

Numerical Solution of Hybrid Fuzzy Differential Equations by Adams Fifth Order Predictor-Corrector Method

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Abstract

In this paper we study numerical methods for hybrid fuzzy differential equations by an application of the Adams fifth order predictor-corrector method for fuzzy differential equations. We prove a convergence result and give numerical examples to illustrate the theory.

Keywords: Hybrid systems; Fuzzy differential equations; Adams fifth order predictor-corrector method.

1 Introduction

Hybrid systems are devoted to modelling, design and validation of interactive systems of computer programs and continuous systems. That is, control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics can be modelled by hybrid systems. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named as hybrid fuzzy differential systems. For analytical results on stability properties and comparison theorems we refer to [16, 17, 22].

In the last few years, many works have been performed by several authors in numerical solutions of fuzzy differential equations [1, 2, 3, 18, 19]. Pederson and Sambandam [19, 22] have investigated the numerical solution of hybrid fuzzy differential equations by using Runge-Kutta and Euler methods. Recently, the numerical solutions of fuzzy differential equations by predictor-corrector method has been studied in [19, 21, 22]. Kanagarajan and sambath[15] studied the numerical solution of fuzzy differential equations by improved predictor-corrector method.

The structure of this paper organized as. In Section 2. we bring definitions to fuzzy valued functions. In Section 3 we define hybrid fuzzy differential systems. In Sections 4, 5 and 6 we apply the Adams-Basforth, Adams-Moulton and Adams fifth order predictor-corrector methods for solving hybrid fuzzy differential equations. In Section 7, we give converge and stability results. The proposed algorithm is illustrated by solving some examples in Section 8.

2. Preliminaries

Let $P_K(\mathcal{R}^n)$ denote the family of all non-empty, compact, convex subsets of \mathcal{R}^n . If $\alpha, \beta \in \mathcal{R}^n$ and $A, B \in P_K(\mathcal{R}^n)$, then $\alpha(A + B) = \alpha A + \alpha B$, $\alpha(\beta A) = (\alpha\beta)A$, $1A = A$ and if $\alpha, \beta \leq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$.

Denote by E^n the set of $u : \mathcal{R}^n \rightarrow [0, 1]$ such that u satisfies (i) – (iv) mentioned below:

- (i) u is normal, that is, there exists an $x_0 \in \mathcal{R}^n$ such that $u(x_0) = 1$,
- (ii) u is fuzzy convex, that is, for $x, y \in \mathcal{R}^n$ and $0 \leq \lambda \leq 1$.

$$u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)],$$

- (iii) u is upper semi continuous,
- (iv) $[u]^0 = \text{the closure of } [x \in \mathcal{R}^n : u(x) > 0]$ is compact.

For $0 < \alpha \leq 1$, we denote $[u]^\alpha = [x \in \mathcal{R}^n : u(x) \geq \alpha]$. Then from (i) to (iv), it follows that

α -level sets $[u]^\alpha \in p_k(\mathcal{R}^n)$ for $0 \leq \alpha \leq 1$. An example of a $u \in E^1$ is given by

$$u(x) = \begin{cases} 4x - 3 & \text{if } x \in (0.75, 1], \\ -2x + 3 & \text{if } x \in (1, 1.5), \\ 0 & \text{if } x \notin (0.75, 1.5), \end{cases} \quad (1)$$

The α -level sets are given by

$$[u]^\alpha = [0.75 + 0.25\alpha, 1.5 - 0.5\alpha].$$

Let I be a real interval. A mapping $y : I \rightarrow E$ is called a fuzzy process and its α -level set is denoted by $[y(t)]^\alpha = [\underline{y}^\alpha(t), \bar{y}^\alpha(t)]$, $t \in I$, $\alpha \in (0, 1]$.

Triangular fuzzy numbers are those fuzzy sets in E which are characterized by an ordered triple $(x^l, x^c, x^r) \in \mathcal{R}^3$ with $x^l \leq x^c \leq x^r$ such that $[U]^0 = [x^l, x^r]$ and $[U]^1 = [x^c]$, then

$$[U]^\alpha = [x^c - (1 - \alpha)(x^c - x^l), x^c + (1 - \alpha)(x^r - x^c)] \quad (2)$$

for any $\alpha \in I$.

Definition 2.1. Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(0) = y_0, \quad (3)$$

where $f : [a, b] \times R^n \rightarrow R^n$. An m -step method for solving the initial-value problem is one whose difference equation for finding the approximation $y(t_{i+1})$ at the mesh point t_{i+1} can be represented by the following equation :

$$y_{i+1} = \sum_{j=0}^{m-1} a_{m-j-1} y_{i-j} + h \sum_{j=0}^{m-1} b_{m-j-1} f(t_{i-j+1}, y_{i-j+1}) \quad (4)$$

for $i = m-1, m, \dots, N-1$, such that $a = t_0 \leq t_1 \leq \dots \leq t_N = b$, $h = \frac{(b-a)}{N} = t_{i+1} - t_i$ and $a_0, \dots, a_{m-1}, b_0, \dots, b_{m-1}$ are constants with the starting values

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_{m-1} = \alpha_{m-1}.$$

When $b_m = 0$, the method is known as explicit, since Equation (3) gives y_{i+1} explicit in terms of previously determined values. When $b_m \neq 0$, the method is known as implicit, since y_{i+1} occurs on both sides of Equation (3) and is specified only implicitly.

A especial case of multistep method is . Here, we set

$$y_{i+1} = y_{i-1} + h \sum_{m=0}^q k_m \nabla^m f(t_i, y_i), \quad q = 0, 1, 2, \dots, \quad (5)$$

where the constants

$$k_m = (-1)^m \int_{-1}^1 \begin{pmatrix} -s \\ k \end{pmatrix} ds$$

are independent of $f, t = t_0 + sh$, $\nabla f(t_i, y_i)$ is the first backward difference of the $f(t, y(t))$ at the point of $t = t_i$ and higher backward difference are defined by $\nabla^k f(t, y_i) = \nabla(\nabla^{k-1} f(t_i, y_i))$. The special case $q = 4$ of Adams-Bashforth and Adams Moulton are:

Adams-Bashforth five-Step method:

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_3 = \alpha_3, \quad y_4 = \alpha_4,$$

$$y_{i+1} = y_i + \frac{h}{720} [1901f(t_i, y_i) - 2774f(t_{i-1}, y_{i-1}) + 2616f(t_{i-2}, y_{i-2}) - 1274f(t_{i-3}, y_{i-3}) + 251f(t_{i-4}, y_{i-4})],$$

where $i = 4, 5, \dots, N-1$.

Adams-Moulton four-step method :

$$y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_3 = \alpha_3, \\ y_{i+1} = y_i + \frac{h}{720} [251f(t_{i+1}, y_{i+1}) + 646f(t_i, y_i) - 264f(t_{i-1}, y_{i-1}) + 106f(t_{i-2}, y_{i-2}) - 19f(t_{i-3}, y_{i-3})].$$

where $i = 4, 5, \dots, N - 1$.

