A Class Of Diameter Six Trees with Graceful Labeling

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Abstract

Here we denote a *diameter six tree* by $(a_0;\ a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots,\ b_n;\ c_1, c_2, \ldots, c_r)$, where a_0 is the center of the tree; $a_i,\ i = 1, 2, \ldots, m$, $b_j,\ j = 1, 2, \ldots, n$, and $c_k,\ k = 1, 2, \ldots, r$ are the vertices of the tree adjacent to a_0; each a_i is the center of a diameter four tree, each b_j is the center of a star, and each c_k is a pendant vertex. Here we give graceful labelings to some new classes of diameter six trees $(a_0; a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots,\ b_n;\ c_1, c_2, \ldots, c_r)$ in which we find diameter four trees consisting of four different combinations of odd, even, and pendant branches with the total number of branches odd. Here by a branch we mean a star, i.e. we call a star an odd branch if its center has an odd degree and an even branch if its center has an even degree.

Keywords: graceful labeling, diameter six tree, component moving transformation, transfers of the first and second types, BD8TF

1 Introduction

Definition 1.1. A *diameter six tree* is a tree which has a representation of the form
$(a_0;\ a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots, b_n;\ c_1, c_2, \ldots, c_r)$, where a_0 is the center of the tree; $a_i,\ i = 1, 2, \ldots, m$, $b_j,\ j = 1, 2, \ldots, n$, and $c_k,\ k = 1, 2, \ldots, r$ are the vertices of the tree adjacent to a_0; each a_i is the center of a diameter four tree, each b_j is the center of a star, and each c_k is a pendant vertex. We observe that in a diameter six tree with above representation $m \geq 2$, i.e. there should be at least two (vertices) a_i s adjacent to c which are the centers of diameter four trees. Here we use the notation D_6 to denote a diameter six tree. In the literature [1, 3, 5, 7, 6, 8, 9, 10, 17, 23] we find that all trees up to diameter five are graceful. As far as diameter six trees are concerned only *banana trees* are graceful [2, 4, 5, 7, 9, 11, 12, 13, 19, 20, 17, 22, 21, 24]. From literature [4] a *banana tree* is a tree obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). Chen et.al. [4] conjectured that banana trees are graceful. Bhat-Nayak and Deshmukh [2], Murugan and Arumugam [13, 12] and Vilfred [22, 21] gave graceful labelings to different classes of banana trees. Sethuraman and Jesintha [9, 11, 19, 20]) proved that all banana trees and extended banana trees (graphs obtained by joining a vertex to one leaf of each of any number of stars by a path of length of at least two) are graceful. Here we give graceful labelings to some new classes of diameter six trees $a_1, a_2, \ldots, a_m;\ b_1, b_2, \ldots,\ b_n;\ c_1, c_2, \ldots, c_r)$ in which the branches of some diameter four tree are all even, whereas the branches of the remaining diameter four trees are all odd. Here by a branch we mean a star, i.e. we call a star an odd branch if its center has an odd degree and an even branch if its center has an degree.

2 Preliminaries

Definition 2.1. [14, 15, 16] For an edge $e = \{u, v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a component incident on the vertex v. If a and b are vertices of a tree T, $u(T)$ is a component incident on a, and $b \notin u(T)$ then deleting the edge $\{a, u\}$ from T and making b and u adjacent is termed as *the component* $u(T)$ has been transferred or moved from a to b. In this paper by the label of the component " $u(T)$ " we mean the label of the vertex u. Let T be a tree and a and b be two vertices of T. By $a \to b$ *transfer* we mean that some components from a have been moved to b. If we consider successive transfers $a_1 \to a_2$, $a_2 \to a_3$, $a_3 \to a_4$, $\ldots$ we simply write $a_1 \to a_2 \to a_3 \to a_4 \ldots$ transfer. In the transfer $a_1 \to a_2 \to \ldots \to a_{n-1} \to a_n$, each vertex a_i, $i = 1, 2, \ldots, n-1$ is called a vertex of transfer. Let T be a labelled tree with a labeling f. We consider the vertices of T whose labels form the sequence $(a, b, a-1, b+1, a-2, b+2)$ (respectively, $(a, b, a+1, b-1, a+2, b-2)$). Let a be adjacent to some vertices having labels different from the above labels. The $a \longrightarrow b$ transfer is called *a transfer of the first type* if the labels of the transferred components constitute a set of consecutive integers. The $a \longrightarrow b$ transfer is called *a transfer of the second type* if the labels of the transferred components can be divided into two segments, where each segment is a set of consecutive integers. A sequence of eight transfers of the first type $a \to b \to a-1 \to b+1 \to a \to b \to a-1 \to b+1 \to a-2$ (respectively, $a \to b \to a+1 \to b-1 \to a \to b \to a+1 \to b-1 \to a+2$), is called a *backward double 8 transfer of the first type* or $BD8TF$ a to $a-2$ (respectively, a to $a+2$).

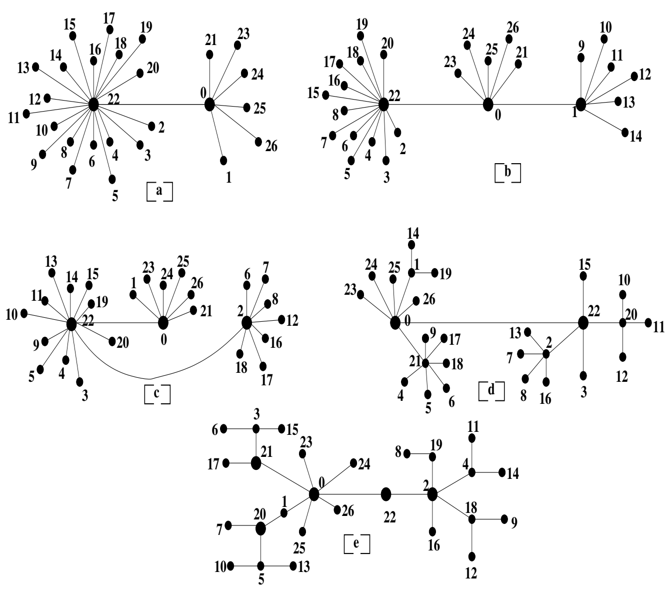


Figure 1: The graceful trees in (b), (c), d), and (e) are obtained from the graceful tree in (a) by applying transfers of the first type $22 \to 1$, the transfer of second type $22 \to 2$, BD8TF 22 to 20, and a sequence of transfers consisting of $22 \to 1 \to 21$ transfers of the first type, followed by the BD8TF 21 to 19, followed by $19 \to 4$ transfer of the first type, and finally $4 \to 18 \to 5$ transfers of the second type, respectively.

