On Semi-Symmetric Metric Connection in a Generalized (k,μ) Space Forms

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Abstract – **In this paper we study some properties of curvature tensor on semi-symmetric metric connection in a generalized (k,μ) space forms. As a consequence of these results we investigate the conditions for a generalized (k,μ) space forms to be h-projectively semi-symmetric, ϕ-projectively semi-symmetric and Ricci semi-symmetric with respect to semi-symmetric metric connection. In all these cases the manifold becomes an η-Einstein manifold.**

Keywords **- generalized (k,μ) space forms, Ricci semi-symmetric, projective curvature tensor, h-projectively semi-symmetric, ϕ-projectively semi-symmetric, η-Einstein manifolds.**

1. INTRODUCTION

In 1924, Friedmann and Schouten [7] introduced the idea of semi-symmetric linear connection on a differentiable manifold. A linear connection $\tilde{\nabla}$ in an n-dimensional differentiable manifold M is said to be semi-symmetric connection if its torsion $\tilde{T}$ is of the form

$$\tilde{T}(X,Y) = u(X)Y - u(Y)X, \tag{1}$$

where u is a 1-form. The connection $\tilde{\nabla}$ is a metric connection if there is a Riemannian metric g in M such that $\tilde{\nabla}g = 0$, otherwise it is non-metric. In 1930, H.A.Hayden [8]defined a semi-symmetric metric connection on a Riemannian manifold and this was further developed by K.Yano[18]. In [3], Agashe and Chafle introduced a semi-symmetric non-metric connection on a Riemannian manifold and this was further studied by U.C.De and D.Kamilya[15], J.Sengupta, U.C.De and T.Q.Binh[11], S.C.Biswas and U.C.De[10], B.B.Chaturvedi and P.N.Pandey[4] and others. In[9], Sharfuddin and Hussian defined a semi-symmetric metric connection in an almost contact manifold by identifying the 1-form u in(1) with the contact form η, that is by setting

$$\tilde{T}(X,Y) = \eta(X)Y - \eta(Y)X. \tag{2}$$

U.C.De and J.Sengupta[16] investigated the curvature tensor of an almost contact metric manifold that admits a type of semi-symmetric metric connection and studied the properties of curvature tensor,conformal curvature tensor and projective curvature tensor. M.M.Tripathi[12] studied the semi-symmetric metric connection in a Kenmotsu manifolds. In [13], the semi-symmetric non metric connection in a Kenmotsu manifold was studied by M.M.Tripathi and N.Nakkar. Also in [14], M.M.Tripathi proved the existence of a new connection and showed, under particular cases, this connection reduces to semi-symmetric connections, which are not introduced so far. On the other hand, A generalized Sasakian space form was defined by Carriazo et al. in [1], as an almost contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3, \tag{3}$$

where f_1, f_2, f_3 are some differentiable functions on M and

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y$$
$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z \qquad (4)$$
$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,$$

for any vector fields X,Y,Z on M. In [6] , the authors have defined a generalized (k,μ) space form as an almost contact metric manifold (M,ϕ,ξ,η,g) whose curvature tensor can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \qquad (5)$$

where $f_1, f_2, f_3, f_4, f_5, f_6$ are differentiable functions on M and R_1,R_2,R_3 are tensors defined above and

$$R_4(X,Y)Z = g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,$$
$$R_5(X,Y)Z = g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX,$$
$$R_6(X,Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,$$

for any vector fields $X,Y,Z,$ where $2h = L_\xi\phi$ and L is the usual Lie derivative.This manifold was denoted by $M(f_1, f_2, f_3, f_4, f_5, f_6)$.

Natural examples of generalized (k,μ) space forms are (k,μ) space forms and generalized Sasakian space forms. The authors in[1] proved that contact metric generalized (k,μ) space forms are generalized (k,μ) spaces and if dimension is greater than or equal to 5, then they are (k,μ) spaces with constant $\phi-$sectional curvature $2f_6 -1.$ They gave a method of constructing examples of generalized (k,μ) space forms and proved that generalized (k,μ) space forms with trans-Sasakian structure reduces to generalized Sasakian space forms. Further in [2], it is proved that under D_a-homothetic deformation generalized (k,μ) space form structure is preserved for dimension 3, but not in general.

