Hopf Bifurcation Analysis for the Pest-Predator Models Under Insecticide Use with Time Delay

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***Abstract* — We consider a delayed Pest-predator model under insecticide use. First, the paper considers the stability and local Hopf bifurcation for a modified Pest-predator model with time delay. In succession, using the normal form theory and center manifold argument, we obtain some explicit results which determine the stability, direction and other properties of bifurcation periodic solutions.**

***Key words*— Hopf bifurcation, Stability, time delay, Pest-predator model, Center manifold, Normal form**
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I. INTRODUCTION

With the rapid development of chemistry, many pesticides are applied in the world. However, insecticide pollution is also recognized as a major health hazard to human beings and to natural enemies. Thus it is required that we should combine pesticide efficacy tests with biological control research, so that the effects on the pest and the natural enemies are considered as a unified whole. Many researchers have been devoting to study the Pest-predator model. In this paper, our research is based on the Pest-predator models under insecticide use. However, here we consider the model with time delay.

Let x(t), y(t) denote the density of the pest and predator (natural enemies) at the time t, respectively. We could have the Pest-predator model as followings

$$\begin{cases} \dot{x}(t) = x(t)(r_1 - a_{11}x(t) - a_{12}y(t)), \\ \dot{y}(t) = y(t)(-r_2 + a_{21}x(t)), \end{cases} \tag{1.1}$$

Where $r_1 > 0$ is the intrinsic growth rate of pest, $r_2 > 0$ is the death rate of predator, $a_{12} > 0$ is the coefficient of intraspecific competition, $a_{21} > 0$ is the product of the per-capital rate of predication and the rate of converting pest into predator.

By introducing the time delay, the above system (1.1) can be written as the following form:

$$\begin{cases} \dot{x}(t) = x(t)(r_1 - a_{11}x(t) - a_{12}y(t) - Ex(t-\tau)), \\ \dot{y}(t) = y(t)(-r_2 + a_{21}x(t)), \end{cases} \tag{1.2}$$

where τ is positive, E > 0 is the proportionality constant which represents the rate of mortality due to applied pesticide, and the other parameters are the same as (1.1).

II. LOCAL STABILITY AND HOPF BIFURCATION

In this section, we are devoted in investing the stability of the positive equilibrium and by applying the Hopf bifurcation theorem, we give the conditions of the Hopf bifurcation.

Clearly, system (1.2) has the unique positive equilibrium at $E^* = (x^*, y^*)$, where

$$x^* = \frac{a_{21}x^*(r_1 - E) - a_{11}r_2}{a_{21}a_{12}}, y^* = \frac{r_2}{a_{21}},$$

when $a_{21}x^*(r_1 - E) - a_{11}r_2 > 0$. And, then linearizate the system (1.2) at E^*.

Letting $x_1(t) = x(t) - x^*, x_2(t) = y(t) - y^*$, we have

$$\begin{cases} \dot{x}_1(t) = (-a_{11}x^* + E)x_1(t) - a_{12}x^* x_2(t) - Ex_1(t-\tau) \\ \qquad - a_{11}x_1^2(t) - a_{12}x_1(t)x_2(t), \\ \dot{x}_2(t) = a_{21}y^* x_1(t) + a_{21}x_1(t)x_2(t). \end{cases} \tag{2.1}$$

The linearization part of the system (2.1) at the equilibrium (0,0) is as below:

$$\begin{cases} \dot{x}_1(t) = (-a_{11}x^* + E)x_1(t) - a_{12}x^* x_2(t) - Ex_1(t-\tau), \\ \dot{x}_2(t) = a_{21}y^* x_1(t). \end{cases} \tag{2.2}$$

And the characteristic equation is

$$\lambda^2 + (a_{11}x^* - E)\lambda + a_{12}a_{21}x^* y^* + E\lambda e^{-\lambda\tau} = 0 \tag{2.3}$$

It is clear that the Eq.(2.3) is the following second degree exponential polynomial equation

$$\lambda^2 + p\lambda + r + s\lambda e^{-\lambda\tau} = 0 \tag{2.4}$$

In order to investigate the stability of the positive equilibrium of system (1.2), we need to study the distribution of roots of Eq.(2.4). Obviously, $\lambda = 0$ is not a root of Eq.(2.4).