Theorem 2.2. [14, 15, 16] In a graceful labeling f of a graceful tree T, let a and b be the labels of two vertices. Let a be attached to a set A of vertices (or components) having labels $n, n+1, n+2, \ldots, n+p$ (different from the above vertex labels), which satisfy $(n+1+i)+(n+p-i) = a+b$, $i \geq 0$ (respectively, $(n+i)+(n+p-1-i) = a+b$, $i \geq 0$). Then the following hold.

(a) By making a transfer $a \to b$ of first type we can keep an odd number of components at a from the set A and move the rest to b, and the resultant tree thus formed will be graceful.

(b) If A contains an even number of elements, then by making a sequence of transfers of the second type $a \to b \to a-1 \to b+1 \to a-2 \to b+2 \to \ldots$ (respectively, $a \to b \to a+1 \to b-1 \to a+2 \to b-2 \to \ldots$), an even number of elements from A can be kept at each vertex of the transfer, and the resultant tree thus formed is graceful.

(c) By a BD8TF a to $b+1$ (respectively, $b-1$), we can keep an even number of elements from A at $a, b, a-1$, and $b+1$ (respectively, $a, b, a+1$, and $b-1$), and move the rest to $a-2$ (respectively, $a+2$). The resultant tree formed in each of the above cases is graceful.

(d) Consider the transfer $R: a \to b \to a-1 \to b+1 \to \ldots \to z$ (respectively, $a \to b \to a+1 \to b-1 \to \ldots \to z$), with $z = a-p_1$ or $b+p_2$ (respectively, $a+r_1$ or $b-r_2$), such that R is partitioned as $R: T_1 \to T_2 \to T_3 \to \ldots \to T_n$, where each T_i, $1 \leq i \leq n$, is either a transfer of the first type or BD8TF. Construct a tree T^* from T by carrying out the transfer R by successively carrying out the transfers T_1, T_2, T_3, $\ldots$ T_n. The tree T^* is graceful.

(e) Consider the transfer $R': a \to b \to a-1 \to b+1 \to \ldots \to \ldots$ (respectively, $a \to b \to a+1 \to b-1 \to \ldots \to \ldots$), such that R' is partitioned as $R': T'_1 \to T'_2$, where T'_1 is sequence of transfers consisting of the transfers of the first type and BD8TF and T_2' is a sequence of transfer of the second type. The tree T^{**} obtained from T by making the transfer R' is graceful.

Lemma 2.3. [10] If g is a graceful labeling of a tree T with n edges then the labeling g_n defined as $g_n(x) = n - g(x)$, for all $x \in V(T)$, called the *inverse transformation* of g is also a graceful labeling of T.

3 Results

Theorem 3.1. (a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$, with degree of a_i, b_j are even $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. If the branches incident on the center a_i, $i = 1, 2, \ldots, m_1, m_1 < m$ are odd and branches incident on the center $a_i, i = m_1+1, m_1+2, \ldots, m$ are even branches. Then D_6 has a graceful labeling.
(b) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$, with degree of a_i, b_j are even for $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$. If the branches incident on the center a_i, $i = 1, 2, \ldots, m_1,\ m_1 < m$ are odd and branches incident on the center $a_i, i = m_1+1, m_1+2, \ldots, m$ are even branches. Then D_6 has a graceful labeling.
(c) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; c_1, c_2, \ldots, c_r\}$, with m odd, degree of a_i, are even for $i = 1, 2, \ldots, m$. If the branches incident on the center a_i, $i = 1, 2, \ldots, m_1, m_1 < m$ are odd and branches incident on the center $a_i,\ i = m_1+1, m_1+2,\ \ldots,\ m$ are even branches. Then D_6 has a graceful labeling.
(d) $D_6 = \{a_0; a_1, a_2, \ldots, a_m\}$, with either m is odd or $m_o \geq 2$, degree of a_i are even for $i = 1, 2, \ldots, m$. If the branches incident on the center a_i, $i = 1, 2, \ldots, m_1, m_1 < m$ are odd and branches incident on the center $a_i,\ i = m_1+1, m_1+2,\ \ldots,\ m$ are even branches. Then D_6 has a graceful labeling.

Proof: (a) Case - I Let $m+n$ be odd. Let $|E(D_6)| = q$ and $deg(a_0) = m+n = 2k+1$. Let us remove the pendant vertices adjacent to a_0 and represent the new graceful tree by $D_6^{(1)}$. Consider the graceful tree G as represented in Figure 2.

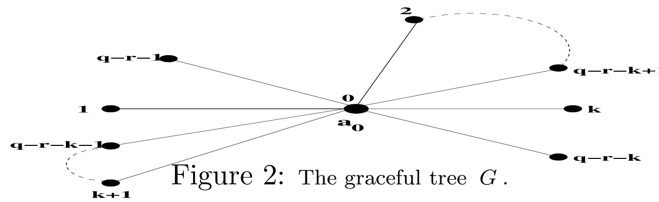


Figure 2: The graceful tree G.

Let $A = \{k+1, k+2, \ldots, q-k-r-1\}$. Observe that $(k+i)+(q-r-k-i) = q-r$. Consider the sequence of transfer $T_1 : q-r \to 1 \to q-r-1 \to 2 \to q-r-2 \to \ldots \to k \to q-r-k \to k+1$ of the first type of the vertex levels in the set A. Observe that the transfer T_1 and the set A satisfy the properties of Theorem 2.2. We execute the transfer T_1 by keeping an odd number of elements of A at each vertex of the transfer. In the transfer T_1, the first m vertices are designated as the vertices $a_1, a_2, \ldots, a_m$, respectively, and the remaining n vertices are designated as the vertices b_1, b_2, $\ldots$, b_n. Observe that

$$a_i = \begin{cases} q-r-\frac{i-1}{2} & \text{if } i \text{ is odd} \\ \frac{i}{2} & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad b_j = \begin{cases} \begin{cases} q-r-\frac{m+j-1}{2} & \text{if } j \text{ is odd} \\ \frac{m+j}{2} & \text{if } i \text{ is even} \end{cases} & \text{if } m \text{ is even} \\ \begin{cases} \frac{m+j}{2} & \text{if } j \text{ is odd} \\ q-r-\frac{m+j-1}{2} & \text{if } j \text{ is even} \end{cases} & \text{if } m \text{ is odd} \end{cases}$$