Definition 2.2. Associated with the difference equation

$$y_{i+1} = a_{m-1}y_i + a_{m-2}y_{i-1} + \dots + a_0y_{i+1-m} + hF(t_i, h, y_{i+1}, y_i, \dots, y_{i+1-m}), \\ y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_{m-1} = \alpha_{m-1}, \quad (6)$$

the characteristic polynomial of the method is defined by

$$p(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \dots - a_1\lambda - a_0.$$

If $|\lambda_i| \leq 1$ for each $i = 1, 2, \dots, m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Theorem 2.1. [11] A multistep method of the form (4) is stable if and only if it satisfies the roots condition.

Notations used in this paper are as follows:

A tilde is placed over a symbol to denote a fuzzy set so $\tilde{\alpha}_1, \tilde{f}(t), \dots$

An arbitrary fuzzy number with an ordered pair of functions $(\underline{u}(\alpha), \bar{u}(\alpha))$. $0 \leq \alpha \leq 1$, which satisfy the following requirements is represented.

1. $\underline{u}(\alpha)$ is a bounded left continuous non-decreasing function over $[0, 1]$,
2. $\bar{u}(\alpha)$ is a bounded right continuous non-increasing function over $[0, 1]$,
3. $\underline{u}(\alpha) \leq \bar{u}(\alpha)$, $0 \leq \alpha \leq 1$.

Definition 2.3. The supremum metric d_∞ on E is defined by

$$d_\infty(U, V) = \sup \{d_H([U]^\alpha, [V]^\alpha) : \alpha \in I\},$$

and (E, d_∞) is a complete metric space.

Definition 2.4. A mapping $F : T \rightarrow E$ is a Hukuhara differentiable at $t_0 \in T \subseteq R$ if for some $h_0 > 0$ the Hukuhara differences $F(t_0 + \Delta t) \sim_h F(t_0)$, $F(t_0) \sim_h F(t_0 - \Delta t)$, exist in E for all $0 < \Delta t < h_0$ and if there exist an $F'(t_0) \in E$ such that

$$\lim_{\Delta t \rightarrow 0^+} d_\infty \left(\frac{F(t_0 + \Delta t) -_h F(t_0)}{\Delta t, F'(t_0)} \right) = 0$$

and

$$\lim_{\Delta t \rightarrow 0^+} d_\infty \left(\frac{F(t_0) -_h F(t_0 - \Delta t)}{\Delta t, F'(t_0)} \right) = 0$$

the fuzzy set $F'(t_0)$ is called the Hukuhara derivative of F at t_0 .

Recall that $U \sim_h V = W \in E$ are defined on level sets, where $[U]^\alpha \sim_h [V]^\alpha = [W]^\alpha$ for all $\alpha \in I$. By consideration of definition of the metric d_∞ , all the level set mappings $[F(\cdot)]^\alpha$ are Hukuhara differentiable at t_0 with Hukuhara derivatives $[F'(t_0)]^\alpha$ for each $\alpha \in I$ when $F : T \rightarrow E$ is Hukuhara differentiable at t_0 with Hukuhara derivative $F'(t_0)$.

Definition 2.5. The fuzzy integral $\int_a^b y(t)dt$, $0 \leq a \leq b \leq 1$, is defined by

$$\left[\int_a^b y(t)dt \right]^\alpha = \left[\int_a^b \underline{y}^\alpha(t)dt, \int_a^b \bar{y}^\alpha(t)dt \right],$$

provided the Lebesgue integrals on the right exist.

Definition 2.6. If $F : T \rightarrow E$ is Hukuhara differentiable and its Hukuhara derivative F' is integrable over $[0,1]$, then

$$F(t) = F(t_0) + \int_{t_0}^t F'(s)ds,$$

for all values of t_0, t where $0 \leq t_0 \leq t \leq 1$.

Definition 2.7. A mapping $y : I \rightarrow E$ is called a fuzzy process. We denote

$$[y(t)]^\alpha = [\underline{y}^\alpha(t), \bar{y}^\alpha(t)], \quad t \in I, \quad \alpha \in (0, 1].$$

The Seikkala derivative $y'(t)$ of a fuzzy process y is defined by $[y(t)]^\alpha = [\underline{y}^\alpha(t), \bar{y}^\alpha(t)]$, $t \in I$, $\alpha \in (0, 1]$ provided the equation defines a fuzzy number $y'(t) \in E$.

Definition 2.8. If $y : I \rightarrow E$ is Seikkala differentiable and its Seikkala derivative y' is integrable over $[0, 1]$, then

$$y(t) = y(t_0) + \int_{t_0}^t y'(s)ds,$$

for all values of t_0, t where $t_0, t \in I$.

Theorem 2.2. [12] Let $(t_i, \tilde{u}_i), i = 0, 1, 2, \dots, n$ be the observed data and suppose that each of the $\tilde{u}_i = (u_i^l, u_i^c, u_i^r)$ is an element of E . Then for each $t \in [t_0, t_n]$, $\tilde{f}(t) = (f^l(t), f^c(t), f^r(t)) \in E$.

$$f^l(t) = \sum_{l_i(t) \geq 0} l_i(t)u_i^l + \sum_{l_i(t) < 0} l_i(t)u_i^r,$$

$$f^c(t) = \sum_{i=0} l_i(t)u_i^c,$$

$$f^r(t) = \sum_{l_i(t) \geq 0} l_i(t)u_i^r + \sum_{l_i(t) < 0} l_i(t)u_i^l,$$

such that $l_i(t) = \prod_{j \neq i}^n \frac{(t - t_j)}{(t_i - t_j)}$.

3. The Hybrid Fuzzy Differential System

Consider the hybrid fuzzy differential system

$$\begin{cases} x'(t) = f(t, x(t), \lambda_k(x_k)), & t \in [t_k, t_{k+1}], \\ x(t_k) = x_k, \end{cases} \quad (7)$$

where $0 \leq t_0 < t_1 < \dots < t_k < \dots, t_k \rightarrow \infty$, $f \in C[R_+ \times E_1 \times E_1, E_1]$, $\lambda_k \in C[E_1, E_1]$. Here we assume the existence and uniqueness of the hybrid system hold on each $[t_k, t_{k+1}]$. To be specific the system would look like:

$$x'(t) = \begin{cases} x'_0(t) = f(t, x_0(t), \lambda_0(x_0)), & x_0(t_0) = x_0, & t_0 \leq t \leq t_1, \\ x'_1(t) = f(t, x_1(t), \lambda_1(x_1)), & x_1(t_1) = x_1, & t_1 \leq t \leq t_2, \\ \vdots \\ x'_k(t) = f(t, x_k(t), \lambda_k(x_k)), & x_k(t_k) = x_k, & t_k \leq t \leq t_{k+1}. \\ \vdots \end{cases}$$

By the solution of Equation (7) we mean the following function:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t_0) = x_0, & t_0 \leq t \leq t_1, \\ x_1(t_1) = x_1, & t_1 \leq t \leq t_2, \\ \vdots \\ x_k(t_k) = x_k, & t_k \leq t \leq t_{k+1}, \\ \vdots \end{cases}$$

We note that the solution of (7) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in E^1$ and $k = 0, 1, 2, \dots$

4. Adams-Bashforth methods

In this section, for hybrid fuzzy differential equation (7), we develop the Adams-Bashforth method via an application of the Adams-Bashforth method for fuzzy differential equations [4] when f and λ_k in equation (7) can be obtained via the Zadeh extension principle from $f \in C[R_+ \times R \times R, R]$ and $\lambda_k \in C[R; R]$. We assume that the existence and uniqueness of solutions of equation (7) hold for each $[t_k, t_{k+1}]$.