If $i\omega$ ($\omega > 0$) is a root of Eq.(2.4), then

$$-\omega^2 + pi\omega + r + si\omega(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Separating the real and imaginary parts, we get

$$\begin{cases} p\omega + s\omega\cos(\omega\tau) = 0, \\ s\omega\sin(\omega\tau) = \omega^2 - r, \end{cases} \tag{2.5}$$

which lead to

$$\omega^4 - (s^2 - p^2 + 2r)\omega^2 + r^2 = 0. \tag{2.6}$$

Let

$$\Delta = (s^2 - p^2 + 2r)^2 - 4r^2$$

It is easy to see that:

if at least one of the following conditions is satisfied:

(P1) $\Delta < 0$;

(P2) $\Delta > 0, s^2 - p^2 + 2r < 0, r^2 > 0$;

(P3) $\Delta = 0, s^2 - p^2 + 2r \le 0$,

then Eq.(2.6) has no positive root;

if $\Delta > 0, s^2 - p^2 + 2r > 0, r^2 > 0$ or $\Delta = 0, s^2 - p^2 + 2r > 0$ holds, then Eq.(2.6) has two positive roots

$$\omega_\pm = \frac{\sqrt{2}}{2}(s^2 - p^2 + 2r \pm \sqrt{\Delta})^{\frac{1}{2}}, \tag{2.7}$$

if $r^2 < 0$ or $r^2 = 0$; $s^2 - p^2 + 2r > 0$ holds, then Eq.(2.6) has one positive root ω:

But, it is clear that this can't happen since r > 0 forever.

Then from (2.5), we can determine

$$\tau_j^\pm = \frac{1}{\omega_\pm}\cos^{-1}(-\frac{p}{s}) \pm \frac{2j\pi}{\omega_\pm}, j = 0,1,\cdots, \tag{2.8}$$

at which Eq.(2.4) has a pair of purely imaginary $i\omega_\pm$.

Denote by

$$\lambda(\tau)=\alpha(\tau)+i\omega(\tau)$$

the root of Eq.(2.4), such that

$$\alpha(\tau_j^{\pm})=0,\omega(\tau_j^{\pm})=\omega_{\pm}.$$

Substituting $\lambda(\tau)$ into (2.4) and differentiating it with respect to τ yields

$$[\frac{d\lambda}{d\tau}]^{-1}=\frac{(2\lambda+p)e^{\lambda\tau}}{s\lambda^2}+\frac{1}{\lambda^2}-\frac{\tau}{\lambda},$$

then we get

$$\mathrm{Re}[\frac{d\lambda}{d\tau}]^{-1}_{\tau=\tau_j^{\pm}}=\frac{\omega_{\pm}^2}{\Gamma}(\pm\sqrt{\Delta}),$$

where $\Gamma=s^2\omega_{\pm}^4>0.$ Thus, when $\Delta\neq 0,$ we have

$$sign\{\mathrm{Re}[\frac{d\lambda}{d\tau}]_{\tau=\tau_j^+}\}=sign\{\mathrm{Re}[\frac{d\lambda}{d\tau}]^{-1}_{\tau=\tau_j^+}\}=sign\{\frac{\omega_+^2\sqrt{\Delta}}{\Gamma}\}>0, \quad (2.9)$$

and

$$sign\{\mathrm{Re}[\frac{d\lambda}{d\tau}]_{\tau=\tau_j^-}\}=sign\{\mathrm{Re}[\frac{d\lambda}{d\tau}]^{-1}_{\tau=\tau_j^-}\}=sign\{\frac{\omega_-^2\sqrt{\Delta}}{\Gamma}\}<0. \quad (2.10)$$

Then the following results about the distribution of root of Eq.(2.4) are got.

Lemma 2.1

i)When at least one of the conditions (P_1)--(P_3) is satisfied, then all the roots of Eq.(2.4) have negative real parts for all $\tau\geq 0$.

(ii)When $\Delta>0,r>0$ and $2Ea_{11}x^*-a_{11}^2x^{*2}+2a_{12}a_{21}x^*y^*>0,$ then there exists a $k\in N$ such that when $\tau\in(\tau_{j-1}^-,\tau_j^+), j=0,1,\cdots k,$ all roots of Eq.(2.4) have negative real parts, where $\tau_{-1}^-=0,$ and when $\tau\in(\tau_j^+,\tau_j^-), j=0,1,\cdots k-1,$ and $\tau>\tau_k^+,$ Eq.(2.4) has at least one root with positive real part.

We can obtain the following lemma by applying the lemma 2.1.

Lemma 2.2

(1)If at least one of the conditions (P_1)--(P_3) is hold, then system (2.1) is asymptotically stable at the positive equilibrium E^*.