Let A_1 be the set of vertex labels of A which have come to the vertex $k+1$ after the transfer T_1. Since each transfer in T_1 is a transfer of 1st type, the elements of A_1 are the consecutive integers. Next consider the transfer $T_2 : k+1 \to q-r-k-1 \to k+2 \to q-r-k-2 \to k+3 \to q-r-k-3 \to \ldots, \to r$, where

$$r = \begin{cases} k+k_1+1; & \text{if } m \text{ is odd} \\ q-r-k-k_1; & \text{if } m \text{ is even} \end{cases}, \quad k_1 = \sum_{i=1}^{m} deg(a_i)$$

Observe that the vertices of transfer T_2 and the elements of A_1 satisfy the hypothesis of Theorem 2.2. Excluding a_0 let the sum of number of neighbours of a_1, a_2, $\ldots$, a_{m_1} be s_1 and the sum of number of neighbours of a_{m_1+1}, a_{m_1+2}, $\ldots$, a_m be s_2. As per Theorem 2.2 the first s_1 vertices of T_2 lie on a_i, $1 \leq i \leq m_1$ and they are the centers of odd branches and the remaining s_2-1 vertices of T_2 lie on $a_i,\ m_1+1 \leq i \leq m$ and they are the centers of even branches. So T_2 consists of s_1 transfers of the first type (for keeping desired odd number of vertices of A_1 at each vertex of transfer) followed by $s_2 - 1$ transfers of the second type (for keeping desired even number of vertices of A_1 at each vertex of transfer) so that we get back the tree $D_6^{(1)}$ and by Theorem 2.2 it is graceful. Attach the vertices c_1, c_2, $\ldots$, c_r to a_0 and assign them the labels $q-r+1$, $q-r+2$, $q-r+3$, $\ldots$, q so as to get back D_6 with a graceful labeling.

Case - II Let $m + n$ be even. Let us construct a tree G_6 from D_6 by removing the vertices $b_n, c_1, c_2, \ldots, c_r$. Obviously G_6 is a diameter six tree with center a_0 having odd degree. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof for Case -I by replacing n with $n-1$ and $q-r$ with q_1 and give a graceful labeling to G_6. Observe that the vertex a_0 in the graceful tree G_6 gets the label 0. Attach $b_n, c_1, c_2, \ldots, c_r$ to a_0 and assign the labels $q_1+1, q_1+2, \ldots, q_1+r, q_1+r+1$ to them. Obviously, the tree $G_6 \cup \{b_n, c_1, c_2, \ldots, c_r\}$ is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+r+1} to $G_6 \cup \{b_n, c_1, c_2, \ldots, c_r\}$ so that the label of the vertex b_n becomes 0. By Lemma 2.3, g_{q_1+r+1} is a graceful labeling of $G_6 \cup \{b_n, c_1, c_2, \ldots, c_r\}$. Let there be p pendant vertices adjacent to b_n in D_6. Now attach these vertices to b_n and assign labels $q_1+r+2, q_1+r+3, \ldots, q_1+r+p+1$ to them. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6.

(b) Proof follows if we set $r = 0$ in the proof involving part (a).

(c) Proof follows if we set $n = 0$ in the proof involving part (a).

(d) Case - I If m is odd then the proof follows immediately from the same involving part (b) by setting $n = 0$.

Case - II Let m be even. Let us designate the vertices a_1, a_2, $\ldots$, a_{m_1} such that $deg(a_1) \leq deg(a_2) \geq deg(a_3) \geq \ldots \geq deg(a_{m_1})$, i.e. the degree a_{m_1} is minimum among all the neighbours of a_0 which are the centers of diameter four trees containing only odd branches. Excluding a_0 let there be $2p_i + 1$ neighbours of a_i, $i = 1, 2, \ldots, m_1$ in D_6. Remove a_{m_1} and all the components incident on it, i.e. construct the tree $D_6 \setminus \{a_{m_1}\}$. Make any $2p_{m_1}$ neighbours of a_{m_1} adjacent to the vertex a_2. The resultant tree thus formed from D_6 is obviously a diameter six tree and let it be denoted by G_6. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof involving Case - I of part (a) by setting $n = 0$ and $r = 0$ and replacing m_1 with $m_1 - 1$ and $q - r$ with q_1 and give a graceful labeling to G_6. We observe that the vertex a_2 gets label 1, and the $2(p_2 + p_{m_1}) + 1$ neighbours of a_2 get the labels $q_1 - x$, $x + 1 + i$, $q_1 - x - i$, $x = k + p_1 + 1$, $i = 1, 2, \ldots, p_2 + p_{m_1}$. While labeling G_6 we allot labels $x+i+2$, q_1-x-i, $i = 1, 2, \ldots, p_{m_1}$ to $2p_{m_1}$ neighbours of a_{m_1} that were shifted to a_2 while constructing G_6. Next we attach the vertex a_{m_1} to a_0 and assign label $q_1 + 1$ to a_{m_1}. Now we move the vertices $x+i+2, q_1-x-i, i = 1, 2, \ldots, p_{m_1}$, to a_{m_1}. Since $(x+i+2)+(q_1-x-i) = q_1+2 = 1+(q_1+1)$, for $i = 1, 2, \ldots, p_{m_1}$, by Theorem 2.2 the resultant tree, say G_1 thus formed is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+1} to G_1 so that the label of the vertex a_{m_1} becomes 0. By Lemma 2.3, g_{q_1+1} is a graceful labeling of G_1. Now attach one remaining vertex to a_{m_1} and assign the label q_1+2 to it. Let this graceful labeling of the new tree, say G_2 thus formed be g_1. Let there be p neighbours of q_1+2 in D_6. Apply inverse transformation ${g_1}_{q_1+2}$ to G_2 so that the label of the vertex q_1+2 of G_2 becomes 0. By Lemma 2.3, g_{q_1+2} is a graceful labeling of G_2. Now attach the p pendant vertices adjacent to the vertex labelled 0 and assign them the labels $q_1+3, q_1+4, \ldots, q_1+p+2$. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6. ■

Example 3.2. The diameter six tree in Figure 3 (a) is a graceful diameter six of the type in Theorem 3.1(b). Here $q = 79$, $m = 6$, and $n = 3$.

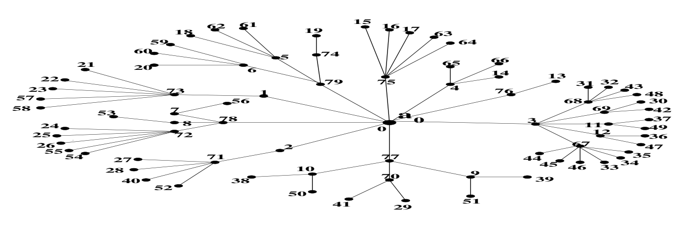


Figure 3: A diameter six tree of the type in Theorem 3.1(b) with a graceful labeling.