For fixed r , we replace each interval $[t_k, t_{k+1}]$ by a set of $N_k + 1$ discrete equally spaced grid points, $t_k = t_{k,0} < t_{k,1} < \dots < t_{k,N} = t_{k+1}$ (including the endpoints) at which the exact solution $\tilde{x}(t)$ is approximated by some $\tilde{y}_k(t)$.

Fix $k \in \mathbb{Z}^+$. The fuzzy initial value problem

$$\left. \begin{aligned} x'_k(t) &= f(t, x_k(t), \lambda_k(x_k)), & t_k \leq t \leq t_{k+1}, \\ \tilde{x}(t_k) &= \tilde{x}_k, \end{aligned} \right\} \quad (8)$$

can be solved by Adams-Bashforth five-step method. Let the fuzzy initial values be $\tilde{y}(t_{i-1}), \tilde{y}(t_i), \tilde{y}(t_{i+1}), \tilde{y}(t_{i+2}), \tilde{y}(t_{i+3})$, that is $\tilde{f}(t_{i-1}, y(t_{i-1}), \lambda_k(y_k)), \tilde{f}(t_i, y(t_i), \lambda_k(y_k)), \tilde{f}(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)), \tilde{f}(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)), \tilde{f}(t_{i+3}, y(t_{i+3}), \lambda_k(y_k))$, which are triangular fuzzy numbers and are shown by

$$\begin{aligned} &\{f^l(t_{i-1}, y(t_{i-1}), \lambda_k(y_k)), f^c(t_{i-1}, y(t_{i-1}), \lambda_k(y_k)), f^r(t_{i-1}, y(t_{i-1}), \lambda_k(y_k))\}, \\ &\{f^l(t_i, y(t_i), \lambda_k(y_k)), f^c(t_i, y(t_i), \lambda_k(y_k)), f^r(t_i, y(t_i), \lambda_k(y_k))\}, \\ &\{f^l(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)), f^c(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)), f^r(t_{i+1}, y(t_{i+1}), \lambda_k(y_k))\}, \\ &\{f^l(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)), f^c(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)), f^r(t_{i+2}, y(t_{i+2}), \lambda_k(y_k))\}, \\ &\{f^l(t_{i+3}, y(t_{i+3}), \lambda_k(y_k)), f^c(t_{i+3}, y(t_{i+3}), \lambda_k(y_k)), f^r(t_{i+3}, y(t_{i+3}), \lambda_k(y_k))\}. \end{aligned}$$

Also integrate equation (8) from t_{i+3} to t_{i+4} we get

$$\tilde{y}(t_{i+4}) = \tilde{y}(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}} \tilde{f}(t, y_k(t), \lambda_k(y_k)) dt. \quad (9)$$

By fuzzy interpolation of $\tilde{f}(t_{i-1}, (t_{i-1}), \lambda_k(y_k)), \tilde{f}(t_i, (t_i), \lambda_k(y_k)), \tilde{f}(t_{i+1}, (t_{i+1}), \lambda_k(y_k)), \tilde{f}(t_{i+2}, (t_{i+2}), \lambda_k(y_k)), \tilde{f}(t_{i+3}, (t_{i+3}), \lambda_k(y_k))$, we have;

$$\begin{aligned} f^l(t, y_k(t), \lambda_k(y_k)) &= \sum_{j=i-1, l_j(t) \geq 0}^{i+3} l_j(t) f^l(t_{k,j}, y(t_{k,j}), \lambda_k(y_k)) \\ &+ \sum_{j=i-1, l_i(t) < 0}^{i+2} l_j(t) f^r(t_{k,j}, y(t_{k,j}), \lambda_k(y_k)), \end{aligned}$$

$$\begin{aligned}
 f^c(t, y_k(t), \lambda_k(y_k)) &= \sum_{j=i-1}^{i+3} l_j(t) f^c(t_{k,j}, y(t_{k,j}), \lambda_k(y_k)), \\
 f^r(t, y_k(t), \lambda_k(y_k)) &= \sum_{j=i-1, l_j(t) \geq 0}^{i+3} l_j(t) f^r(t_{k,j}, y(t_{k,j}), \lambda_k(y_k)) \\
 &+ \sum_{j=i-1, l_j(t) < 0}^{i+3} l_j(t) f^l(t_{k,j}, y(t_{k,j}), \lambda_k(y_k)),
 \end{aligned}$$

for $t_{i+3} \leq t \leq t_{i+4}$:

$$\begin{aligned}
 l_{i-1}(t) &= \frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})(t_{i-1}-t_{i+3})} \geq 0, \\
 l_i(t) &= \frac{(t-t_{i-1})(t-t_{i+1})(t-t_{i+2})(t-t_{i+3})}{(t_i-t_{i-1})(t_i-t_{i+1})(t_i-t_{i+2})(t_i-t_{i+3})} \leq 0, \\
 l_{i+1}(t) &= \frac{(t-t_{i-1})(t-t_i)(t-t_{i+2})(t-t_{i+3})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})} \geq 0, \\
 l_{i+2}(t) &= \frac{(t-t_{i-1})(t-t_i)(t-t_{i+1})(t-t_{i+3})}{(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})} \leq 0, \\
 l_{i+3}(t) &= \frac{(t-t_{i-1})(t-t_i)(t-t_{i+1})(t-t_{i+2})}{(t_{i+3}-t_{i-1})(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})} \geq 0,
 \end{aligned}$$

therefore the following results will be obtained:

$$\begin{aligned}
 f^l(t, y(t), \lambda_k(x_k)) &= l_{i-1}(t) f^l(t_{i-1}, y(t_{i-1}), \lambda_k(x_k)) + l_i(t) f^r(t_i, y(t_i), \lambda_k(x_k)) \\
 &+ l_{i+1}(t) f^l(t_{i+1}, y(t_{i+1}), \lambda_k(x_k)) + l_{i+2}(t) f^r(t_{i+2}, y(t_{i+2}), \lambda_k(x_k)) \\
 &+ l_{i+3}(t) f^l(t_{i+3}, y(t_{i+3}), \lambda_k(x_k)), \tag{10}
 \end{aligned}$$

