Coupled Fixed Point Results in Partially Ordered Complete Metric Space

Savitri[1], Nawneet Hooda[2]

Department of Mathematics, DCR University of Science & Technology, Murthal, India

***Abstract*— In this paper we prove some coupled fixed point results in partially ordered complete metric space.**

***Keywords*— Partially ordered set, Cauchy sequence, coupled fixed point, complete metric space, control function.**

I. Introduction and Mathematical Preliminaries

Fixed point theory in partially ordered metric spaces has greatly developed in recent times. The fixed points of mappings in partially ordered metric space are of vital use in many mathematical problems in applied and pure mathematics. The first step in this direction was taken by Ran and Reurings [1]. After that Nieto and Lopez [9], Agarwal et al. [12], O'Regan and Petrusel [8] and Lakshmikantham and Ciric [16] established marvellous fixed point results for different type of mappings. In 2011, Binayak Choudhury and Metiya [13] proved some multivalued and singlevalued fixed point results in partially ordered metric spaces. In 1987 Guo and Lakshmikantham [6] introduced the notion of coupled fixed point. Bhaskar and Lakshmikantham [15] reconsidered the concept of coupled fixed point in partially ordered metric space in 2006.

In this paper we have established some coupled fixed point theorems for mappings satisfying certain different conditions.

To begin, we first recall the definitions and notation that will be needed in the sequel.

Definition 1.1. A partially ordred set is a set X with a binary operation $\leq$ denoted by $(X,\leq)$ such that for all $p,q,r\in X$

(i) $p\leq p$ (reflexivity),

(ii) $p\leq q$ and $q\leq p\Rightarrow p=q$ (anti-symmetry),

(iii) $p\leq q$ and $q\leq r\Rightarrow p=r$ (transitivity).

Definition 1.2. An element $(x,y)\in X^2$ is called a coupled fixed point of the mapping $F:X^2\to X$ if $F(x,y)=x$ and $F(y,x)=y$.

Definition 1.3. A function $\psi:[0,\infty)\to[0,\infty)$ is called an Altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous,

(ii) $\psi(t)=0$ if and only if $t=0$.

This control function has been greatly used in metric fixed point theory. Its use and importance can be viewed in the works of Khan et al. [11], Sastry and Babu [10], Dutta and Choudhury [4], Choudhury [2], Doric [5], Choudhury and Das [3], Mihet [7] and Naidu [14].

Notation. Let $(X,\leq)$ be a partially ordered set. We endow the product space X^2 with the partial order $\leq$ defined by:

For $(x,y),(u,v)\in X^2$, $(x,y)\leq(u,v)\Leftrightarrow x\leq u, y\geq u$.

II. Main Results

Theorem 2.1. Let $(X,d,\leq)$ be a partially ordered complete metric space, $\xi:[0,\infty)\to[0,\infty)$ is an Altering distance function and $\beta\in(0,1)$. Let $F:X^2\to X$ be a mapping such that the following conditions are satisfied:

(i) there exists $(x_0,y_0)\in X^2$ such that $(x_0,y_0)\leq(F(x_0,y_0),F(y_0,x_0))$,

(ii) for $(x_1,y_1),(x_2,y_2)\in X^2$,

$$(x_1,y_1)\leq(x_2,y_2)\ \Rightarrow F(x_1,y_1)\leq F(x_2,y_2)\ \text{and}\ F(y_1,x_1)\geq F(y_2,x_2),$$

(iii) If $(x_n, y_n) \to (x, y)$ is non-decreasing in first co-ordinate and non-increasing in second coordinate then

$$x_n \le x,\ y_n \ge y \quad \text{for all } n,$$

(iv) $\xi(d(F(x_1,y_1),F(x_2,y_2))) \le \beta\xi\left(\max\left\{d(x_1,x_2),d(x_1,F(x_1,y_1)),d(x_2,F(x_2,y_2)),\frac{d(x_2,F(x_1,y_1))+d(x_1,F(x_2,y_2))}{2}\right\}\right)$

for either $x_1 \le x_2$ and $y_1 \ge y_2$ or $x_1 \ge x_2$ and $y_1 \le y_2$.

Then F has a coupled fixed point.