In this paper we study the semi-symmetric metric connection in generalized (k,μ) space form. Section 2 is devoted to preliminaries. In section 3, we find the expression for curvature tensor(resp. Ricci tensor) with respect to the semi-symmetric metric connection and investigate relations between curvature tensor (resp.Ricci tensor) with respect to Levi-civita connection. In sections 4,5,6 respectively we investigate the conditions for a generalized (k,μ) space forms to be Ricci semi-symmetric, ϕ-projectively semi-symmetric and h-projectively semi-symmetric with respect to semi-symmetric metric connection. In all these cases the manifold becomes an η-Einstein manifold.

2. PRELIMINARIES

A (2n+1)-dimensional Riemannian manifold (M,g) is said to be an almost contact metric manifold if it admits a tensor field ϕ of type (1,1), a vector field ξ, and a 1-form η satisfying

$$\phi^2 = -I + \eta\otimes\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\circ\phi = 0, \qquad (6)$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad (7)$$

$$g(X,\phi Y) = -g(\phi X,Y), g(X,\phi X) = 0, g(X,\xi) = \eta(X). \qquad (8)$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Phi$,

where $\Phi(X,Y) = g(X,\phi Y)$ is the fundamental 2-form of M.

It is well known that on a contact metric manifold (M,ϕ,ξ,η,g), the tensor h is defined by $2h = L_\xi\phi$ which is symmetric and satisfies the following relations.

$$h\xi = 0, h\phi = -\phi h, trh = 0, \eta\circ h = 0, \qquad (9)$$

$$\nabla_X \xi = -\phi X - \phi hX, (\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{10}$$

In a $(2n+1)$-dimensional (k,μ)-contact metric manifold, we have [5]

$$h^2 = (k-1)\phi^2, k \leq 1, \tag{11}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{12}$$

$$\begin{aligned}(\nabla_X h)(Y) = \; & [(1-k)g(X,\phi Y) + g(X, h\phi Y)]\xi + \eta(Y)h(\phi X + \phi hX) \\ & - \mu\eta(X)\phi hY.\end{aligned} \tag{13}$$

Definition 1: *A contact metric manifold M is said to be*

(i) Einstein if $S(X,Y) = \lambda g(X,Y)$, where λ is a constant and S is the Ricci tensor,

(ii) η-Einstein if $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$, where α and β are smooth functions on M.

In a $(2n+1)$-dimensional generalized (k,μ) space-form, the following relations hold.

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY], \tag{14}$$

$$\begin{aligned}QX = \; & [2nf_1 + 3f_2 - f_3]X + [(2n-1)f_4 - f_6]hX \\ & -[3f_2 + (2n-1)f_3]\eta(X)\xi,\end{aligned} \tag{15}$$

$$\begin{aligned}S(X,Y) = \; & [2nf_1 + 3f_2 - f_3]g(X,Y) + [(2n-1)f_4 - f_6]g(hX,Y) \\ & -[3f_2 + (2n-1)f_3]\eta(X)\eta(Y),\end{aligned} \tag{16}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{17}$$

$$r = 2n[(2n+1)f_1 + 3f_2 - 2f_3], \tag{18}$$

for any vector fields X,Y,Z where Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of $M(f_1,\ldots,f_6)$.

The relation between the associated functions $f_i, i = 1,\ldots,6$ of $M(f_1,\ldots,f_6)$ was recently discussed by Carriazo et al. [6]. Let M be a (2n+1)-dimensional generalized (k,μ) space-form and ∇ be Levi-Civita connection on M. A linear connection $\tilde{\nabla}$ on M is said to be semi-symmetric if the torsion tensor $\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$ satisfies

$$\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y \tag{19}$$

for all $X,Y \in TM$. A semi-symmetric connection $\tilde{\nabla}$ is called semi-symmetric metric connection, if it further satisfies $\tilde{\nabla}g = 0$.

A semi-symmetric metric connection $\tilde{\nabla}$ in a generalized (k,μ) space-form can be defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{20}$$

where ∇ is the Levi-Civita connection on M ([17],[9]).