Notation 3.3. Let $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ be diameter six tree. For next couple of results we will consistently use the following notations.

n_e = Number of stars adjacent to a_0 with center having odd degree.

n_o = Number of stars adjacent to a_0 with center having even degree, i.e. $n = n_e + n_o$.

m_o^o = Number of diameter four trees adjacent to a_0 containing only odd branches and centers having even degree.

m_o^e = Number of diameter four trees adjacent to a_0 containing only even branches and centers having even degree.

m_e^o = Number of diameter four trees adjacent to a_0 containing only odd branches and centers having odd degree.

m_e^e = Number of diameter four trees adjacent to a_0 containing only even branches and centers having odd degree, i.e. $m = m_o^o + m_o^e + m_e^o + m_e^e$.

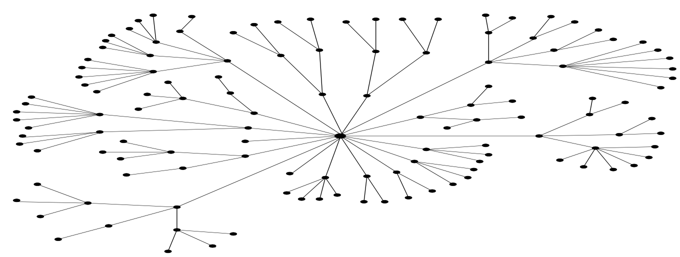
Theorem 3.4. If $m_e^o \cong 0 \, mod \, 4$, $m_e^e \cong 0 \, mod \, 4$, and $n_e \cong 0 \, mod \, 4$, $m + n$ is odd then

(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.

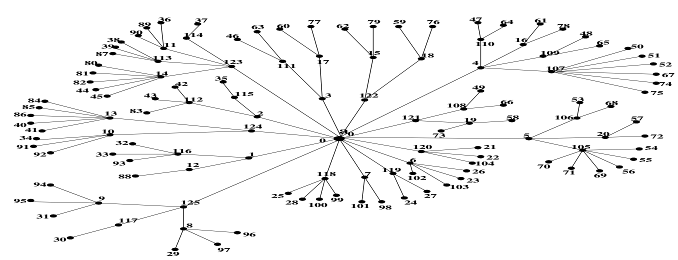
(b) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ has a graceful labeling.

Proof: Let us first prove part (a). Let $|E(D_6)| = q$ and $deg(a_0) = m + n = 2k + 1$. Remove r pendant vertices adjacent to a_0 and denote the new diameter six tree thus formed by $D_6^{(1)}$. Form the graceful tree as G (Figure 2), the set A, and the transfer T_1 as in the proof involving Case - I of Theorem 3.1(a). Suppose that the vertices $a_1, a_2, \ldots, a_{m_o^o}$ are the centers of m_o^o diameter four trees adjacent to a_0 with even degree and each is attached to an odd branches, $a_{m_o^o+1}, a_{m_o^o+2}, \ldots, a_{m_o}$ are the centers of m_o^e diameter four trees adjacent to a_0 with even degree and each is attached to odd branches, $a_{m_o+1}, a_{m_o+2}, \ldots, a_{m_o+m_e^e}$ are the centers of m_e^e diameter four trees adjacent to a_0 with odd degree and each is attached to even branches, and $a_{m_o+m_e^e+1}, a_{m_o+m_e^e+2}, \ldots, a_m$ are the centers of m_o^e diameter four trees adjacent to a_0 with odd degree and each is adjacent to odd branches. Suppose that the centers $b_1, b_2, \ldots, b_{n_e}$ are the centers of stars adjacent to a_0 with odd degree and $b_{n_e+1}, b_{n_e+2}, \ldots, b_n$ are the centers of stars adjacent to a_0 with even degree. Here T_1 consists of m_o^o successive transfers of the first type, followed by $\frac{m_e^o+m_e^e}{4}$ successive BD8TF, followed by $m_o^e + n_o$ successive transfers of the first type, and finally $\frac{n_e}{4}$ successive BD8TF. Carry out the transfer T_1 by keeping desired number of elements of A at each vertex of the transfer. Then repeat the remaining procedure in the proof of Theorem 3.1 and get the results. The proof for part(b) follows by setting $r = 0$ in the proof involving part (a). ■

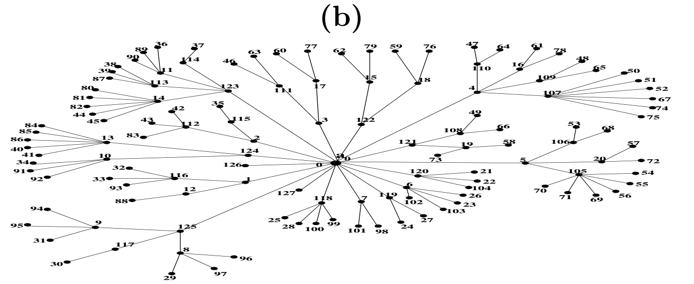
Example 3.5. The diameter six tree in Figure 4 (a) is a diameter six of the type in Theorem 3.4. Here $q = 127$, $m = 10$, and $n = 5$. We first form the graceful diameter six tree G_6 as in Figure (b) by removing all the pendant vertices adjacent to a_0. Finally, the graceful tree D_6 in Figure (c) which is obtained from the graceful tree in (b) by attaching two pendant vertices to a_0 and assigning them the labels 126 and 127.



(a)



(b)



(c)

Figure 4: A diameter six tree of the type in Theorem 3.4(b) with a graceful labeling.

Theorem 3.6. If $m_e^o \cong 0 \, mod \, 4$, $m_e^e \cong 0 \, mod \, 4$, $n_e \cong 0 \, mod \, 4$, $m+n$ is even, and $n_o \geq 1$ then
(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.
(b) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ has a graceful labeling.

Theorem 3.7. If $m_e^o \cong 0 \, mod \, 4$, $m_e^e \cong 0 \, mod \, 4$, and $n_e \cong 1 \, mod \, 4$, and $m+n$ is even then
(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.
(b) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ has a graceful labeling.