Proof. Let $F(x_0, y_0) = x_1$ and $F(y_0, x_0) = y_1$, then by hypothesis (i) we have $(x_0, y_0) \le (x_1, y_1)$

$$\Rightarrow x_0 \le x_1 \text{ and } y_0 \ge y_1. \tag{2.1}$$

Now by hypothesis (ii) and using equation (2.1), we have

$$F(x_0,y_0) \le F(x_1,y_1) \text{ and } F(y_0,x_0) \ge F(y_1,x_1).$$

Let $F(x_1,y_1) = x_2$ and $F(y_1,x_1) = y_2$.

Then we have

$$x_1 \le x_2 \text{ and } y_1 \ge y_2. \tag{2.2}$$

Again using hypothesis (ii) and equation (2.2), we have

$$F(x_1,y_1) \le F(x_2,y_2) \text{ and } F(y_1,x_1) \ge F(y_2,x_2).$$

Continuing like this we can construct a monotone non-decreasing sequence $\{x_n\}$ and monotone non-increasing sequence $\{y_n\}$ in X that is

$$x_1 \le x_2 \le x_3 \le \ldots \le x_{n-1} \le x_n \le \ldots,$$
$$y_1 \ge y_2 \ge y_3 \ge \ldots \ge y_{n-1} \ge y_n \ge \ldots,$$

such that $F(x_n,y_n) = x_{n+1}$ and $F(y_n,x_n) = y_{n+1}$ for all n.

If there exist a positive integer ℓ such that

$$x_\ell = x_{\ell+1} \text{ and } y_\ell = y_{\ell+1}.$$

Then (x_ℓ, y_ℓ) is a coupled fixed point of F.

Hence we assume that either $x_n \ne x_{n+1}$ or $y_n \ne y_{n+1}$ for all n.

First we assume that $x_n \ne x_{n+1}$ for all n.

Now since $x_n \le x_{n+1}$ and $y_n \ge y_{n+1}$, using the hypothesis (iv), we have

$$\xi(d(x_{n+1},y_{n+2})) = \xi(d(F(x_n,y_n),F(x_{n+1},y_{n+1})))$$
$$\le \beta\xi\left(\max\left\{d(x_n,x_{n+!}),d(x_n,F(x_n,y_n)),d(x_{n+1},F(x_{n+1},y_{n+1})),\frac{d(x_{n+1},F(x_n,y_n))+d(x_n,F(x_{n+1},y_{n+1}))}{2}\right\}\right)$$

$$\Rightarrow \quad \xi(d(x_{n+1},y_{n+2})) \le \beta\xi\left(\max\left\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\right\}\right). \tag{2.3}$$

$$(\because\ \frac{d(x_n,x_{n+2})}{2} \le \max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\})$$

Assume

$$d(x_n,x_{n+1}) \le d(x_{n+!},x_{n+2}) \text{ for some } n \in N.$$

Then

$$\xi(d(x_{n+1},x_{n+2})) \le \beta\xi(d(x_{n+1},x_{n+2}))$$
$$\Rightarrow \quad d(x_{n+1},x_{n+2}) = 0$$
$$\Rightarrow \quad x_{n+1} = x_{n+2}$$

which gives a contradiction to our assumption that $x_n \ne x_{n+1}$ for all n.

So $d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1})$ for all n and $\{d(x_n,x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers.

$\Rightarrow$ $\exists$ a real number $p \ge 0$ such that $d(x_n,x_{n+1}) \to p$ and $n \to \infty$.

Taking the limit $n \to \infty$ and using continuity of ψ, we have

$$\psi(p) \le \beta\psi(p). \tag{2.4}$$

$\Rightarrow$ $p = 0$ ($\because$ otherwise equation (2.4) will lead to a contradiction)

Hence $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$ (2.5)

Similar if we assume $y_n \neq y_{n+1}$ for all n then we will arrived a contradiction and have

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 . \tag{2.6}$$

Now we will show that $\{x_n\}$ is a Cauchy sequence in X .

If possible, let $\{x_n\}$ is not a Cauchy sequence in X .

