Asymptotic Behavior of Solutions for Forced Non Linear Delay Impulsive Differential Equations

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***Abstract*— We establish new sufficient conditions for asymptotic behavior of the solutions of non linear forced neutral delay differential equations with constant impulsive jumps. Our results improve and generalize some known results in the literature.**

***Keywords* – Impulsive neutral delay differential equation, forced term, asymptotic behavior.**

I. Introduction

Consider the following non linear forced neutral delay differential equations with constant impulsive jumps

$$[x(t)-\sum_{i=1}^{m}p_i(t)x(t-\tau_i)]'+\sum_{j=1}^{n}q_j(t)f(x(t-\sigma_j))=r(t), \quad t\neq t_k$$

$$x(t_k^+)-x(t_k)=\beta_k, \quad k=1,2,3,... \qquad (1)$$

where p_i, q_j, $r \in C([t_0, \infty), R)$, τ_i, $\sigma_j \geq 0$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$, $f \in C(R, R)$ and $xf(x) > 0$ for $x \neq 0$, $0 \leq \tau_1 < \tau_2 < ... < \tau_m$, $0 \leq \sigma_1 < \sigma_2 < ... < \sigma_n$, $PC(R_+, R)$ denotes the set of all functions $\phi : R_+ \to R$ such that ϕ is continuous on $[0, t_1] \cup [t_k < t \leq t_{k+1}]$, $k = 1, 2, 3, ...$ with $\phi(t_k^+)=\lim_{t\to t_k^+}\phi(t)$ exists for $k = 1, 2, 3, ...$; the sequence $\{t_k\}$, $k = 1, 2, ...$, is impulsive time which satisfies $0 < t_1 < t_2 < ... < t_k < ....$ $\lim_{k\to\infty} t_k=\infty$ and $\{\beta_k\}$, $k = 1, 2, ...$ are constant impulsive perturbation sequence.

Let $\phi \in C([t_0 - d, t_0], R)$, $d = \max\{\tau_m, \sigma_n\}$. By a solution of system (1) satisfying the initial condition ϕ, we mean any function $x : [t_0 - d, \infty) \to R$ for which the following conditions hold:

(i) If $t \in [t_0, \infty)$ and $t \neq t_k$, $t \neq t_k + \tau_i$, $t \neq t_k + \sigma_j$, $k = 1, 2, ..., i = 1, 2, ..., m$, $j = 1, 2, ..., n$, $x(t)-\sum_{i=1}^{m}p_i(t)x(t-\tau_i)$ is continuous and differentiable and satisfies the first equation of system (1);

(ii) For $k = 1, 2, ...$, $x(t_k^+)$, $x(t_k^-)$ exist, $x(t_k^-)=x(t_k)$, and satisfy the second equation of system (1).

As usual, a solution x(t) of system (1) is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise it is called oscillatory.

The asymptotic behavior of solutions of equations (1) has been studied in [6] as a special case when $p_i(t) = 0$. In [2, 4, 7], the authors have studied the asymptotic behavior of solutions by taking impulses of the form $x(t_k^+)=b_k x(t_k)$. The purpose of this paper is to study the asymptotic behavior of the solutions of system (1) by taking constant impulsive jumps. Our results are generalized and improved the known results [2, 4, 6, 7, 9].

II. Main Results

First we introduce the following conditions:

$$|f(x)|\leq M|x| \quad \text{for} \quad x\in R \qquad (2)$$

where M is a positive constant,

$$R_1(t)=\int_t^\infty r(s)ds \quad \text{exists on} \quad [t_0,\infty) \qquad (3)$$

$$\lim_{k\to\infty}\beta_k^+=0 \qquad (4)$$

$$R(t)=\begin{cases}\int_t^\infty r(s)ds, & t\in[0,t_1]\cup(t_k,t_{k+1}],\\ \int_{t_k}^\infty r(s)ds+\beta_{k-1}^+, & t=t_k, k=1,2,3,...\end{cases} \qquad (5)$$