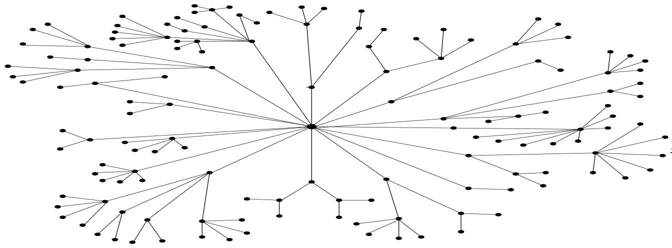
Proof of Theorems 3.6 and 3.7: Let us designate the vertex b_n as the center of a star with even degree if D_6 is a diameter six tree in Theorem 3.6 and as the center of a star with odd degree if D_6 is a diameter six tree in Theorem 3.7. Construct a tree G_6 from D_6 by removing the vertices c_1, c_2, $\ldots$, c_r, and the star with center b_n as we have done in the proof of Case - II of Theorem 3.1(a). Then we proceed as in the proof involving Case - II of Theorem 3.1(a) so as to get back D_6 with a graceful labeling. ■

Theorem 3.8. If $m_e^o \cong 1 \, mod \, 4$, $m_e^e \cong 0 \, mod \, 4$, and $n_e \cong 0 \, mod \, 4$, $m+n$ is even, $m_o^o + m_e^o \geq 3$, and the degree of center of at least one diameter four tree whose center has odd degree consists of only odd branches ≥ 4, then
(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.

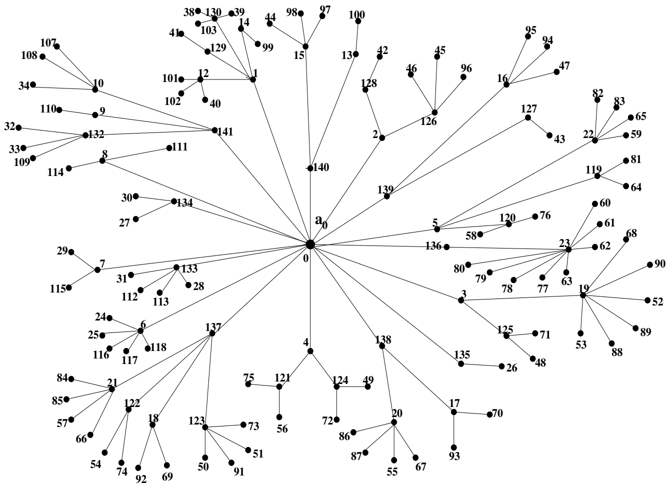
(b) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ has a graceful labeling.

Proof: Let us first prove the part (a). Let us designate the vertices a_1, a_2, $\ldots$, a_m in such a way that a_2 is the center of a diameter four tree which contains only odd branches and $deg(a_2) \geq 4$. Excluding a_0 let there be $2p_i + 1$ neighbours of a_i, for $i = 1, 2, \ldots, m_o^o$ and there be $2p_i$ neighbours of a_i, for $i = m_o^o + 1, m_o^o + 2, \ldots, m_o^o + m_e^o$ in D_6. Remove c_1, c_2, $\ldots$, c_r, and the diameter four tree with center $a_{m_o^o + m_e^o}$, i.e. construct the tree $D_6 \setminus \{c_1, c_2, \ldots, c_r, a_{m_o^o+m_e^o}\}$. Make all, say $2p_{m_o^o+m_e^o}$ neighbours of $a_{m_o^o+m_e^o}$ adjacent to the vertex a_2. The resultant tree thus formed from D_6 is obviously a diameter six tree and let it be denoted by G_6. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof of Theorem 3.4 (a) by replacing m_e^o with $m_e^o - 1$ and q with q_1 and give a graceful labeling to G_6. Observe that the vertex a_2 gets label 1, and the $2(p_2 + p_{m_o^o+m_e^o}) + 1$ (or $2(p_2 + p_{m_o^o+m_e^o})$) neighbours of a_2 get the labels $q_1 - x$, $x+1+i$, $q_1 - x - i$, $x = k + p_1 + 1$, $i = 1, 2, \ldots, p_2 + p_{m_o^o+m_e^o}$ (or $q_1 - x$, $x+1+i$, $q_1 - x - i$, $x = k + p_1 + 1$, $i = 1, 2, \ldots, p_2 + p_{m_o^o+m_e^o}$, and one more vertex). While labeling G_6 we allot labels $x+i+2$, $q_1 - x - i$, $i = 1, 2, \ldots, p_{m_o^o+m_e^o}$ to $2p_{m_o^o+m_e^o}$ neighbours of $a_{p_{m_o^o+m_e^o}}$ that were shifted to a_2 while constructing G_6. Next we attach the vertex $a_{p_{m_o^o+m_e^o}}$ to a_0 and assign label $q_1 + 1$ to $a_{p_{m_o^o+m_e^o}}$. Now we move the vertices $x+i+2, q_1-x-i, i = 1, 2, \ldots, p_{p_{m_o^o+m_e^o}}$, to $a_{p_{m_o^o+m_e^o}}$. Since $(x+i+2)+(q_1-x-i) = q_1+2 = 1+(q_1+1)$, for $i = 1, 2, \ldots, p_{p_{m_o^o+m_e^o}}$, by Theorem 2.2 the resultant tree, say G_1 thus formed is graceful with a graceful labeling, say g. Finally, attach the pendant vertices c_1, c_2, $\ldots$, c_r to a_0 and assign them the labels $q_1 + 2, q_1 + 3, \ldots, q_1 + r + 1$. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6. The proof of part (b) follows if we set $r = 0$ in the proof of part (b). ■

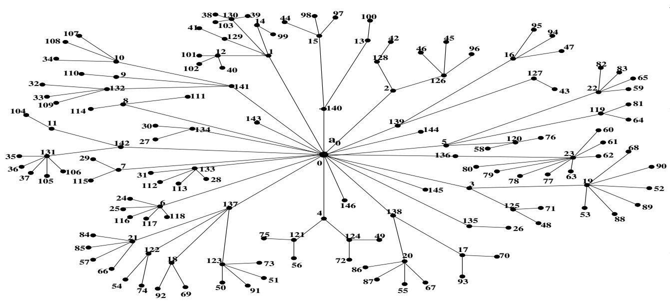
Example 3.9. The diameter six tree in Figure 5 (a) is a diameter six of the type in Theorem 3.8. Here $q = 149$, $m = 12$, and $n = 7$. We first form the graceful diameter six tree G_6 as in Figure (b) by removing all the pendant vertices and one star adjacent to a_0. Figure (c) represents the tree obtained from the graceful tree in (b). Here we attach five vertices to a_0 and assign them the labels 142, 143, 144, 145, and 146. Shift the components 11 and 131 incident on 1 to the vertex whose label is 142. The graceful tree in Figure (d) is obtained by applying inverse transformation to the graceful tree in Figure (c). Finally, the graceful tree D_6 in Figure (e) is obtained from the graceful tree in Figure (d) when we attach three vertices to the vertex labelled 0 and assign them the labels 147, 148, and 148.