Then there exists an $\varepsilon > 0$ such that

$$d(x_{n(t)}, x_{m(t)}) \geq \varepsilon \text{ for all } t \in N,\ m(t) > n(t) > t .$$

If $m(t)$ is the smallest such natural number, then we have

$$d(x_{n(t)}, x_{m(t)}) \geq \varepsilon \tag{2.7}$$

and

$$d(x_{n(t)}, x_{m(t)-1}) < \varepsilon . \tag{2.8}$$

Then

$$\begin{aligned} d(x_{n(t)}, x_{m(t)}) &\leq d(x_{n(t)}, x_{m(t)-1}) + d(x_{m(t)-1}, x_{m(t)}) \\ &< \varepsilon + d(x_{m(t)-1}, x_{m(t)}) . \quad \text{(by using (2.8))} \end{aligned} \tag{2.9}$$

Combining equations (2.7) and (2.9) and taking limit as $t \to \infty$ and then using (2.5) we have

$$\lim_{t\to\infty} d(x_{n(t)}, x_{m(t)}) = \varepsilon . \tag{2.10}$$

Now

$$d(x_{n(t)}, x_{m(t)}) \leq d(x_{n(t)}, x_{n(t)+1}) + d(x_{n(t)+1}, x_{m(t)+1}) + d(x_{m(t)+1}, x_{m(t)}) .$$

Taking limit as $t \to \infty$ and using equations (2.5) and (2.10) we get

$$\lim_{t\to\infty} d(x_{n(t)+1}, x_{m(t)+1}) \geq \varepsilon . \tag{2.11}$$

Also

$$d(x_{n(t)+1}, x_{m(t)+1}) \leq d(x_{n(t)+1}, x_{n(t)}) + d(x_{n(t)}, x_{m(t)}) + d(x_{m(t)}, x_{m(t)+1}) .$$

Again taking limit as $t \to \infty$ and using equations (2.5) and (2.10), we get

$$\lim_{t\to\infty} d(x_{n(t)+1}, x_{m(t)+1}) \leq \varepsilon . \tag{2.12}$$

Combining (2.11) and (2.12), we have

$$\lim_{t\to\infty} d(x_{n(t)+1}, x_{m(t)+1}) = \varepsilon . \tag{2.13}$$

Again

$$d(x_{n(t)}, x_{m(t)}) \leq d(x_{n(t)}, x_{m(t)+1}) + d(x_{m(t)+1}, x_{m(t)})$$

and

$$d(x_{n(t)}, x_{m(t)+1}) \leq d(x_{n(t)}, x_{m(t)}) + d(x_{m(t)}, x_{m(t)+1}) .$$

Taking limit as $t \to \infty$ and using (2.5) and (2.10), we get

$$\lim_{t\to\infty} d(x_{n(t)}, x_{m(t)+1}) = \varepsilon . \tag{2.14}$$

Again

$$d(x_{m(t)}, x_{n(t)}) \leq d(x_{m(t)}, x_{n(t)+1}) + d(x_{n(t)+1}, x_{n(t)})$$

and

$$d(x_{m(t)}, x_{n(t)+1}) \leq d(x_{m(t)}, x_{n(t)}) + d(x_{n(t)}, x_{n(t)+1}) .$$

Taking limit as $t \to \infty$ and using (2.5) and (2.10), we get

$$\lim_{t\to\infty} d(x_{m(t)}, x_{n(t)+1}) = \varepsilon . \tag{2.15}$$

Now

$$m(t) > n(t) \Rightarrow x_{m(t)} \geq x_{n(t)} \text{ and } y_{m(t)} \leq y_{n(t)} \text{ for all } t \in N .$$

So using hypothesis (iv), we have

$$\begin{aligned} \xi(d(x_{n(t)+1}, x_{m(t)+1})) &= \xi(d(F(x_{n(t)}, y_{n(t)}), F(x_{m(t)}, y_{m(t)}))) \\ &\leq \beta\xi\Big(\max\Big\{d(x_{n(t)}, x_{m(t)}), d(x_{n(t)}, F(x_{n(t)}, y_{n(t)})), d(x_{m(t)}, F(x_{m(t)}, y_{m(t)})), \\ &\qquad \frac{d(x_{n(t)}, F(x_{m(t)}, y_{m(t)})) + d(x_{n(t)}, F(x_{m(t)}, y_{m(t)}))}{2}\Big\}\Big) \end{aligned}$$

$$\leq \beta\xi\left(\max\left\{d(x_{n(t)},x_{m(t)}),d(x_{n(t)},x_{n(t)+1}),d(x_{m(t)},x_{m(t)+1}),\frac{d(x_{n(t)},x_{m(t)+1})+d(x_{n(t)},x_{m(t)+1})}{2}\right\}\right).$$