$$\begin{aligned}
 f^c(t, y(t), \lambda_k(x_k)) &= l_{i-1}(t) f^c(t_{i-1}, y(t_{i-1}), \lambda_k(x_k)) + l_i(t) f^c(t_i, y(t_i), \lambda_k(x_k)) \\
 &+ l_{i+1}(t) f^c(t_{i+1}, y(t_{i+1}), \lambda_k(x_k)) + l_{i+2}(t) f^c(t_{i+2}, y(t_{i+2}), \lambda_k(x_k)) \\
 &+ l_{i+3}(t) f^c(t_{i+3}, y(t_{i+3}), \lambda_k(x_k)), \tag{11}
 \end{aligned}$$

$$\begin{aligned}
 f^r(t, y(t), \lambda_k(x_k)) &= l_{i-1}(t) f^r(t_{i-1}, y(t_{i-1}), \lambda_k(x_k)) + l_i(t) f^l(t_i, y(t_i), \lambda_k(x_k)) \\
 &+ l_{i+1}(t) f^r(t_{i+1}, y(t_{i+1}), \lambda_k(x_k)) + l_{i+2}(t) f^l(t_{i+2}, y(t_{i+2}), \lambda_k(x_k)) \\
 &+ l_{i+3}(t) f^r(t_{i+3}, y(t_{i+3}), \lambda_k(x_k)). \tag{12}
 \end{aligned}$$

From (4.3) and (4.5) it follows that:

$$\tilde{y}^\alpha(t_{i+4}) = [\underline{y}^\alpha(t_{i+4}), \bar{y}^\alpha(t_{i+4})],$$

where

$$\underline{y}^\alpha(t_{i+4}) = \underline{y}^\alpha(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}} \{ \alpha f^c(t, y(t), \lambda_k(y_k)) + (1-\alpha) f^l(t, y(t), \lambda_k(y_k)) \} dt \tag{13}$$

$$\bar{y}^\alpha(t_{i+4}) = \bar{y}^\alpha(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}} \{ \alpha f^c(t, y(t), \lambda_k(y_k)) + (1-\alpha) f^r(t, y(t), \lambda_k(y_k)) \} dt, \tag{14}$$

If equations (4.6) and (4.7) are used in (4.9) and (4.7),(4.8) in (4.10)

$$\begin{aligned}\underline{y}^\alpha(t_{i+4}) &= \underline{y}^\alpha(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}} \left\{ \alpha[l_{i-1}(t)f^c(t_{i-1}, y(t_{i-1}), \lambda_k(y_k)) + l_i(t)f^c(t_i, y(t_i), \lambda_k(y_k)) \right. \\ &\quad + l_{i+1}(t)f^c(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) + l_{i+2}(t)f^c(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) \\ &\quad + l_{i+3}(t)f^c(t_{i+3}, y(t_{i+3}), \lambda_k(y_k))] + (1-\alpha)[l_{i-1}(t)f^r(t_{i-1}, y(t_{i-1}), \lambda_k(y_k)) \\ &\quad + l_i(t)f^l(t_i, y(t_i), \lambda_k(y_k)) + l_{i+1}(t)f^r(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) \\ &\quad \left. + l_{i+2}(t)f^l(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) + l_{i+3}(t)f^l(t_{i+3}, y(t_{i+3}), \lambda_k(y_k))] \right\} dt, \\ \overline{y}^\alpha(t_{i+4}) &= \overline{y}^\alpha(t_{i+3}) + \int_{t_{i+3}}^{t_{i+4}} \left\{ \alpha[l_{i-1}(t)f^c(t_{i-1}, y(t_{i-1}), \lambda_k(y_k)) + l_i(t)f^c(t_i, y(t_i), \lambda_k(y_k)) \right. \\ &\quad + l_{i+1}(t)f^c(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) + l_{i+2}(t)f^c(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) \\ &\quad + l_{i+3}(t)f^c(t_{i+3}, y(t_{i+3}), \lambda_k(y_k))] + (1-\alpha)[l_{i-1}(t)f^l(t_{i-1}, y(t_{i-1}), \lambda_k(y_k)) \\ &\quad + l_i(t)f^r(t_i, y(t_i), \lambda_k(y_k)) + l_{i+1}(t)f^l(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) \\ &\quad \left. + l_{i+2}(t)f^r(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) + l_{i+3}(t)f^r(t_{i+3}, y(t_{i+3}), \lambda_k(y_k))] \right\} dt.\end{aligned}$$

The following results will be obtained by integration:

$$\begin{aligned}\underline{y}^\alpha(t_{i+4}) &= \underline{y}^\alpha(t_{i+3}) + \frac{1901h}{720} [\alpha f^c(t_{i+3}, y(t_{i+3}), \lambda_k(y_k)) + (1-\alpha)f^l(t_{i+3}, y(t_{i+3}), \lambda_k(y_k))] \\ &\quad - \frac{2774h}{720} [\alpha f^c(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) + (1-\alpha)f^r(t_{i+2}, y(t_{i+2}), \lambda_k(y_k))] \\ &\quad + \frac{2616h}{720} [\alpha f^c(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) + (1-\alpha)f^l(t_{i+1}, y(t_{i+1}), \lambda_k(y_k))] \\ &\quad - \frac{1274h}{720} [\alpha f^c(t_i, y(t_i), \lambda_k(y_k)) + (1-\alpha)f^r(t_i, y(t_i), \lambda_k(y_k))] \\ &\quad + \frac{251h}{720} [\alpha f^c(t_{i-1}, y(t_{i-1}), \lambda_k(y_k)) + (1-\alpha)f^r(t_{i-1}, y(t_{i-1}), \lambda_k(y_k))], \\ \overline{y}^\alpha(t_{i+4}) &= \overline{y}^\alpha(t_{i+3}) + \frac{1901h}{720} [\alpha f^c(t_{i+3}, y(t_{i+3}), \lambda_k(y_k)) + (1-\alpha)f^r(t_{i+3}, y(t_{i+3}), \lambda_k(y_k))] \\ &\quad - \frac{2774h}{720} [\alpha f^c(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) + (1-\alpha)f^l(t_{i+2}, y(t_{i+2}), \lambda_k(y_k))] \\ &\quad + \frac{2616h}{720} [\alpha f^c(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) + (1-\alpha)f^r(t_{i+1}, y(t_{i+1}), \lambda_k(y_k))] \\ &\quad - \frac{1274h}{720} [\alpha f^c(t_i, y(t_i), \lambda_k(y_k)) + (1-\alpha)f^l(t_i, y(t_i), \lambda_k(y_k))] \\ &\quad + \frac{251h}{720} [\alpha f^c(t_{i-1}, y(t_{i-1}), \lambda_k(y_k)) + (1-\alpha)f^l(t_{i-1}, y(t_{i-1}), \lambda_k(y_k))].\end{aligned}$$