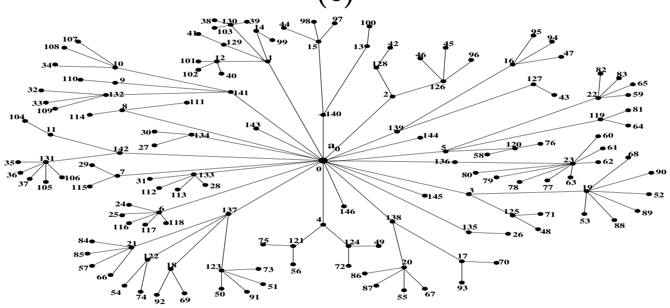
(a)



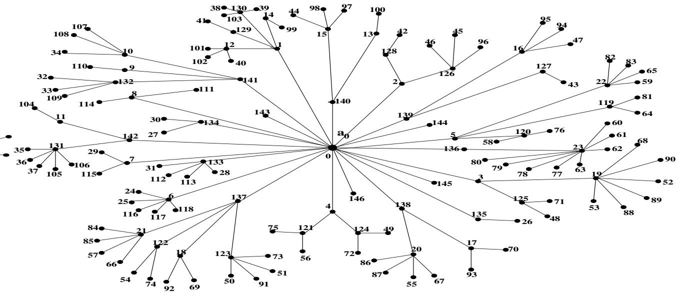
(b)



(c)



(d)



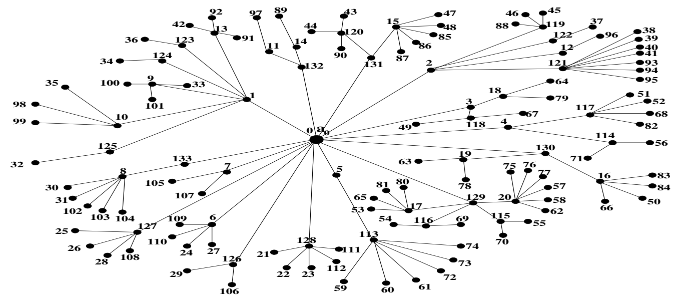
(e)

Figure 5: A diameter six tree of the type in Theorem 3.8 with a graceful labeling.

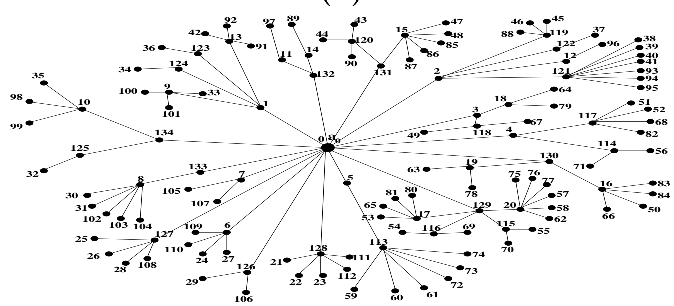
Theorem 3.10. If $m_e^o \cong 0 \ mod\ 4$, $m_e^e \cong 0 \ mod\ 4$, and $n_e \cong 0 \ mod\ 4$, $m+n$ is even, $m_o^o \geq 1$, $m_o^o + m_e^o \geq 3$, and the degree of the center of at least one diameter four tree consisting of only odd branches ≥ 4, then $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ has a graceful labeling.

Proof: Let us designate the vertices a_1, a_2, $\ldots$, a_m in such a way that a_2 is the center of a diameter four tree which contains only odd branches and $deg(a_2) \geq 4$. Excluding a_0 let there be $2p_i + 1$ neighbours of a_i, $i = 1, 2, \ldots, m_o^o - 1$ and there be $2p_i$ neighbours of a_i, $i = m_o^o, m_o^o + 1, m_o^o + 2, \ldots, m_o^o + m_e^o - 1$ in D_6. Remove one diameter four tree containing only odd branches. Let us designate this vertex as $a_{m_o^o+m_e^o}$. Excluding a_0 let there be $2p_{m_o^o+m_e^o} + 1$ neighbours of $a_{m_o^o+m_e^o}$. Make any $2p_{m_o^o+m_e^o}$ neighbours of $a_{m_o^o+m_e^o}$ adjacent to the vertex a_2. The resultant tree thus formed from D_6 is obviously a diameter six tree and let it be denoted by G_6. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof of Theorem 3.4 (a) by replacing m_e^o with $m_o^o - 1$ and q with q_1 and give a graceful labeling to G_6. We observe that the vertex a_2 gets label 1, and the $2(p_2 + p_{m_o^o+m_e^o}) + 1$ neighbours of a_2 get the labels $q_1 - x$, $x + 1 + i$, $q_1 - x - i$, $x = k + p_1 + 1$, $i = 1, 2, \ldots, p_2 + p_{m_o^o+m_e^o}$. While labeling G_6 we allot labels $x+i+2$, $q_1 - x - i$, $i = 1, 2, \ldots, p_{m_o^o+m_e^o}$ to $2p_{m_o^o+m_e^o}$ neighbours of $a_{p_{m_o^o+m_e^o}}$ that were shifted to a_2 while constructing G_6. Next we attach the vertex $a_{p_{m_o^o+m_e^o}}$ to a_0 and assign label $q_1 + 1$ to $a_{p_{m_o^o+m_e^o}}$. Now we move the vertices $x+i+2, q_1-x-i, i = 1, 2, \ldots, p_{p_{m_o^o+m_e^o}}$, to $a_{p_{m_o^o+m_e^o}}$. Since $(x+i+2)+(q_1-x-i) = q_1+2 = 1+(q_1+1)$, for $i = 1, 2, \ldots, p_{p_{m_o^o+m_e^o}}$, by Theorem 2.2 the resultant tree, say G_1 thus formed is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+1} to G_1 so that the label of the vertex $a_{p_{m_o^o+m_e^o}}$ becomes 0. By Lemma 2.3, g_{q_1+1} is a graceful labeling of G_1. Now attach one remaining vertex to $a_{p_{m_o^o+m_e^o}}$ and assign the label $q_1 + 2$ to it. Let this graceful labeling of the new tree, say G_2 thus formed be g_1. Let there be p neighbours of $q_1 + 2$ in D_6. Apply inverse transformation $g_{1_{q_1+2}}$ to G_2 so that the label of the vertex q_1+2 of G_2 becomes 0. By Lemma 2.3, g_{q_1+2} is a graceful labeling of G_2. Now attach the p pendant vertices adjacent to the vertex labelled 0 and assign them the labels $q_1 + 3, q_1 + 4, \ldots, q_1 + p + 2$. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6. ∎