Taking limit $t\to\infty$ and using previous equations as required, we get

$$\xi(\varepsilon)\leq\beta\xi(\varepsilon),$$

which is not possible by definition of ξ. Hence $\{x_n\}$ is a Cauchy sequence in X.

Similarly $\{y_n\}$ is also a Cauchy sequence in X. But X is given to be a complete space. So $\exists\ x,y\in X$ such that

$$x_n\to x \text{ and } y_n\to y. \tag{2.16}$$

Then by hypothesis (iii) $x_n\leq x$, $y_n\geq y$ for all $n\in N$.

So using hypothesis (iv), we get

$$\xi(d(x_{n(t)+1},F(x,y))=\xi(d(F(x_n,y_n),F(x,y)))$$

$$\leq\beta\xi\left(\max\left\{d(x_n,x),d(x_n,F(x_n,y_n)),d(x,F(x,y)),\frac{d(x,F(x_n,y_n))+d(x_n,F(x,y))}{2}\right\}\right).$$

Taking limit as $n\to\infty$ and using (2.5) and (2.16), we get

$$\xi(d(x,F(x,y)))\leq\beta\xi(d(x,F(x,y)))$$

$\Rightarrow\quad d(x,F(x,y))=0$

$$\Rightarrow\quad F(x,y)=x \tag{2.17}$$

and

$$\xi(d(y_{n+1},F(y,x))=\xi(d(F(y_n,x_n),F(y,x)))$$

$$\leq\beta\xi\left(\max\left\{d(y_n,y),d(y_n,F(y_n,x_n)),d(y,F(y,x)),\frac{d(y,F(y_n,x_n))+d(y_n,F(y,x))}{2}\right\}\right)$$

$$=\beta\xi\left(\max\left\{d(y_n,y),d(y_n,y_{n+1}),d(y,F(y,x)),\frac{d(y,y_{n+1})+d(y_n,F(y,x))}{2}\right\}\right).$$

Now taking limit as $n\to\infty$ and using (2.6) and (2.16), we get

$$\xi(d(y,F(y,x)))\leq\beta\xi(d(y,F(y,x)))$$

$\Rightarrow\quad d(y,F(y,x))=0$

$$\Rightarrow\quad F(y,x)=y. \tag{2.18}$$

Thus from (2.17) and (2.18) we have $F(x,y)=x$, $F(y,x)=y$

$\Rightarrow\quad (x,y)$ is the coupled fixed point of F.

In the next theorem we will show that if hypothesis (iii) is removed from Theorem 2.1 and F is considered as a continuous a continuous mapping then still there exists a coupled fixed point.

Theorem 2.2. Let $(X,d,\leq)$ be a partially ordered complete metric space, ξ is an Altering distance function and $\beta\in(0,1)$. Let $F:X^2\to X$ be continuous mapping such that the following conditions are satisfied:

(i) there exists $(x_0,y_0)\in X^2$ such that $(x_0,y_0)\leq(F(x_0,y_0),F(y_0,x_0))$,

(ii) for $(x_1,y_1),(x_2,y_2)\in X^2$,
$(x_1,y_1)\leq(x_2,y_2)\Rightarrow F(x_1,y_1)\leq F(x_2,y_2)$ and $F(y_1,x_1)\geq F(y_2,x_2)$,

(iii) $\xi(d(F(x_1,y_1),F(x_2,y_2))\leq\beta\xi\left(\max\left\{d(x_1,x_2),d(x_1,F(x_1,y_1)),d(x_2,F(x_2,y_2)),\frac{d(x_2,F(x_1,y_1))+d(x_1,F(x_2,y_2))}{2}\right\}\right)$

for either $x_1\leq x_2$ and $y_1\geq y_2$ or $x_1\geq x_2$ and $y_1\leq y_2$.