(2)If $\Delta>0,r>0$ and $2Ea_{11}x^*-a_{11}^2x^{*2}+2a_{12}a_{21}x^*y^*>0,$ then there are $k\in N$ such that the stability of $E^*$of the system (2.1) switches from stability to unstability. That is, the positive equilibrium $E^*$of the system (2.1) is asymptotically stable for $\tau\in\bigcup_{j=0}^{k}(\tau_{j-1}^-,\tau_j^+),$ where $\tau_{-1}^-=0,$ and unstable for $\tau\in\bigcup_{j=0}^{k-1}(\tau_j^+,\tau_j^-),$ and $\tau>\tau_k^+$..

(3)If and then system(2.1) undergoes a Hopf bifurcation at

III. DIRECTION AND STABILITY OF THE BIFURCATING PERIODIC SOLUTIONS

In this section, we derive explicit formulate for computing the direction of the Hopf bifurcation and the stability of bifurcation periodic solution at critical values τ_0 by using the normal form theory and center manifold reduction.

Letting $x_1(t)=x(t)-x^*, x_2(t)=y(t)-y^*, \overline{x_i(t)}=x_i(t\tau), \tau=\tau_0+\mu,$ and dropping the bars for simplification of notation, system (2.1) is transformed into a FDE as

$$\dot{x}(t) = l_\mu(x_t) + f(\mu, x_t), \tag{3.1}$$

where x(t) is a vector $(x_1(t), x_2(t))^T, x_t = x(t+\theta)$ for $\theta \in [-1,0]$, with

$$l_\mu \varphi = B_1\varphi(0) + B_2\varphi(-1), \tag{3.2}$$

Where

$$B_1 = \begin{pmatrix} E - a_{11}x^* & -a_{12}x^* \\ a_{21}y^* & 0 \end{pmatrix}, B_2 = \begin{pmatrix} -E & 0 \\ 0 & 0 \end{pmatrix},$$

$$f(\mu,\varphi) = \begin{pmatrix} -a_{11}\varphi_1^2(0) - a_{12}\varphi_1(0)\varphi_2(0) \\ a_{21}\varphi_1(0)\varphi_2(0) \end{pmatrix}. \tag{3.3}$$

Using the Riesz representation theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-1,0]$, such that

$$l_\mu \varphi = \int_{-1}^{0} d\eta(\theta,\mu)x(t), \varphi \in C. \tag{3.4}$$

In fact, we can choose

$$\eta(\theta,\mu) = B_1\delta(\theta) + B_2\delta(\theta+1), \tag{3.5}$$

where $\delta(\theta)$ is Dirac delta function.

In the next for $\theta \in [-1,0]$, we define

$$A(\mu)\varphi = \begin{cases} \dfrac{d\varphi}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(\theta,\mu)\varphi(\theta), & \theta = 0. \end{cases} \tag{3.6}$$

$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-1,0), \\ f(\mu,\theta), & \theta = 0. \end{cases} \tag{3.7}$$

Then system (3.1) can be rewritten as

$$\dot{x}(t) = A(\mu)x_t + R(\mu)x_t, \tag{3.8}$$

where $x_t(\theta) = x(t+\theta)$.

The adjoint operator A^* of A is defined by

$$A^*(\mu)\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{d\theta}, & s \in (0,1], \\ \int_{-1}^{0} d(\eta^T(t,0)\psi(-t)), & s = 0. \end{cases} \tag{3.9}$$

where η^T is the transpose of the matrix η.

For $\varphi \in C^1[-1,0]$, and $\psi \in C^1[0,1]$, we define

$$\langle \psi, \varphi \rangle = \overline{\psi}(0)\cdot\varphi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi, \tag{3.10}$$

where $\eta(\theta) = \eta(\theta,0)$. As we all know that $\pm i\omega_0$ is an eigenvalue of A(0), so $\pm i\omega_0$ is also an eigenvalue of $A^*(0)$. And assume that $q(\theta)$ and $q^*(s)$ is the eigenvector of A and A^* corresponding to $+i\omega_0$ and $-i\omega_0$, respectively, satisfying with $\langle q^*, q\rangle = 1$ and $\langle q^*, \bar{q}\rangle = 0$.

We can get

$$q(\theta) = \begin{pmatrix} 1 \\ \dfrac{-a_{11}x^* + E - Ee^{-i\omega_0\tau} - i\omega_0}{a_{12}x^*} \end{pmatrix} e^{i\omega_0\theta}, -1 \le \theta < 0, \tag{3.11}$$

$$q^*(s) = D(\frac{i\omega_0}{a_{12}x^*} \quad 1)e^{i\omega_0 s}, 0 < s < 1,$$

Where $\bar{D} = \dfrac{1}{\alpha\alpha^* + \beta\beta^* + E\tau_0\alpha\beta^* e^{-i\tau_0\omega_0}}$.