Thus

$$\begin{aligned}
 \underline{y}^\alpha(t_{i+4}) &= \underline{y}^\alpha(t_{i+3}) + \frac{h}{720} [1901 \underline{f}^\alpha(t_{i+3}, y(t_{i+3}), \lambda_k(y_k)) - 2774 \bar{f}^\alpha(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) \\
 &\quad + 2616 \underline{f}^\alpha(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) - 1274 \bar{f}^\alpha(t_i, y(t_i), \lambda_k(y_k)) \\
 &\quad + 251 \bar{f}^\alpha(t_{i-1}, y(t_{i-1}), \lambda_k(y_k))], \\
 \bar{y}^\alpha(t_{i+4}) &= \bar{y}^\alpha(t_{i+3}) + \frac{h}{720} [1901 \bar{f}^\alpha(t_{i+3}, y(t_{i+3}), \lambda_k(y_k)) - 2774 \underline{f}^\alpha(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) \\
 &\quad + 2616 \bar{f}^\alpha(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) - 1274 \underline{f}^\alpha(t_i, y(t_i), \lambda_k(y_k)) \\
 &\quad + 251 \underline{f}^\alpha(t_{i-1}, y(t_{i-1}), \lambda_k(y_k))].
 \end{aligned} \tag{15}$$

Therefore Adams-Basforth five-step method is obtained as follows:

$$\left\{ \begin{aligned}
 \underline{y}^\alpha(t_{i+4}) &= \underline{y}^\alpha(t_{i+3}) + \frac{h}{720} [1901 \underline{f}^\alpha(t_{i+3}, y(t_{i+3}), \lambda_k(y_k)) - 2774 \bar{f}^\alpha(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) \\
 &\quad + 2616 \underline{f}^\alpha(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) - 1274 \bar{f}^\alpha(t_i, y(t_i), \lambda_k(y_k)) \\
 &\quad + 251 \bar{f}^\alpha(t_{i-1}, y(t_{i-1}), \lambda_k(y_k))], \\
 \bar{y}^\alpha(t_{i+4}) &= \bar{y}^\alpha(t_{i+3}) + \frac{h}{720} [1901 \bar{f}^\alpha(t_{i+3}, y(t_{i+3}), \lambda_k(y_k)) - 2774 \underline{f}^\alpha(t_{i+2}, y(t_{i+2}), \lambda_k(y_k)) \\
 &\quad + 2616 \bar{f}^\alpha(t_{i+1}, y(t_{i+1}), \lambda_k(y_k)) - 1274 \underline{f}^\alpha(t_i, y(t_i), \lambda_k(y_k)) \\
 &\quad + 251 \underline{f}^\alpha(t_{i-1}, y(t_{i-1}), \lambda_k(y_k))], \\
 \underline{y}^\alpha(t_{i-1}) &= \alpha_0, \underline{y}^\alpha(t_i) = \alpha_1, \underline{y}^\alpha(t_{i+1}) = \alpha_2, \underline{y}^\alpha(t_{i+2}) = \alpha_3, \underline{y}^\alpha(t_{i+3}) = \alpha_4, \\
 \bar{y}^\alpha(t_{i-1}) &= \alpha_5, \bar{y}^\alpha(t_i) = \alpha_6, \bar{y}^\alpha(t_{i+1}) = \alpha_7, \bar{y}^\alpha(t_{i+2}) = \alpha_8, \bar{y}^\alpha(t_{i+3}) = \alpha_9.
 \end{aligned} \right. \tag{16}$$

5. Adams-Moulton five step methods

Fix $k \in Z^+$. The fuzzy initial value problem (4.4) can be solved by Adams-Moulton four-step method. The Adams-Moulton four step method is obtained as follows:

$$\left\{ \begin{aligned}
 \underline{y}^\alpha(t_{k,i+4}) &= \underline{y}^\alpha(t_{k,i+3}) + \frac{h}{720} [251 \underline{f}^\alpha(t_{k,i+4}, y(t_{k,i+4}), \lambda_k(y_k)) \\
 &\quad + 646 \underline{f}^\alpha(t_{k,i+3}, y(t_{k,i+3}), \lambda_k(y_k)) - 264 \bar{f}^\alpha(t_{k,i+2}, y(t_{k,i+2}), \lambda_k(y_k)) \\
 &\quad + 106 \underline{f}^\alpha(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)) - 19 \underline{f}^\alpha(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))], \\
 \bar{y}^\alpha(t_{k,i+4}) &= \bar{y}^\alpha(t_{k,i+3}) + \frac{h}{720} [251 \bar{f}^\alpha(t_{k,i+4}, y(t_{k,i+4}), \lambda_k(y_k)) \\
 &\quad + 646 \bar{f}^\alpha(t_{k,i+3}, y(t_{k,i+3}), \lambda_k(y_k)) - 264 \underline{f}^\alpha(t_{k,i+2}, y(t_{k,i+2}), \lambda_k(y_k)) \\
 &\quad + 106 \bar{f}^\alpha(t_{k,i+1}, y(t_{k,i+1}), \lambda_k(y_k)) - 19 \underline{f}^\alpha(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))], \\
 \underline{y}^\alpha(t_{k,i}) &= \alpha_0, \underline{y}^\alpha(t_{k,i+1}) = \alpha_1, \underline{y}^\alpha(t_{k,i+2}) = \alpha_2, \underline{y}^\alpha(t_{k,i+3}) = \alpha_3 \\
 \bar{y}^\alpha(t_{k,i}) &= \alpha_4, \bar{y}^\alpha(t_{k,i+1}) = \alpha_5, \bar{y}^\alpha(t_{k,i+2}) = \alpha_6, \bar{y}^\alpha(t_{k,i+3}) = \alpha_7
 \end{aligned} \right. \tag{17}$$

6. Adams fifth order predictor-corrector method.

The following algorithm is based on Adams-Bashforth five-step method as a predictor and also an iteration of Adams-Moulton four-step method as a corrector.

ALGORITHM:

Fix $k \in Z^+$. To approximate the solution of following fuzzy initial value problem.

$$x'_k(t) = f(t_{k,i}, y(t_{k,i}), \lambda_k(y_k))$$

$$\underline{y}^\alpha(t_{k,i-1}) = \underline{\alpha}_0, \underline{y}^\alpha(t_{k,i}) = \underline{\alpha}_1, \underline{y}^\alpha(t_{k,i+1}) = \underline{\alpha}_2, \underline{y}^\alpha(t_{k,i+2}) = \underline{\alpha}_3, \underline{y}^\alpha(t_{k,i+3}) = \underline{\alpha}_4.$$

$$\overline{y}^\alpha(t_{k,i-1}) = \overline{\alpha}_0, \overline{y}^\alpha(t_{k,i}) = \overline{\alpha}_1, \overline{y}^\alpha(t_{k,i+1}) = \overline{\alpha}_2, \overline{y}^\alpha(t_{k,i+2}) = \overline{\alpha}_3, \overline{y}^\alpha(t_{k,i+3}) = \overline{\alpha}_4.$$

positive integer N_k is chosen.