Then F has a coupled fixed point.

Proof. From the proof of the Theorem 2.1, we find that $\{x_n\}$ is a monotone non-decreasing Cauchy sequence converging to $x\in X$ and $\{y_n\}$ is monotone non-increasing Cauchy sequence converging to $y\in X$.

That is

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y .$$

Then continuity of F implies that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(y, x) .$$

Hence we find $(x, y) \in X^2$ such that $F(x, y) = x$ and $F(y, x) = y$

$\Rightarrow$ (x, y) is the coupled fixed point of F.

Theorem 2.3. Let $(X, d, \leq)$ be a finally ordered complete metric space and $\xi : [0,\infty) \to [0,\infty)$ is an Altering distance function. Let $F : X^2 \to X$ be a mapping such that the following conditions are satisfied:

(i) there exists $(x_0, y_0) \in X^2$ such that $(x_0, y_0) \in (F(x_0, y_0), F(y_0, x_0))$,

(ii) for $(x_1, y_1), (x_2, y_2) \in X^2$,
$(x_1, y_1) \leq (x_2, y_2) \Rightarrow F(x_1, y_1) \leq F(x_2, y_2)$ and $F(y_1, x_1) \geq F(y_2, x_2)$,

(iii) If $(x_n, y_n) \to (x, y)$ is monotone non-decreasing in first co-ordinate and monotone non-increasing in second co-ordinate then $x_n \leq x$, $y_n \geq y$ for all n,

(iv) $$\xi(d(F(x_1, y_1), F(x_2, y_2)) \leq \xi\left(\max\left\{ d(x_1, x_2), d(x_1, F(x_1, y_1)), d(x_2, F(x_2, y_2)), \frac{d(x_2, F(x_1, y_1)) + d(x_1, F(x_2, y_2))}{2} \right\}\right) - \eta\left(\max\left\{d(x_1, x_2), d(x_2, F(x_2, y_2))\right\}\right)$$

for either $(x_1, y_1) \leq (x_2, y_2)$ or $(x_2, y_2) \leq (x_1, y_1)$, where $\eta : [0,\infty) \to [0,\infty)$ is any continuous function with $\eta(t) = 0$ iff $t = 0$. Then F has a coupled fixed point.

Proof. We will construct the same sequence $\{x_n\}$ and $\{y_n\}$ as in Theorem 2.1.

Now, if there exist a positive integer ℓ such that $x_\ell = x_{\ell+1}$ and $y_\ell = y_{\ell+1}$ then (x_ℓ, y_ℓ) is a coupled fixed point of F. Hence we assume that either $x_n \neq x_{n+1}$ or $y_n \neq y_{n+1}$ for all $n \geq 0$. First we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$.

Now since $x_n \leq x_{n+1}$, $y_n \geq y_{n+1}$, so using the hypothesis (iv), we have

$$\begin{aligned}
\xi(d(x_{n+1}, x_{n+2})) &= \xi(d(F(x_n, y_n), F(x_{n+1}, y_{n+1}))) \\
&\leq \xi\left(\max\left\{ d(x_n, x_{n+1}), d(x_n, F(x_n, y_n)), d(x_{n+1}, F(x_{n+1}, y_{n+1})), \frac{d(x_{n+1}, F(x_n, y_n)) + d(x_n, F(x_{n+1}, y_{n+1}))}{2} \right\}\right) \\
&\quad -\eta\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, F(x_{n+1}, y_{n+1}))\right\}\right) \\
&= \xi\left(\max\left\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})}{2} \right\}\right) \\
&\quad -\eta\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right) \\
&\leq \xi\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right) - \eta\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right) \\
&\qquad \left(\because \frac{d(x_n, x_{n+2})}{2} \leq \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right)
\end{aligned}$$

$$\Rightarrow \quad \xi\left(d(x_{n+1}, x_{n+2})\right) \leq \xi\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right) - \eta\left(\max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}\right) \tag{2.19}$$

Suppose $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, for some positive integer n.