In the remainder of this section, by using the same notation as in Hassard et al.[1], we first construct the coordinate for describing the center manifold C_0 at $\mu = 0$. Letting x_t be the solution of (3.1) with $\mu = 0$, we define

$$z(t) = \langle q^*, x_t\rangle,$$

$$W(t,\theta) = x_t - 2\,\mathrm{Re}[z(t)q(t)]. \tag{3.12}$$

On the center manifold C_0 we have

$$W(t,\theta) = W(z, \bar{z}, t), \tag{3.13}$$

where

$$W(z,\bar{z},t) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots, \tag{3.14}$$

z and $\bar{z}$ are local coordinate for C_0 in the direction of q and q^*. If x_t is real, we will deal with real solution only. Since $\mu = 0$,

$$\dot{z}(t) = \langle q^*, \dot{x}(t)\rangle = \langle q^*, A(\mu)x_t + R(\mu)x_t\rangle = \langle q^*, Ax_t\rangle + \langle q^*, Rx_t\rangle$$

$$= i\omega_0 z + \bar{q}^*(0)\cdot f(0, W(t,0) + 2\,\mathrm{Re}[z(t)q(0)]). \tag{3.15}$$

The (3.15) can also be substituted by

$$\dot{z}(t) = i\omega_0 z + g(z,\bar{z}), \tag{3.16}$$

that is,

$$g(z,\bar{z}) = \bar{q}^*(0)\cdot f(0, W(t,0) + 2\,\mathrm{Re}[z(t)q(0)])$$

$$= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{6} + \cdots. \tag{3.17}$$

From (3.12), differentiating it with respect to t can get

$$\dot{W} = \dot{x}_t - \dot{z}(t)q(\theta) - \dot{\bar{z}}(t)\bar{q}(\theta)$$

$$= \dot{x}(t) - 2\,\mathrm{Re}[\dot{z}(t)q(\theta)]$$

$$= \begin{cases} AW - 2\,\mathrm{Re}[\bar{q}^*(0)f(z,\bar{z})q(\theta)], & \theta \in [-2\tau, 0), \\ AW - 2\,\mathrm{Re}[\bar{q}^*(0)f(z,\bar{z})q(\theta)] + f_0(z,\bar{z}), & \theta = 0, \end{cases} \tag{3.18}$$

$$= AW + H(z,\bar{z},\theta)$$

Where $H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots.$

Since $x_t(\theta) = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$ and Eq.(3.11), we have

$$\varphi_1(0) = W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + z + \bar{z} + o(|(z,\bar{z})|^3),$$

$$\varphi_2(0) = W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \frac{-a_{11}x^* + E - Ee^{-i\omega_0\tau} - i\omega_0}{a_{12}x^*}z$$
$$+ \frac{-a_{11}x^* + E - Ee^{-i\omega_0\tau} + i\omega_0}{a_{12}x^*}\bar{z} + o(|(z,\bar{z})|^3).$$

From $g(z,\bar{z}) = \bar{q}^*(0) f_0(z,\bar{z}) = \bar{D}\begin{pmatrix} \frac{-i\omega_0}{a_{12}x^*} & 1 \end{pmatrix}\begin{pmatrix} -a_{11}\varphi_1^2(0) - a_{12}\varphi_1(0)\varphi_2(0) \\ a_{21}\varphi_1(0)\varphi_2(0) \end{pmatrix}$ and substitute $\varphi_1(0), \varphi_2(0)$ into it, then by comparing coefficients with Eq.(3.17), we can obtain

$$g_{20} = 4\bar{D}a_{11}\frac{i\omega_0}{a_{12}x^*} + 2D(\frac{i\omega_0}{x^*} + a_{21})\frac{-a_{11}x^* + E - Ee^{-i\omega_0\tau} - i\omega_0}{a_{12}x^*},$$

$$g_{02} = 2D(\frac{i\omega_0}{x^*} + a_{21})\frac{-a_{11}x^* + E - Ee^{-i\omega_0\tau} + i\omega_0}{a_{12}x^*},$$

$$g_{11} = 2\bar{D}a_{11}\frac{i\omega_0}{a_{12}x^*} + \bar{D}(\frac{i\omega_0}{x^*} + a_{21})\frac{2(-a_{11}x^* + E - Ee^{-i\omega_0\tau})}{a_{12}x^*},$$

$$g_{21} = \bar{D}(\frac{i\omega_0}{x^*} + a_{21})[12W_{11}^{(2)}(0) + 3W_{20}^{(2)}(0) + (3W_{20}^{(1)}(0) + 6W_{11}^{(1)}(0))\cdot$$
$$\frac{-a_{11}x^* + E - Ee^{-i\omega_0\tau} - i\omega_0}{a_{12}x^*}] + \bar{D}a_{11}\frac{i\omega_0}{a_{12}x^*}(6W_{20}^{(1)}(0) + 12W_{11}^{(1)}(0)).$$