where $\beta_k^+=\max\{\beta_k,0\}$, $k \in Z_+ \cup \{0\}$, and $\beta_0 = 0$ and

$$t_{k+1}-t_k>\tau_m, \quad k=1,2,.... \qquad (6)$$

Theorem 2.1.
Let the conditions (2), (3), (4), (5) and (6) hold. Assume that there exists a constant $\sigma \in [0, \sigma_n]$ satisfying, the following conditions for sufficiently large t:

$$\sum_{j=1}^{n}q_j(t-\sigma+\sigma_j)\geq 0 \quad (7) \qquad \int_0^\infty\sum_{j=1}^{n}q_j(t-\sigma+\sigma_j)\,dt=\infty, \quad (8)$$

and

$$\sum_{\sigma_j<\sigma}\int_{t-\sigma}^{t-\sigma_j}q_j^-(s+\sigma_j)\,ds+\sum_{\sigma_j>\sigma}\int_{t-\sigma_j}^{t-\sigma}q_j^+(s+\sigma_j)\,ds < \frac{1-\sum_{i=1}^{m}p_i^+(t)}{M}, \qquad (9)$$

where $q_j^+(t)=\max\{q_j(t),0\}$, $q_j^-(t)=\max\{-q_j(t),0\}$, $p_i^+(t)=\max\{p_i(t),0\}$. Then every non oscillatory solution of system (1) tends to zero as $t\to\infty$.

Proof.

Choose a positive integer N such that (6)-(9) hold for $t \ge t_N$. Let x(t) be a non-oscillatory solution of system (1). Without loss of generality, we suppose that $x(t) > 0$, $x(t-\tau_i) > 0$ and $x(t-\sigma_j) > 0$, $t \ge t_N$, i = 1, 2, ..., m, j = 1, 2, 3, ..., n. For all $t \ge t_N$, define

$$\beta(t) = \begin{cases} \beta^{+}_{N^t}; & t > t_{N+1} \\ 0; & t \in [t_N, t_{N+1}] \end{cases} \quad (10)$$

where N^t corresponds to the largest subscript of impulsive point in (t_N, t).

Set

$$z(t) = x(t) - \sum_{i=1}^{m} p_i(t)x(t-\tau_i) - \int_{t-\sigma_j}^{t-\sigma} \sum_{j=1}^{n} q_j(s+\sigma_j)f(x(s))ds + R(t) - \beta(t) \quad (11)$$

Then for $t \ge t_N$, $t \ne t_k$, $t \ne t_k + \tau_i$, $t \ne t_k + \sigma_j$, i = 1, 2, ..., m, j = 1, 2, ..., n, k = 1, 2, 3, ..., we can choose Δt sufficiently small such that there is no impulsive point in $(t, t+\Delta t)$. Then we have $\operatorname{Lt}_{\Delta t \to 0} \frac{\beta(t+\Delta t) - \beta(t)}{\Delta t} = 0$, we have

$$z'(t) = -\sum_{j=1}^{n} q_j(t-\sigma+\sigma_j)f(x(t-\sigma)) \quad (12)$$

while for $t = t_k$, k = N+1, N+2, ..., we get $R(t_k^+) - R(t_k) = -\beta^{+}_{k-1}$, moreover

$$z(t_k^+) - z(t_k) = x(t_k^+) - x(t_k) + R(t_k^+) - R(t_k) - \beta(t_k^+) + \beta(t_k)$$
$$= \beta_k - \beta^{+}_{k-1} - \beta^{+}_k + \beta^{+}_{k-1} \le 0,$$

by (10). Therefore, from the above discussion, we have z(t) is decreasing on $[t_N, \infty)$. Set $L = \lim_{t\to\infty} z(t)$. We claim that $L \in R$. Otherwise, $L = -\infty$ and by conditions (2), (3), (9) and (11), x(t) must be unbounded. In fact, suppose that there exists a constant c such that $x(t) \le c$. Then from (11), (2), (3), we have