Then (2.19) implies

$$\xi(d(x_{n+1}, x_{n+2})) \leq \xi(d(x_{n+1}, x_{n+2})) - \eta(d(x_{n+1}, x_{n+2}))$$

$\Rightarrow$ $\eta(d(x_{n+1}, x_{n+2})) \leq 0$

$\Rightarrow$ $d(x_{n+1}, x_{n+2}) = 0$

$\Rightarrow \quad x_{n+1} = x_{n+2}$ which is contradiction to our assumption that $x_n \neq x_{n+1}$ for all n

$\Rightarrow \quad d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all n (2.20)

and $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers

$\Rightarrow$ there exist a real number $p \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = p. \tag{2.21}$$

Now using (2.19) and (2.20), we have

$$\xi(d(x_{n+1}, x_{n+2})) \leq \xi(d(x_n, x_{n+1})) - \eta(d(x_n, x_{n+1})).$$

Taking limit $n \to \infty$ and using (2.21)

$$\xi(p) \leq \xi(p) - \eta(p) \qquad (\because \ \xi, \eta \text{ are continuous mappings})$$

which will lead to a contradiction unless $p = 0$

$$\Rightarrow \quad \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.22}$$

Similarly if we assume $y_n \neq y_{n+1}$ for all n. Then we will arrive a contraction and we get

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \tag{2.23}$$

Now we will show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X.

Firstly, if $\{x_n\}$ is not a Cauchy sequence in X, then using the same arguments as in Theorem 2.1, we have

$$\lim_{n\to\infty} d(x_{n(t)}, x_{m(t)}) = \varepsilon, \tag{2.24}$$

$$\lim_{n\to\infty} d(x_{n(t)+1}, x_{m(t)+1}) = \varepsilon, \tag{2.25}$$

$$\lim_{n\to\infty} d(x_{n(t)}, x_{m(t)+1}) = \varepsilon, \tag{2.26}$$

$$\lim_{n\to\infty} d(x_{m(t)}, x_{n(t)+1}) = \varepsilon. \tag{2.27}$$

Now $m(t) > n(t) \Rightarrow x_{m(t)} \geq x_{n(t)}$ and $y_{m(t)} \leq y_{n(t)}$ for all $t \in N$.

Using condition (iv), we get

$$\begin{aligned}
\xi(d(x_{n(t)+1}, x_{m(t)+1})) &\leq \xi(d(F(x_{n(t)}, y_{n(t)}), F(x_{m(t)}, y_{m(t)}))) \\
&\leq \xi\left(\max\left\{d(x_{n(t)}, x_{m(t)}), d(x_{n(t)}, F(x_{n(t)}, y_{n(t)})), d(x_{m(t)}, F(x_{m(t)}, y_{m(t)})), \frac{d(x_{m(t)}, F(x_{n(t)}, y_{n(t)})) + d(x_{n(t)}, F(x_{m(t)}, y_{m(t)}))}{2}\right\}\right) \\
&\quad -\eta\left(\max\left\{d(x_{n(t)}, x_{m(t)}), d(x_{m(t)}, F(x_{m(t)}, y_{m(t)}))\right\}\right) \\
&= \xi\left(\max\left\{d(x_{n(t)}, x_{m(t)}), d(x_{n(t)}, x_{n(t)+1}), d(x_{m(t)}, x_{m(t)+1}), \frac{d(x_{m(t)}, x_{n(t)+1}) + d(x_{n(t)}, x_{m(t)+1})}{2}\right\}\right) \\
&\quad -\eta\left(\max\left\{d(x_{n(t)}, x_{m(t)}), d(x_{m(t)}, x_{m(t)+1})\right\}\right)
\end{aligned}$$

Taking $t \to \infty$ and using (2.21)-(2.27), we have

$$\xi(\varepsilon) \leq \xi(\varepsilon) - \eta(\varepsilon)$$

which is not possible by definition of η.

Hence $\{x_n\}$ is a Cauchy sequence in X.

Similarly $\{y_n\}$ is also a Cauchy sequence in X.

But X is given to be complete so $\exists \ x, y \in X$ such that

$$x_n \to x \text{ and } y_n \to y \text{ as } n \to \infty. \tag{2.28}$$

Then by hypothesis (iii) $x_n \leq x$ and $y_n \leq y$ for all n.

