n^m x\right)\right) \\
&\le \frac{1}{n^6} \sum_{i=0}^{l-1} \frac{1}{n^{6i}} \varphi(\mathrm{n}^i \mathrm{x}, 0) + \frac{1}{n^6} \sum_{i=0}^{m-1} \frac{1}{n^{6i}} \varphi(\mathrm{n}^i \mathrm{x}, 0) \\
&\le \frac{1}{n^6} \sum_{i=m}^{n-1} \frac{1}{n^{6i}} \varphi(\mathrm{n}^i \mathrm{x}, 0)
\end{aligned} \tag{2.6}$$

for all $x \in Y$. Taking the limit as $m \to \infty$ in (2.6) and considering (2.1), it follows that the sequence $\{\frac{1}{n^{6m}} f(n^m x)\}$ is a Cauchy sequence for each $x \in Y$. Since X is complete, we can define $S : Y \to X$ by

$$S\left(x\right) = \lim_{m \to \infty} \frac{1}{n^{6m}} f\left(n^m x\right). \tag{2.7}$$

To show that S satisfies (1.8), replacing $(x; y)$ by $(n^m x,\ n^m y)$ in (2.2) and dividing by n^{6m}, we obtain

$$P\left(\frac{1}{n^{6m}} D_s f\left(n^m x, n^m y\right)\right) \le \frac{1}{n^{6m}} \varphi\left(n^m x, n^m y\right) \tag{2.8}$$

for all $x; y \in Y$, for all positive integer n. Using (2.1) and (2.5) in (2.8), we see that S satisfies (1.8), for all $x, y \in Y$. Taking limit $m \to \infty$ in (2.5), we arrive (2.3). Now, we have to show that S is the unique sextic mapping. Let $s : Y \to X$ be another sextic mapping which satisfies (1.8) and the inequality (2.3). Then we have

$$P\left(s(x)-S(x)\right)=P\left(\frac{1}{n^{6m}}s\left(n^{m}x\right)-\frac{1}{n^{6m}}S\left(n^{m}x\right)\right)$$
$$\leq P\left(\frac{1}{n^{6m}}s\left(n^{m}x\right)-\frac{1}{n^{6m}}f\left(n^{m}x\right)\right)+P\left(\frac{1}{n^{6m}}f\left(n^{m}x\right)-\frac{1}{n^{6m}}S\left(n^{m}x\right)\right)$$
$$\leq 2\sum_{i=0}^{\infty}\frac{1}{n^{6(n+i)}}\varphi(\mathrm{n}^{m+i}\mathrm{x},0)$$
$$\leq 2\sum_{i=0}^{\infty}\frac{1}{n^{6i}}\varphi(\mathrm{n}^{i}\mathrm{x},0) \tag{2.9}$$

for all $x \in Y$. Allowing $n \to \infty$ in (2.9), we see that S is unique. This completes the proof.

Corollary 2.2. Let $f : Y \to X$ be a mapping satisfying

$$\mathrm{P}\ (\mathrm{D_s f}(x,y)) \leq \in \tag{2.10}$$

for all $x,y \in Y$. Then there exists a unique sextic mapping $S : Y \to X$ satisfying (1.8) and

$$P\left(f(x)-S(x)\right)\leq\frac{\in}{n^{6}-1} \tag{2.11}$$

for every $x \in Y$.

Proof. This Corollary can be proved by choosing $\varphi(x; y) = \in$, for all $x; y \in Y$ in Theorem 2.1.

Corollary 2.3. Let $f : Y \to X$ be a mapping and let there exist real numbers p < 6 and $k \geq 0$ such that

$$P\left(D_s f(x,y)\right) \leq k\ (\|x\|^p + \|y\|^p) \tag{2.12}$$

for all $x,y \in Y$. Then there exists a unique sextic mapping $S : Y \to X$ satisfying (1.8) and

$$P\left(f(x)-S(x)\right)\leq\frac{k}{n^{6}\,(1-n^{p-6})}\|x\|^{p} \tag{2.13}$$

for every $x \in Y$.

Proof. It is easy to prove this Corollary by taking $\varphi(x, y) = k(\|x\|^p + \|y\|^p)$, for all $x, y \in Y$ in Theorem 2.1.

3. Counter-examples

We present below a counter-example to show that the functional equation (1.8) is not stable for $p = 6$ in Corollary 2.3.

Example 3.1. Let $\phi : R \to R$ be a mapping defined by

$$\phi(x) = \begin{cases} \mu x^6 & \text{for } x \in (-1,1) \\ \mu & \text{otherwise} \end{cases}$$

Where $\mu > 0$ is a constant, and defined a mapping $f : R \to R$ by

$$f(x) = \sum_{k=0}^{\infty} \frac{\phi(n^k x)}{n^{6k}}, \text{ for all } x \in R.$$

Then the mapping f satisfies the inequality

$$|D_S f(x, y)| \le \frac{2\mu n^6 (2n^6 - n^4 - n^2 + 4)}{n^6 - 1}\left(|x|^6 + |y|^6\right) \tag{3.1}$$

for all $x \in R$. Therefore there do not exist a sextic mapping $S : R \to R$ and a constant $\lambda > 0$ such that

$$|f(x) - S(x)| \le \lambda |x|^6 \tag{3.2}$$

for all $x \in R$.