$$z(t) \ge x(t) - c\sum_{i=1}^{m} p_i^{+}(t) - Mc\sum_{\sigma_j<\sigma} q_j^{-}(s+\sigma)ds$$
$$- Mc\sum_{\sigma_j>\sigma} q_j^{+}(s+\sigma)ds + R(t) - \beta(t) > -\infty,$$

which contradicts $L = -\infty$. Thus x(t) is unbounded. Choose $t^* \ge t_N + \tau_n + \sigma_m$ such that $z(t^*) - R(t^*) + \beta(t^*) < 0$ and $x(t^*) = \max\{x(t) : t_N \le t \le t^*\}$.

Hence

$$0 > z(t^*) - R(t^*) + \beta(t^*)$$
$$\ge x(t^*) - \sum_{i=1}^{m} p_i^{+}(t)x(t^*) - M\int_{t-\sigma}^{t-\sigma_j} \sum_{\sigma_j<\sigma} q_j^{-}(s+\sigma_j)x(s)ds$$
$$- M\int_{t-\sigma_j}^{t-\sigma} \sum_{\sigma_j>\sigma} q_j^{+}(s+\sigma_j)x(s)ds + R(t^*) - \beta(t^*) - R(t^*) + \beta(t^*)$$
$$\ge x(t^*)\left[1 - M\left(\sum_{\sigma_j<\sigma}\int_{t-\sigma}^{t-\sigma_j} q_j^{-}(s+\sigma_j)ds + \sum_{\sigma_j>\sigma}\int_{t-\sigma_j}^{t-\sigma} q_j^{+}(s+\sigma_j)ds\right) - \sum_{i=1}^{m} p_i^{+}(t)\right]$$
$$\ge 0,$$

which is a contradiction. Thus $L \in R$.

We are now in a position to prove that

$$\lim_{t\to\infty} x(t) = 0 \quad (13)$$

Integrating (12) from t_N to t and letting $t \to \infty$, we find

$$L - Z(t_N) \le \int_{t_N}^{\infty} \sum_{j=1}^{n} q_j(t-\sigma+\sigma_j)f(x(t-\sigma))dt \quad (14)$$

which together with (8) and (14) implies that

$$\liminf_{t\to\infty} f(x(t)) = 0. \quad (15)$$

Since x(t) is bounded, it follows that

$$\liminf_{t\to\infty} x(t) = 0 \quad (16)$$

From (14), we have

$$\lim_{t\to\infty} \int_{t-\sigma_j}^{t-\sigma} q_j(s+\sigma_j)f(x(s))ds = 0$$

Then integrating (12) we get,

$$\lim_{t\to\infty}\left[x(t) - \sum_{i=1}^{m} p_i(t)x(t-\tau_i)\right] = L \in \mathbf{R},$$

which implies that

$$\lim_{t\to\infty} x(t) \quad \text{exists.} \quad (17)$$

By (16) and (17) we obtain (13) which is our claim.

The proof is complete. □

Theorem 2.2.

Let the conditions (2), (3) hold. Assume that

$$\lim_{k\to\infty} \beta_k = 0, \quad (18)$$

and there exists constant $\sigma \in [0, \sigma_n]$ and functions $Q_1(t)$, $Q_2(t)$ such that for sufficiently large t,

$$\sum_{j=1}^{n} q_j(t-\sigma+\sigma_j) \ne 0 \quad (19) \qquad \limsup_{t\to\infty} \sum_{i=1}^{m} |p_i(t)| < 1/2 \quad (20)$$

$$\sum_{j=1}^{n} \int_{t-\sigma-\sigma_j}^{t-\sigma} |q_j(s+\sigma_j)| ds \le Q_1(t) \quad (21)$$

$$\sum_{j=1}^{n} \int_{t-\sigma}^{t-\sigma_j} \operatorname{sgn}(\sigma-\sigma_j)|q_j(s+\sigma_j)| ds \le Q_2(t) \quad (22)$$

and

$$\limsup_{t\to\infty} Q_1(t) + \limsup_{t\to\infty} Q_2(t) < \frac{1 - 2\limsup_{t\to\infty}\sum_{i=1}^{m}|p_i(t)|}{M} \quad (23)$$