Now we can use condition (iv) and get

$$\begin{aligned}
\xi(d(x_{n+1}, F(x, y)) &= \xi(d(F(x_n, y_n), F(x, y))) \\
&\leq \xi\left(\max\left\{d(x_n, x), d(x_n, F(x_n, y_n)), d(x, F(x, y)), \frac{d(x, F(x_n, y_n)) + d(x_n, F(x, y))}{2}\right\}\right) \\
&\quad -\eta\left(\max\left\{d(x_n, x), d(x, F(x, y))\right\}\right).
\end{aligned}$$

Taking limit $n \to \infty$ and using (2.22) and (2.28), we get

$$\xi(d(x, F(x,y)) \le \xi(d(x, F(x,y))) - \eta(d(x, F(x,y))) \quad (\because \ \xi, \eta \text{ are continuous})$$

which will lead to a contradiction unless $d(x, F(x,y)) = 0$

$$\Rightarrow \quad F(x,y) = x. \tag{2.29}$$

Also

$$\begin{aligned}\xi(d(y_{n+1}, F(y,x)) &= \xi(d(F(y_n, x_n), F(y,x))) \\ &\le \xi\left(\max\left\{d(y_n, y), d(y_n, F(y_n, x_n)), d(y, F(y,x)), \frac{d(y, F(y_n, x_n)) + d(y_n, F(y,x))}{2}\right\}\right) \\ &\quad -\eta\left(\max\left\{d(y_n, y), d(y, F(y,x))\right\}\right) \\ &= \psi\left(\max\left\{d(y_n, y), d(y_n, y_{n+1}), d(y, F(y,x)), \frac{d(y, y_{n+1}) + d(y_n, F(y,x))}{2}\right\}\right).\end{aligned}$$

Taking limit as $n \to \infty$ and using (2.23) and (2.28), we get

$$\xi(d(y, F(y,x)) \le \xi(d(y, F(y,x))) - \eta(d(y, F(y,x)))$$

which will lead to a contradiction unless

$$d(y, F(y,x)) = 0$$

$$\Rightarrow \quad F(y,x) = y. \tag{2.30}$$

Thus we get $(x,y) \in X^2$ such that $F(x,y) = x$ and $F(y,x) = y$. (using (2.29) and (2.30))

Hence (x,y) is a coupled fixed point of F.

In the next theorem, we will show that if (iii) hypothesis of Theorem 2.3 is replaced by continuity of F, then still F has a coupled fixed point.

Theorem 2.4. Let $(X, d, \le)$ be a partially ordered complete metric space and $\xi : [0,\infty) \to [0,\infty)$ be an Altering distance function. Let $F : X^2 \to X$ be a continuous mapping such that the following conditions holds:

(i) there exists $(x_0, y_0) \in X^2$ such that $(x_0, y_0) \le (F(x_0, y_0), F(y_0, x_0))$,

(ii) for $(x_1, y_1), (x_2, y_2) \in X^2$,

$(x_1, y_1) \le (x_2, y_2) \Rightarrow F(x_1, y_1) \le F(x_2, y_2)$ and $F(y_1, x_1) \ge F(y_2, x_2)$

(iii) $$\begin{aligned}\xi(d(F(x_1, y_1), F(x_2, y_2)) &\le \xi\left(\max\left\{d(x_1, x_2), d(x_1, F(x_1, y_1)), d(x_2, F(x_2, y_2)), \frac{d(x_2, F(x_1, y_1)) + d(x_1, F(x_2, y_2))}{2}\right\}\right) \\ &\quad -\eta\left(\max\left\{d(x_1, x_2), d(x_2, F(x_2, y_2))\right\}\right)\end{aligned}$$

for either $(x_1, y_1) \le (x_2, y_2)$ or $(x_2, y_2) \le (x_1, y_1)$, where $\eta : [0,\infty) \to [0,\infty)$ is any continuous function with $\eta(t) = 0$ iff $t = 0$. Then F has a coupled fixed point.

Proof. From the proof of Theorem 2.1, we find sequence $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Then continuity of F implies that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x,y)$$

And

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(y,x).$$

Hence we find $(x,y) \in X^2$ such that

$$F(x,y) = x \text{ and } F(y,x) = y$$

$\Rightarrow$ (x,y) is the coupled fixed point of F.

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