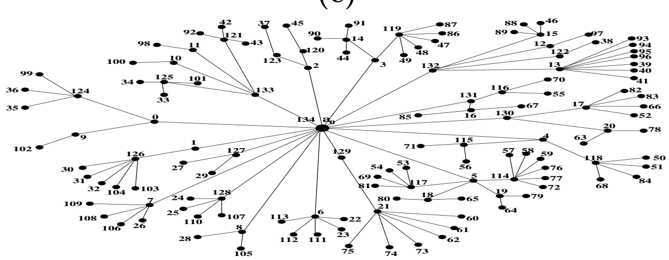
Example 3.11. The diameter six tree in Figure 6 (a) is a diameter six of the type in Theorem 3.10. Here $q = 140$, $m = 10$, and $n = 6$. We first form the graceful diameter six tree G_6 as in Figure (b) by removing one diameter four tree and one star adjacent to a_0 and shifting all except one branches of the removed diameter four tree to the vertex labelled **1** in G_6. Figure (c) represents the tree obtained from the graceful tree in (b) by attaching a new vertex to a_0, assigning it the label 134, and shifting the branches with centers with labels 10 and 125 from the vertex 1 to the vertex 134. The graceful tree in Figure (d) is obtained by applying inverse transformation to the graceful tree in Figure (c). The graceful tree in Figure (e) is obtained from the graceful tree in Figure (d) when we attach one vertex to the vertex labelled 0 and assign the label 135. The graceful tree in Figure (f) is obtained by applying inverse transformation to the graceful tree in Figure (e). Finally, the graceful tree D_6 in Figure (g) is obtained from the graceful tree in Figure (f) when we attach five vertices to the vertex labelled 0 and assign them the labels 136, 137, 138, 139, and 148.



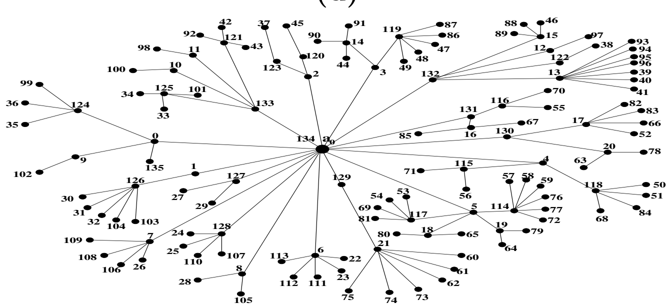
(b)



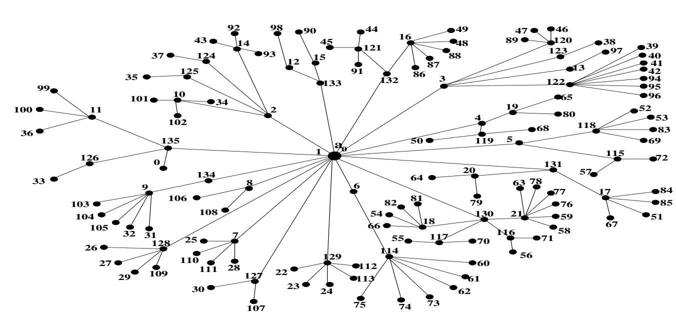
(c)



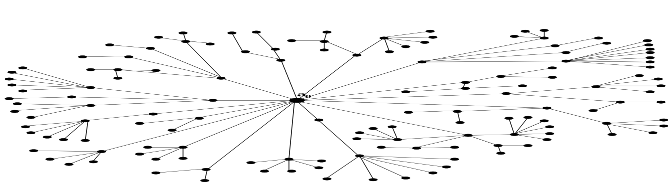
(d)



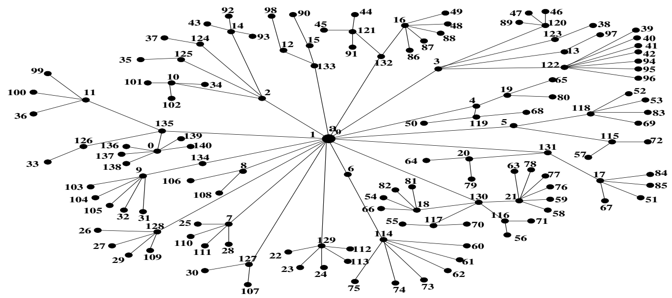
(e)



(f)



(a)



(g)

Figure 6: A diameter six tree of the type in Theorem 3.10 with a graceful labeling.

Theorem 3.12. If $m_e^o \cong 1 \ mod \ 4$, $m_e^e \cong 0 \ mod \ 4$, $n_e \cong 1 \ mod \ 4$, and $m+n$ is odd then
(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.
(b) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ has a graceful labeling.

Theorem 3.13. If $m_e^o \cong 1 \ mod \ 4$, $m_e^e \cong 0 \ mod \ 4$, $n_e \cong 0 \ mod \ 4$, $n_o \geq 1$, and $m+n$ is odd then
(a) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_r\}$ has a graceful labeling.
(b) $D_6 = \{a_0; a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n\}$ has a graceful labeling.