Proof. $|f(x)| = \left|\sum_{k=0}^{\infty} \frac{\phi(n^k x)}{n^{6k}}\right| \le \sum_{k=0}^{\infty} \frac{|\phi(n^k x)|}{|n^{6k}|} \sum_{k=0}^{\infty} \frac{\mu}{n^{6k}} = \mu(1 - \frac{1}{n^6})^{-1} = \frac{\mu n^6}{n^6 - 1}$.

Hence f is bounded by $\frac{\mu n^6}{n^6-1}$. If $(|x|^6 + |y|^6) \ge 1$, then the left hand side of (3.1) is less than $2\mu (2n^6 - n^4 - n^2 + 4)$. Now, suppose that $0 < (|x|^6 + |y|^6) < 1$. Then there exists a positive integer k such that

$$\frac{1}{n^{6k}} \le (|x|^6 + |y|^6) < \frac{1}{n^{6(k-1)}}. \tag{3.3}$$

Hence $(|x|^6 + |y|^6 < \frac{1}{n^{6(k-1)}}$ implies

$$n^{6(k-1)} x^6 < 1, \; n^{6(k-1)} y^6 < 1$$

$$\text{or} \qquad n^{k-1}x < 1,\ n^{k-1}y < 1$$

and consequently

$$n^{k-1}x,\ n^{k-1}y,\ n^{k-1}(x+y),\ n^{k-1}(x-y),\ n^{k-1}(nx+y),\ n^{k-1}(nx-y),$$
$$n^{k-1}(x+ny),\ n^{k-1}(x-ny) \in (-1,1).$$

Therefore, for each value of m = 0, 1, 2, . . . , k - 1, we obtain

$$n^m x,\ n^m y,\ n^m(x+y),\ n^m(x-y),\ n^m(nx+y),\ n^m(nx-y),$$
$$n^m(x+ny),\ n^m(x-ny) \in (-1, 1).$$

and

$$\phi(n^m(nx+y)) + \phi(n^m(nx-y)) + \phi(n^m(x+ny)) + \phi(n^m(x-ny))$$
$$-n^4 + n^2\,[\phi(n^m(x+y)) + (n^m(x\,y))]$$
$$-2\,(n^6 - n^4 - n^2 + 1)\,[\phi(n^m x) + \phi(n^m y)] = 0$$

for m = 0, 1, 2, . . . , m – 1. Using (3.3) and the definition of *f*, we obtain

$$\frac{|D_s f(x,y)|}{(|x|^6+|y|^6)}$$
$$\le \sum_{m=0}^{\infty}\frac{1}{n^{6m}(|x|^6+|y|^6)}\Big\{\phi(n^m(nx+y))+\phi(n^m(nx-y))$$
$$+\ \phi(n^m(nx+y))+\phi(n^m(nx-y))$$
$$-\ (n^4+n^2)\ [\phi(n^m(x-y))+\phi(n^m(x-y))]$$
$$-2(n^6-n^4-\mathrm{n}^2+1)[\phi\left(n^m\left(x-y\right)\right)]$$
$$-2(n^6-n^4-\mathrm{n}^2+1)[\phi\left(n^m x\right)+\left(n^m y\right)]\Big\}$$

$$\le \sum_{m=k}^{\infty}\frac{2\mu\left(2n^6-n^4-n^2+4\right)}{n^{6m}\left(|x|^6+|y|^6\right)}$$

$$\le \frac{2\mu\left(2n^6-n^4-n^2+4\right)}{\left(|x|^6+|y|^6\right)}\sum_{m=k}^{\infty}\frac{1}{n^{6m}}$$

$$\le \frac{2\mu\left(2n^6-n^4-n^2+4\right)}{\left(|x|^6+|y|^6\right)}\frac{1}{n^{6k}}\left(1-\frac{1}{n^6}\right)^{-1}$$

$$\leq \frac{2\mu\left(2n^6-n^4-n^2+4\right)}{\left(|x|^6+|y|^6\right)}\frac{1}{n^{6k}}\left(\frac{n^6}{n^6-1}\right)$$

$$\leq \frac{2\mu n^6\left(2n^6-n^4-n^2+4\right)}{\left(n^{6-1}\right)}$$

for all $x,y \in$ R. That is, the inequality (3.1) holds true. We claim that the sextic functional equation (1.8) is not stable for $p = 6$ in Corollary 2.3. Suppose on the contrary that there exist a sextic mapping $S : R \rightarrow R$ and a constant $\lambda > 0$ satisfying (3.2). Since f is bounded and continuous for all $x \in$ R, S is bounded on any open interval containing the origin and continuous at the origin. Thus we obtain that

$$\left|f(x)\right| \leq (\lambda+1)|x|^6. \tag{3.4}$$

But we can choose a positive integer r with $r\mu > \lambda + 1$. If $x \in \left(0, \frac{1}{n^{6(m-1)}}\right)$, then $n^{6m} x \in (0, 1)$ for all $m = 0; 1, 2, \ldots, r - 1$ and therefore

$$\left|f(x)\right| = \sum_{m=0}^{\infty}\frac{\phi\left(n^m x\right)}{n^{6m}} \geq \sum_{m=0}^{r-1}\frac{\mu n^{6m} x^6}{n^{6m}} = \tau\mu x^6 > (\lambda+1)x^6$$

which contradicts (3.4). Therefore, the sextic functional equation (1.8) is not stable for $p = 6$ in Corollary 2.3.

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