Then every oscillatory solution x(t) of system (1) tends to zero as $t \to \infty$.

Proof.

Let $\mu = \limsup_{t\to\infty} |x(t)|$. First we prove $\mu < \infty$. Otherwise $\mu = \infty$. Choose a $t_N \ge t_0$ such that (18)-(23) hold for $t \ge t_N$ and

$$\sup_{t_N+\tau_m+\sigma_n \le s \le t} |x(s)| \ge \sup_{t_N \le s \le t} |x(s)|.$$

Set

$$z(t) = x(t) - \sum_{i=1}^{m} p_i(t)x(t-\tau_i) - \int_{t-\sigma_j}^{t-\sigma} \sum_{j=1}^{n} q_j(s+\sigma_j)f(x(s))ds + R(t) - \beta(t),$$

where R(t) and β(t) are as in (5) and (10). Then (12) holds and for $t \geq t_N + \tau_m + \sigma_n$, we have

$$|z(t)| = |x(t)| - \sum_{i=1}^{m} |p_i(t)||x(t-\tau_i)| - \sum_{j=1}^{n} \int_{t-\sigma_j}^{t-\sigma} \operatorname{sgn}(\sigma - \sigma_n)|q_j(s+\sigma_j)||f(x(s))|ds - |R(t)| + |\beta(t)|$$

$$\geq |x(t)| - \left[\sum_{i=1}^{m} |p_i(t)| + MQ_2(t)\right] \sup_{t_N \leq s \leq t} |x(s)| - |R(t)| - |\beta(t)|,$$

which implies

$$\sup_{t_N+\tau_m+\sigma_n \leq s \leq t} |z(s)| \geq \left[1 - \sup_{t_N+\tau_m+\sigma_n \leq s \leq t} \sum_{i=1}^{m} |p_i(s)| - M \sup_{t_N+\tau_m+\sigma_n \leq s \leq t} Q_2(s)\right] \sup_{t_N \leq s \leq t} |x(s)| - \sup_{t_N \leq s \leq t} |R(s)| - \sup_{t_N \leq s \leq t} |\beta(s)|. \quad (24)$$

Hence $\limsup_{t\to\infty} |z(t)| = \infty$ by (3) and condition (18).

Noticing that x(t) is oscillatory, from (12) we see that z′(t) oscillates. Thus there exists a $\varsigma \geq t_N + \tau_m + \sigma_n$, such that

$$|z(\varsigma)| = \sup_{t_n+\tau_m+\sigma_n \leq s \leq \varsigma} |z(s)| \quad \text{and} \quad z'(\varsigma) = 0.$$

From (12) and (19) we know that $x(\varsigma - \sigma) = 0$. Integrating (12) both sides from $\varsigma - \sigma$ to ς we obtain

$$z(\varsigma) = z(\varsigma - \sigma) - \int_{\varsigma-\sigma}^{\varsigma} \sum_{j=1}^{n} q_j(s-\sigma+\sigma_j)f(x(s-\sigma))ds$$

$$= -\sum_{i=1}^{m} p_i(\varsigma-\sigma)x(\varsigma-\sigma-\tau_i) - \sum_{j=1}^{n} \int_{\varsigma-\sigma-\sigma_j}^{\varsigma-\sigma} q_j(s+\sigma_j)f(x(s))ds + R(\varsigma-\sigma) - \beta(\varsigma-\sigma),$$

which implies

$$|z(\varsigma)| \leq \left[\sum_{i=1}^{m} |p_i(\varsigma-\sigma)| + MQ_1(\varsigma)\right] \sup_{t_N \leq s \leq \varsigma} |x(s)| + |R(\varsigma-\sigma)| + |\beta(\varsigma-\sigma)| \quad (25)$$