Step 1. Let $h = \frac{t_{k+1} - t_k}{N_k}$,

$$\underline{w}^\alpha(t_{k,0}) = \underline{\alpha}_0, \underline{w}^\alpha(t_{k,1}) = \underline{\alpha}_1, \underline{w}^\alpha(t_{k,2}) = \underline{\alpha}_2, \underline{w}^\alpha(t_{k,3}) = \underline{\alpha}_3, \underline{w}^\alpha(t_{k,4}) = \underline{\alpha}_4,$$

$$\overline{w}^\alpha(t_{k,0}) = \overline{\alpha}_0, \overline{w}^\alpha(t_{k,1}) = \overline{\alpha}_1, \overline{w}^\alpha(t_{k,2}) = \overline{\alpha}_2, \overline{w}^\alpha(t_{k,3}) = \overline{\alpha}_3, \overline{w}^\alpha(t_{k,4}) = \overline{\alpha}_4,$$

Step 2. Let $i = 1$,

Step 3. Let

$$\left\{ \begin{array}{l} \underline{w}^{(0)\alpha}(t_{i+4}) = \underline{w}^\alpha(t_{i+3}) + \frac{h}{720}[1901\underline{f}^\alpha(t_{i+3}, w(t_{i+3}), \lambda_k(y_k)) - 2774\underline{f}^\alpha(t_{i+2}, w(t_{i+2}), \lambda_k(y_k)) \\ \quad + 2616\underline{f}^\alpha(t_{i+1}, w(t_{i+1}), \lambda_k(y_k)) - 1274\underline{f}^\alpha(t_i, w(t_i), \lambda_k(y_k)) + 251\underline{f}^\alpha(t_{i-1}, w(t_{i-1}), \lambda_k(y_k))], \\ \overline{w}^{(0)\alpha}(t_{i+4}) = \overline{w}^\alpha(t_{i+3}) + \frac{h}{720}[1901\overline{f}^\alpha(t_{i+3}, w(t_{i+3}), \lambda_k(y_k)) - 2774\overline{f}^\alpha(t_{i+2}, w(t_{i+2}), \lambda_k(y_k)) \\ \quad + 2616\overline{f}^\alpha(t_{i+1}, w(t_{i+1}), \lambda_k(y_k)) - 1274\overline{f}^\alpha(t_i, w(t_i), \lambda_k(y_k)) + 251\overline{f}^\alpha(t_{i-1}, w(t_{i-1}), \lambda_k(y_k))]. \end{array} \right.$$

Step 4. Let $t_{i+4} = t_0 + (i+4)h$.

Step 5. Let

$$\left\{ \begin{array}{l} \underline{w}^\alpha(t_{i+3}) = \underline{y}^\alpha(t_{i+2}) + \frac{h}{720}[251\underline{f}^\alpha(t_{i+3}, w(t_{i+3}), \lambda_k(y_k)) + 646\underline{f}^\alpha(t_{i+2}, w(t_{i+2}), \lambda_k(y_k)) \\ \quad - 264\underline{f}^\alpha(t_{i+1}, w(t_{i+1}), \lambda_k(y_k)) + 106\underline{f}^\alpha(t_i, w(t_i), \lambda_k(y_k)) - 19\underline{f}^\alpha(t_{i-1}, w(t_{i-1}), \lambda_k(y_k))], \\ \overline{w}^\alpha(t_{i+3}) = \overline{w}^\alpha(t_{i+2}) + \frac{h}{720}[251\overline{f}^\alpha(t_{i+3}, w(t_{i+3}), \lambda_k(y_k)) + 646\overline{f}^\alpha(t_{i+2}, w(t_{i+2}), \lambda_k(y_k)) \\ \quad - 264\overline{f}^\alpha(t_{i+1}, w(t_{i+1}), \lambda_k(y_k)) + 106\overline{f}^\alpha(t_i, w(t_i), \lambda_k(y_k)) - 19\overline{f}^\alpha(t_{i-1}, w(t_{i-1}), \lambda_k(y_k))], \end{array} \right.$$

Step 6. $i = i + 1$.

Step 7. If $i \leq N - 3$ go to step 3.

Step 8. Algorithm will be completed and $(\underline{w}^\alpha(t_{k+1}), \overline{w}^\alpha(t_{k+1}))$ approximates real value of $(\underline{x}^\alpha(t_{k+1}), \overline{x}^\alpha(t_{k+1}))$.

7. Convergence and Stability

For a fixed α , to integrate the system (15) in $[t_0, t_1], [t_1, t_2], \dots, [t_k, t_{k+1}], \dots$, we replace each interval by a set of discrete equally spaced grid points $t_1 < t_2 < \dots < t_N = T$ (including the end points) at which the exact solution $(\underline{x}(t, \alpha), \overline{x}(t, \alpha))$ is approximated by some $(\underline{y}(t, \alpha), \overline{y}(t, \alpha))$. The grid points which the solution is calculated are at $[t_k, t_{k+1}]$ at $t_{k,n} = t_k + nh_k, h_k = (t_{k+1} -$

$t_k)/N_k, 0 \leq n \leq N_k$, the exact and approximate solutions are denoted by $x_{k,n}(\alpha) = [\underline{x}_{k,n}(\alpha), \bar{x}_{k,n}(\alpha)]$ and $y_{k,n}(\alpha) = [\underline{y}_{k,n}(\alpha), \bar{y}_{k,n}(\alpha)]$, respectively. However, we will use

$$\underline{y}_{0,0}(\alpha) = \underline{\alpha}_0, \quad \underline{y}_{0,1}(\alpha) = \underline{\alpha}_1, \quad \underline{y}_{0,2}(\alpha) = \underline{\alpha}_2, \quad \underline{y}_{0,3}(\alpha) = \underline{\alpha}_3,$$

$$\bar{y}_{0,0}(\alpha) = \bar{\alpha}_0, \quad \bar{y}_{0,1}(\alpha) = \bar{\alpha}_1, \quad \bar{y}_{0,2}(\alpha) = \bar{\alpha}_2, \quad \bar{y}_{0,3}(\alpha) = \bar{\alpha}_3,$$

and

$$\underline{y}_{k,0}(\alpha) = \underline{y}_{k-1,N_{k-1}-1}(\alpha), \quad \underline{y}_{k,1}(\alpha) = \underline{y}_{k-1,N_{k-1}}(\alpha), \quad \underline{y}_{k,2}(\alpha) = \underline{y}_{k-1,N_{k-1}+1}(\alpha),$$

$$\bar{y}_{k,0}(\alpha) = \bar{y}_{k-1,N_{k-1}-1}(\alpha), \quad \bar{y}_{k,1}(\alpha) = \bar{y}_{k-1,N_{k-1}}(\alpha), \quad \bar{y}_{k,2}(\alpha) = \bar{y}_{k-1,N_{k-1}+1}(\alpha).$$

Theorem 7.1. For arbitrary fixed $\alpha : 0 \leq \alpha \leq 1$ and $k \in Z^+$, the Adams-Moulton four-step approximates equation (15) converge to the exact solutions $\underline{x}(t_{k+1}, \alpha)$, $\bar{x}(t_{k+1}, \alpha)$.