Because there are W_{11} and W_{20} in g_{21}, so we need compute them in the sequel. Differentiating Eq.(3.14) with respect to θ and then comparing coefficients with $\dot{W} = AW + H(z,\bar{z},\theta)$, we can easily get

$$(A - 2i\omega_0)W_{20} = -H_{20}, AW_{11} = -H_{11}. \quad (3.19)$$

Then as $H_{20} = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta)$ and $H_{11} = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta)$, it is not hard to know that

$$\dot{W}_{20} = 2i\omega_0 W_{20} - H_{20}, \dot{W}_{11} = -H_{11}. \quad (3.20)$$

Integrating the above equation, then

$$W_{20} = \frac{1}{i\omega_0}g_{20}q(0)e^{i\omega_0\theta} - \frac{1}{3i\omega_0}\bar{g}_{02}\bar{q}(0)e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta}$$

$$W_{11} = \frac{1}{i\omega_0}g_{11}q(0)e^{i\omega_0\theta} - \frac{1}{i\omega_0}\bar{g}_{11}\bar{q}(0)e^{-i\omega_0\theta} + E_2$$

In fact, when $\theta = 0$, from $H(z,\bar{z},0) = -2\,\mathrm{Re}[\bar{q}^*(0) f(z,\bar{z})q(\theta)] + f_0(z,\bar{z})$ and the definition of A, we can know

$$E=\begin{pmatrix}E^{(1)}\\E^{(2)}\end{pmatrix}=\begin{pmatrix}\dfrac{-2i\omega_0(g_{20}-\overline{g}_{02})+a_{12}x^*(g_{20}M-\overline{g}_{02}\overline{M})}{-2i\omega_0(E-a_{11}x^*-Ee^{-i\omega_0\tau}-2i\omega_0)+a_{12}a_{21}x^*y^*}\\ \dfrac{(E-a_{11}x^*-Ee^{-i\omega_0\tau}-2i\omega_0)(g_{20}M-\overline{g}_{02}\overline{M})-g_{21}y^*(g_{20}-\overline{g}_{02})}{-2i\omega_0(E-a_{11}x^*-Ee^{-i\omega_0\tau}-2i\omega_0)+a_{12}a_{21}x^*y^*}\end{pmatrix}$$

According to the proof above, we can compute the following parameter

$$C_1(0)=\frac{i}{2\omega_0}(g_{20}g_{11}-2|g_{11}|-\frac{1}{3}|g_{02}|^2)+\frac{g_{21}}{2},$$

$$\mu_2=-\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_0)\}},$$

$$\beta_2=2\,\mathrm{Re}\{C_1(0)\},$$

$$T_2=-\frac{\mathrm{Im}\{C_1(0)\}+\mu_2\,\mathrm{Im}\{\lambda'(\tau_0)\}}{\omega_0}.$$

where the sign of μ_2 determine the direction of the Hopf bifurcation: if $\mu_2>0(\mu_2<0)$ then the Hopf bifurcation is forward(backward) and the bifurcating periodic solutions exist for $\tau>\tau_0(\tau<\tau_0)$. The sign of β_2 determine the stability of the Hopf bifurcating periodic solutions: the bifurcating periodic solutions are stable(unstable) if $\beta_2<0(\beta_2>0)$. The sign of T_2 determines the period of the bifurcating periodic solutions: the period increases(decreases) if $T_2>0(T_2<0)$.

IV. CONCLUSION

Firstly, under the condition of $\tau=0$, we discuss the Pest-predator model. We know that the stability of Pest-predator varies with the parameters changing. What's more, we discuss the Pest-predator model with time delay (1.2). By adjusting the parameters _ , we more easily control the Pest-predator populations such that the population tends to our expected results. Although our analysis indicates that the dynamics of the Pest-predator model with time delay can be much more complicated than we may have expected. It is still important to research the Pest-predator population. We just investigate the positive equilibrium, as the (0,0) is not the ideal point. The point (0,0) means the pest and predator both go to extinction, so we hope the population can converges to the positive equilibrium, that is the predator can survival.

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