From (24) and (25) we have

$$\left[1 - \sup_{t_N+\tau_m+\sigma_n \leq s \leq \varsigma} \sum_{i=1}^{m} |p_i(s)| - M \sup_{t_N+\tau_m+\sigma_n \leq s \leq \varsigma} Q_2(s) - \sum_{i=1}^{m} |p_i(\varsigma-\sigma)| - MQ_1(\varsigma)\right]$$

$$\sup_{t_N \leq s \leq \varsigma} |x(s)| \leq \sup_{t_N \leq s \leq \varsigma} |R(s)| + |R(\varsigma-\sigma)| + \sup_{t_N \leq s \leq \varsigma} |\beta(s)| + |\beta(\varsigma-\sigma)|.$$

$$1 - M\left(Q_1(\varsigma) + \sup_{t_N+\tau_m+\sigma_n \leq s \leq \varsigma} Q_2(s)\right) - \sup_{t_N+\tau_m+\sigma_n \leq s \leq \varsigma} \sum_{i=1}^{m} |p_i(s)| - \sum_{i=1}^{m} |p_i(\varsigma-\sigma)|) - \left[\frac{|R(\varsigma-\sigma)| + |\beta(\varsigma-\sigma)|}{\sup_{t_N \leq s \leq \varsigma} |x(s)|}\right] - \left(\frac{\sup_{t_N \leq s \leq \varsigma} |R(s)| + \sup_{t_N \leq s \leq \varsigma} |\beta(s)|}{\sup_{t_N \leq s \leq \varsigma} |x(s)|}\right) \leq 0$$

Letting $\zeta \to \infty$, we find

$$1 - M\left[\limsup_{t\to\infty} Q_1(t) + \limsup_{t\to\infty} Q_2(t)\right] - 2\limsup_{t\to\infty} \sum_{i=1}^{m} |p_i(t)| \leq 0$$

which contradicts (23). Hence $\mu < \infty$.

Next we prove $\mu = 0$. From (24), (3) and (18), we have

$$\lambda = \limsup_{t\to\infty} |z(t)| \geq \mu\left[1 - \limsup_{t\to\infty} \sum_{i=1}^{m} |p_i(t)| - M\limsup_{t\to\infty} Q_2(t)\right]. \quad (26)$$

On the other hand, there exists a sequence $\{\alpha_k\}$ such that $\lim_{t\to\infty} \alpha_k = \infty$, $\lim_{k\to\infty} |z(\alpha_k)| = \lambda$ and $z'(\alpha_k) = 0$, k = 1, 2, 3, ... Similar to (25), we can get

$$|z(\alpha_k)| \leq \left[\sum_{i=1}^{n} |p_i(\alpha_k - \sigma)| + MQ_1(\alpha_k)\right] \sup_{t_N \leq s \leq \alpha_k} |x(s)| + |R(\alpha_k - \sigma)| + |\beta(\alpha_k - \sigma)|. \quad (27)$$

Let $k \to \infty$, we get,

$$\lambda \leq \left[\limsup_{t\to\infty} \sum_{k=1}^{n} |p_i(t)| + M\limsup_{t\to\infty} Q_1(t)\right]\mu. \quad (28)$$

By (26) and (28)

$$\mu\left[1 - 2\limsup_{t\to\infty} \sum_{i=1}^{m} |p_i(t)| - M\limsup_{t\to\infty} Q_1(t) - M\limsup_{t\to\infty} Q_2(t)\right] \leq 0$$

which implies $\mu = 0$. This completes the proof.

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