Proof. It is sufficient to show

$$\lim_{h_0, \dots, h_k} \underline{y}_{k,N_k}(\alpha) = \underline{x}(t_{k+1}, \alpha), \quad \lim_{h_0, \dots, h_k} \bar{y}_{k,N_k}(\alpha) = \bar{x}(t_{k+1}, \alpha)$$

Remark 7.1. The convergence order of Adams-Moulton four-step method is $O(h^4)$.

Theorem 7.2. For arbitrary fixed $\alpha : 0 \leq \alpha \leq 1$ and $k \in Z^+$, the Adams-Bashforth five-step approximates equation (15) converge to the exact solutions $\underline{x}(t_{k+1}, \alpha)$, $\bar{x}(t_{k+1}, \alpha)$.

Remark 7.2. The convergence order of Adams-Bashforth five-step method is $O(h^4)$.

Theorem 7.3. Fix $k \in Z^+$, the Adams-Bashforth four and five-step methods are stable.

Proof. Fix $k \in Z^+$. For Adams-Bashforth four-step method, there exist only one characteristic polynomial $p(\lambda) = \lambda^3 - \lambda^2$ and it is clear that satisfies the root condition by Theorem 7.1; then the method is stable.

Also, for Adams-Bashforth five-step method, there exist only one characteristic polynomial $p(\lambda) = \lambda^4 - \lambda^3$ and it satisfies the root condition, therefore it is a stable.

Theorem 7.4. Fix $k \in Z^+$, the Adams-Moulton four and five-step methods are stable.

Proof. Similar to Theorem 7.3.

8. Numerical Examples

Consider the fuzzy initial value problem,

$$\begin{aligned} \tilde{x}'(t) &= -\tilde{x}(t), \\ \tilde{x} &= [0.75, 1, 1.125]. \end{aligned} \tag{18}$$

We translate the problem (16) into the following system of equations

$$\begin{aligned} x^{l'}(t) &= -x^r(t), \\ x^{c'}(t) &= -x^c(t), \\ x^{r'}(t) &= -x^l(t), \\ x^l(0) &= 0.75, \quad x^c(0) = 1, \quad x^r(0) = 1.125. \end{aligned} \tag{19}$$

and its solution is $x^l(t) = -0.1875e^t + 0.9375e^{-t}$, $x^c(t) = e^{-t}$, $x^r(t) = 0.1875e^t + 0.9375e^{-t}$. By [3], the exact solution of problem (16) is

$$x(t) = [-0.1875e^t + 0.9375e^{-t}, \quad e^{-t}, \quad 0.1875e^t + 0.9375e^{-t}]$$

which compares well with the predictor-corrector method.

Example 8.1 Consider the fuzzy initial value problem,

$$\begin{cases} \tilde{x}'(t) &= -\tilde{x}(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_{k+1}], \quad t_k, t_{k+1}], \quad t_k = k, \quad k = 0, 1, 2, \dots, \\ \tilde{x}(0) &= [0.75, 1, 1.125], \\ \tilde{x}(0.1) &= [-0.1875e^{0.1} + 0.9375e^{-0.1}, \quad e^{0.1}, \quad 0.1875e^{0.1} + 0.9375e^{-0.1}], \\ \tilde{x}(0.2) &= [-0.1875e^{0.2} + 0.9375e^{-0.2}, \quad e^{0.2}, \quad 0.1875e^{0.2} + 0.9375e^{-0.2}], \\ \tilde{x}(0.3) &= [-0.1875e^{0.3} + 0.9375e^{-0.3}, \quad e^{0.3}, \quad 0.1875e^{0.3} + 0.9375e^{-0.3}], \\ \tilde{x}(0.4) &= [-0.1875e^{0.4} + 0.9375e^{-0.4}, \quad e^{0.4}, \quad 0.1875e^{0.4} + 0.9375e^{-0.4}], \end{cases} \quad (20)$$

where

$$m(t) = |\sin(\pi t)|, \quad k = 0, 1, 2, \dots \quad (21)$$

$$\lambda(t) = \begin{cases} 0, & \text{if } k = 0 \\ \mu, & k \in \{1, 2, \dots\} \end{cases} \quad (22)$$

The hybrid fuzzy initial value problem (18) is equivalent to the following system of fuzzy initial value problems:

$$\begin{cases} \tilde{x}_0^{l'}(t) &= -\tilde{x}_0^r(t), \\ \tilde{x}_0^c(t) &= -\tilde{x}_0^c(t), \\ \tilde{x}_0^{r'}(t) &= -\tilde{x}_0^l(t), \quad t \in [0, 1] \\ \tilde{x}(0) &= [0.75, 1, 1.125], \\ \tilde{x}(0.1) &= [-0.1875e^{0.1} + 0.9375e^{-0.1}, \quad e^{-0.1}, \quad 0.1875e^{0.1} + 0.9375e^{-0.1}], \\ \tilde{x}(0.2) &= [-0.1875e^{0.2} + 0.9375e^{-0.2}, \quad e^{-0.2}, \quad 0.1875e^{0.2} + 0.9375e^{-0.2}], \\ \tilde{x}(0.3) &= [-0.1875e^{0.3} + 0.9375e^{-0.3}, \quad e^{-0.3}, \quad 0.1875e^{0.3} + 0.9375e^{-0.3}], \\ \tilde{x}(0.4) &= [-0.1875e^{0.4} + 0.9375e^{-0.4}, \quad e^{-0.4}, \quad 0.1875e^{0.4} + 0.9375e^{-0.4}], \\ \tilde{x}^{l'}(t_i) &= -\tilde{x}_i^r(t_i) + m(t)x_i^l(t_i), \\ \tilde{x}^c(t_i) &= -\tilde{x}_i^c(t_i) + m(t)x_i^c(t_i) \\ \tilde{x}^{r'}(t_i) &= -\tilde{x}_i^l(t_i) + m(t)x_i^r(t_i) \quad t \in [t_i, t_{i+1}], \\ x_i(t_i) &= x_{i-1}(t_i), x_i(t_{i-1}) = x_{i-1}(t_{i-1}), x_i(t_{i-2}) = x_{i-1}(t_{i-2}), \quad i = 1, 2, \dots \end{cases} \quad (23)$$

For $t \in [0, 1]$, the exact solution of equation (18) satisfies

$$x(t) = [-0.1875e^t + 0.9375e^{-t}, \quad e^{-t}, \quad 0.1875e^t + 0.9375e^{-t}]. \quad (24)$$