Proof of Theorems 3.12 and 3.13: Let us designate the vertex a_2 as the center of diameter four tree whose degree ≥ 4. Let there be $2p_m$ branches adjacent to a_m. Construct a tree G_6 from D_6 by removing the vertices c_1, c_2, $\ldots$, c_r, one star with center b_n, where $deg(b_n)$ is odd (respectively, even) for Theorem 3.13 (respectively, for Theorem 3.12), and one diameter four tree with center a_m such that degree of a_m is odd and the branches incident on it are all odd branches. Obviously G_6 is a diameter six tree with center a_0 having odd degree. Let $|E(G_6)| = q_1$. Repeat the procedure in the proof of Theorem 3.4 (a) by replacing m_e^o with $m_e^o - 1$ (i.e. m with $m-1$), n with $n-1$ and q with q_1 and give a graceful labeling to G_6. Observe that the vertex a_0 in the graceful tree G_6 gets the label 0 and the vertex a_2 gets label 1, and the $2(p_2+p_m)+1$ neighbours of a_2 get the labels q_1-x, $x+1+i$, q_1-x-i, $x = k+p_1+1$, $i = 1, 2, \ldots, p_2+p_m$ if $m_o \geq 2$ (q_1-x, $x+1+i$, q_1-x-i, $x = k+p_1+1$, $i = 1, 2, \ldots, p_2+p_m-1$, and one more vertex if $m_o \leq 1$). While labeling G_6 we allot labels $x+i+2$, q_1-x-i, $i = 1, 2, \ldots, p_m$ to $2p_m$ neighbours of a_m that were shifted to a_2 while constructing G_6. Attach vertices c_1, c_2, $\ldots$, c_r, b_n, and and a_m to a_0 and assign the labels q_1+2, q_1+3, $\ldots$, q_1+r+1, q_1+r+2, and q_1+1, respectively. Obviously, the tree $G_6 \cup \{a_m, c_1, c_2, \ldots, c_r, b_n\}$ is graceful with a graceful labeling, say g. Now we move the vertices $x+i+2, q_1-x-i, i = 1, 2, \ldots, p_m$, to a_m. Since $(x+i+2)+(q_1-x-i) = q_1+2 = 1+(q_1+1)$, for $i = 1, 2, \ldots, p_m$, by Theorem 2.2 the resultant tree, say G_1 thus formed is graceful with a graceful labeling, say g. Apply inverse transformation g_{q_1+r+2} to G_1 so that the label of the vertex b_n becomes 0. By Lemma 2.3, g_{q_1+r+2} is a graceful labeling of G_2. Let there be p pendant vertices adjacent to b_n in D_6. Now attach these vertices to b_n and assign labels q_1+r+3, q_1+r+4, $\ldots$, $q_1+r+p+2$ to them. Observe that we finally form the tree D_6 and the labeling mentioned above is a graceful labeling of D_6. Proof of part(b) follows if we set $r = 0$ in the proof of part (a). ■

Example 3.14. The diameter six tree in Figure 7 (a) is a diameter six of the type in Theorem 3.13. Here $q = 139$, $m = 11$, and $n = 6$. We first form the graceful diameter six tree G_6 as in Figure (b) by removing all the pendant vertices and one diameter four tree adjacent to a_0. Figure (c) represents the tree obtained from the graceful tree in (b). First we attach a vertex to a_0 and assign the label 128 and shift the components incident on the vertices 118 and 11 adjacent to the vertex labeled 1 in G_6 to the vertex 128. Then we attach four pendant vertices to a_0 and assign them the labels 129, 130, 131, 132. Next we apply inverse transform and attach the pendant vertices to the vertex labeled 0 which acts as the center of the star which was removed while constructing G_6 from D_6. Assign labels 133, 134, 135, 136, 137, and 138 to the vertices adjacent to 0.

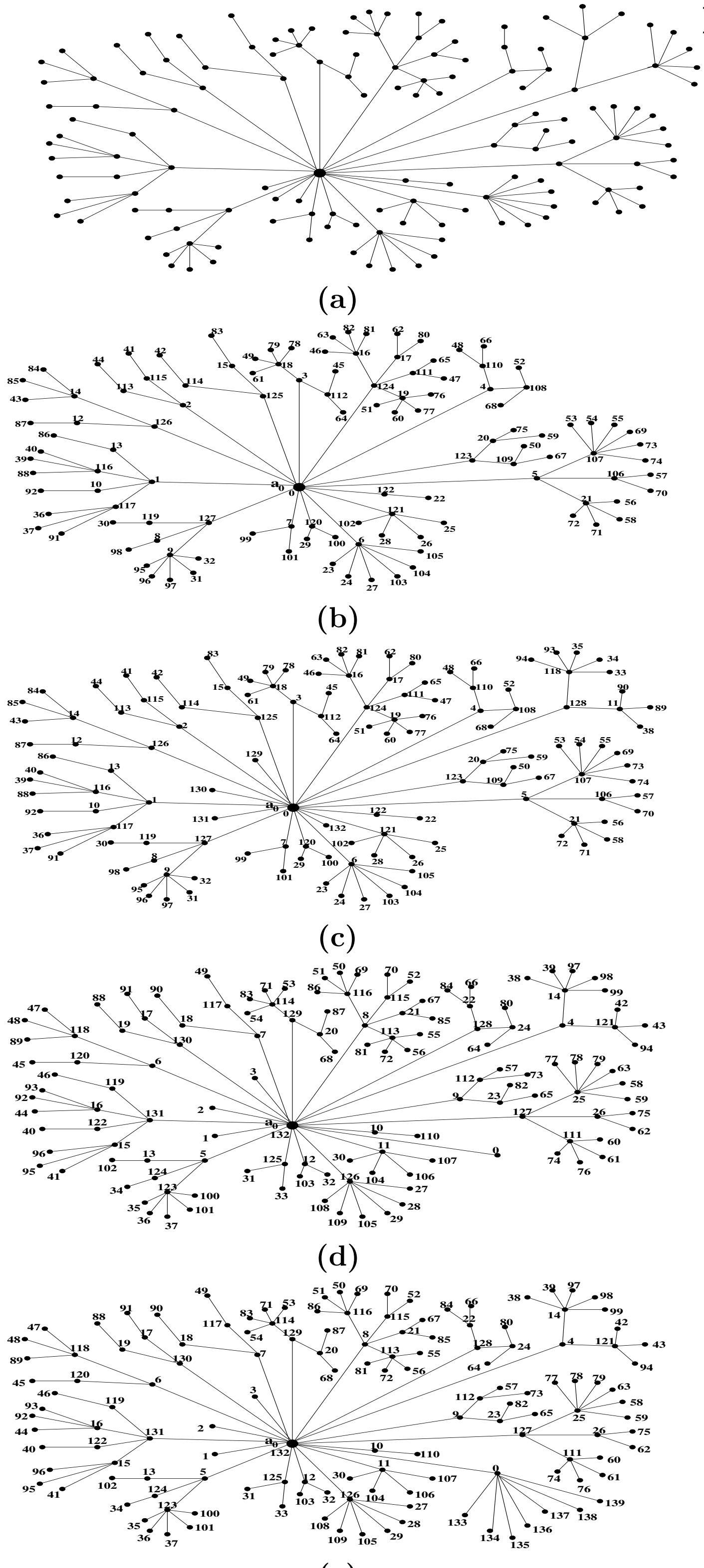


Figure 7: A diameter six tree of the type in Theorem 3.13 with a graceful labeling.

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