For $t \in [1, 2]$, the exact solution of equation (18) satisfies,

$$x(t)^T = \begin{pmatrix} -0.1875 \left[e^t + \frac{1}{1 + \pi^2} (e(\sin(\pi t) + \pi \cos(\pi t)) + \pi e^t) \right] \\ + 0.9375 \left[e^{-t} - \frac{1}{1 + \pi^2} (e^{-1}(\sin(\pi t) - \pi \cos(\pi t)) - \pi e^{-t}) \right] \\ e^{-t} - \frac{1}{1 + \pi^2} (e^{-1}(\sin(\pi t) - \pi \cos(\pi t)) - \pi e^{-t}) \\ 0.1875 \left[e^t + \frac{1}{1 + \pi^2} (e(\sin(\pi t) + \pi \cos(\pi t)) + \pi e^t) \right] \\ + 0.9375 \left[e^{-t} - \frac{1}{1 + \pi^2} (e^{-1}(\sin(\pi t) - \pi \cos(\pi t)) - \pi e^{-t}) \right] \end{pmatrix} \quad (25)$$

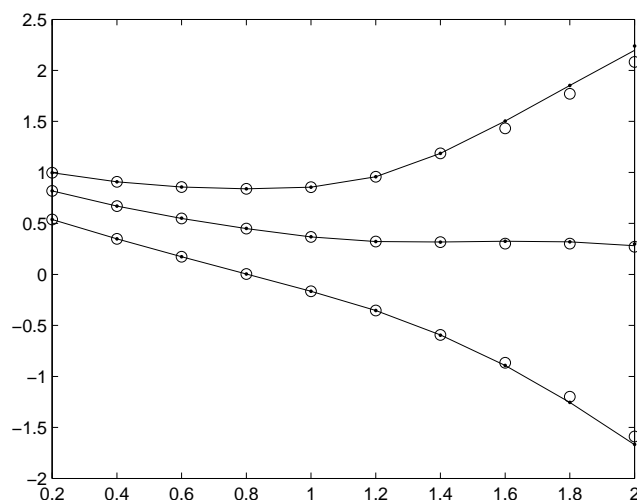


Figure 1: h=0.1

- Exact, o-Predictor corrector order 4, s-Predictor corrector order 5

The results of Example 8.1 on [1,2] are shown in Figure 1.

Example 8.2. Consider the fuzzy initial value problem,

$$\begin{cases} \tilde{x}'(t) &= \tilde{x}(t) + m(t)\lambda_k(x(t_k)), \quad t \in [t_k, t_{k+1}], \quad t_k, t_{k+1}, \quad t_k = k, \quad k = 0, 1, 2, \dots, \\ \tilde{x}(0) &= [0.75, 1, 1.125], \\ \tilde{x}(0.1) &= [0.75e^{0.1}, e^{0.1}, 1.125e^{0.1}], \\ \tilde{x}(0.2) &= [0.75e^{0.2}, e^{0.2}, 1.125e^{0.2}], \\ \tilde{x}(0.3) &= [0.75e^{0.3}, e^{0.3}, 1.125e^{0.3}], \\ \tilde{x}(0.4) &= [0.75e^{0.4}, e^{0.4}, 1.125e^{0.4}], \end{cases} \quad (26)$$

where

$$m(t) = \begin{cases} 2(t(\text{mod}1)), & \text{if } t(\text{mod}1) \leq 0.5 \\ 2(1 - t(\text{mod}1)), & \text{if } t(\text{mod}1) > 0.5, \end{cases} \quad (27)$$

$$\lambda(t) = \begin{cases} 0, & \text{if } k = 0 \\ \mu, & k \in \{1, 2, \dots\}. \end{cases} \quad (28)$$

The hybrid fuzzy initial value problem (24) is equivalent to the following system of fuzzy initial value problems:

$$\begin{cases} \tilde{x}'_0(t) &= \tilde{x}_0(t), \quad t \in [0, 1] \\ \tilde{x}(0) &= [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], \\ \tilde{x}(0.1) &= [(0.75 + 0.25\alpha)e^{0.1}, (1.125 - 0.125\alpha)e^{0.1}], \\ \tilde{x}(0.2) &= [(0.75 + 0.25\alpha)e^{0.2}, (1.125 - 0.125\alpha)e^{0.2}], \\ \tilde{x}(0.3) &= [(0.75 + 0.25\alpha)e^{0.3}, (1.125 - 0.125\alpha)e^{0.3}], \\ \tilde{x}(0.4) &= [(0.75 + 0.25\alpha)e^{0.4}, (1.125 - 0.125\alpha)e^{0.4}], \quad 0 \leq \alpha \leq 1. \\ \tilde{x}'(t_i) &= \tilde{x}_i(t_i) + m(t)x_i(t_i), \quad t \in [t_i, t_{i+1}], \\ x_i(t_i) &= x_{i-1}(t_i), x_i(t_i - 1) = x_{i-1}(t_i - 1), x_i(t_i - 2) = x_{i-1}(t_i - 2), \quad i = 1, 2, \dots \end{cases} \quad (29)$$

In equation(24), $x(t) + m(t)\lambda_k(x(t_k))$ is a continuous function of t, x and $\lambda_k(x(t_k))$. Therefore, by Example 6.1 Of Kaleva [13], for each $k = 0, 1, 2, \dots$, fuzzy initial value problem

$$\begin{cases} \tilde{x}'(t_i) &= \tilde{x}_i(t_i) + m(t)x_i(t_i), \quad t \in [t_i, t_{i+1}], \\ x_i(t_i) &= x_{i-1}(t_k), \\ x(t_k - 1) &= x_{t_k} - 1, \\ x(t_k - 2) &= x_{t_k} - 2, \quad i = 1, 2, \dots \end{cases} \quad (30)$$

has a unique solution $[t_k, t_{k+1}]$.

For $t \in [0, 1]$, the exact solution of equation (24) satisfies

$$x(t) = [0.75e^t, \quad e^t, \quad 1.125e^t]. \quad (31)$$

For $t \in [1, 2]$, the exact solution of equation (24) satisfies,

$$x(t) = x(1)(2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)). \quad (32)$$

The results of Example 8.2 [1,2] are shown in Figure 2.

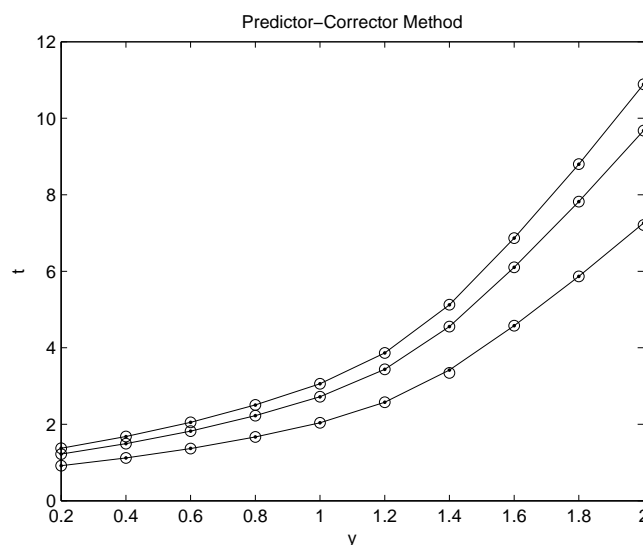


Figure 2: $h=0.1$
- Exact, o-Predictor corrector order 4, .-Predictor corrector order 5

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