3. GENERALIZED (k,μ) SPACE-FORM ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

Let M be a (2n+1)-dimensional generalized (k,μ) space-form. The curvature tensor $\tilde{R}$ of M with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z. \tag{21}$$

From (20) and (21) we have,

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)\beta(X) + g(X,Z)\beta(Y), \tag{22}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the curvature tensor of M with respect to Levi-Civita connection ∇, α is a tensor field of type (0,2) defined by

$$\alpha(X,Y) = (\tilde{\nabla}_X\eta)(Y) + \frac{1}{2}g(X,Y) \tag{23}$$

and

$$\beta(X) = \tilde{\nabla}_X\xi + \frac{1}{2}X. \tag{24}$$

Theorem 1: *Let* M *be a generalized* (k,μ) *space-form with the semi-symmetric metric connection* $\tilde{\nabla}$. *Then* $\alpha(X,Y) = g(\beta(X),Y)$ *for all* $X,Y \in TM$.

Proof: By using the definition of β , (6) and (8), we have

$$\begin{aligned} g(\beta(X),Y) &= g(\tilde{\nabla}_X\xi + \frac{1}{2}X, Y) \\ &= g(\nabla_X\xi + \eta(\xi)X - g(X,\xi)\xi + \frac{1}{2}X, Y) \\ &= -g(\phi X,Y) - g(\phi hX,Y) + \frac{3}{2}g(X,Y) - \eta(X)\eta(Y). \end{aligned} \tag{25}$$

On the other hand from (6) and (10), we obtain

$$\begin{aligned} (\tilde{\nabla}_X\eta)(Y) &= \tilde{\nabla}_X\eta(Y) - \eta(\tilde{\nabla}_X Y) \\ &= -g(\phi X,Y) - g(\phi hX,Y) + g(X,Y) - \eta(X)\eta(Y). \end{aligned} \tag{26}$$

This completes the proof.

From theorem(1) we have the following corollary

Corollary 1: *In a generalized* (k,μ) *space-form, the tensor field* β *of type (1,1) is self-adjoint.*

Theorem 2: *In a (2n+1)-dimensional generalized* (k,μ) *space-form* M *the Ricci tensor* $\tilde{S}$ *and scalar curvature* $\tilde{r}$ *of semi-symmetric metric connection* $\tilde{\nabla}$ *are given by*

$$\tilde{S}(Y,Z) = S(Y,Z) - (2n-1)\alpha(Y,Z) - trace(\alpha)g(Y,Z) \tag{27}$$

and

$$\tilde{r} = r - 4n\, trace(\alpha) \tag{28}$$

where S and r denote the Ricci tensor and scalar curvature of M with respect to Levi-Civita connection ∇ respectively. Consequently , $\tilde{S}$ is symmetric.

Proof: From (22) we have

$$\begin{aligned} g(\tilde{R}(X,Y)Z,U) = \; & g(R(X,Y)Z,U) - \alpha(Y,Z)g(X,U) + \alpha(X,Z)g(Y,U) \\ & - g(Y,Z)g(\beta(X),U) + g(X,Z)g(\beta(Y),U). \end{aligned} \tag{29}$$

Put $X = U = e_i$ and summing up with respect to i. Then (29) implies (27) and (28) follows from (27). Also from (27), it is obvious that $\tilde{S}$ is symmetric.

Lemma 1: *Let* M *be an (2n+1)-dimensional generalized* (k,μ) *space-form with the semi-symmetric metric connection* $\tilde{\nabla}$. *Then*

$$\eta(\tilde{R}(X,Y)Z) = \left(f_1 - f_3 - \frac{1}{2}\right)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)](f_4 - f_6)[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)] - \alpha(Y,Z)\eta(X) + \alpha(X,Z)\eta(Y) \quad (30)$$

$$\tilde{R}(X,Y)\xi = \left(f_1 - f_3 - \frac{1}{2}\right)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY] - \eta(Y)\beta(X) + \eta(X)\beta(Y) \quad (31)$$

$$\tilde{R}(\xi,Y)Z = (f_1 - f_3 - 2)[g(Y,Z)\xi - \eta(Z)Y] + (f_4 - f_6)[g(hY,Z)\xi - \eta(Z)hY] + g(\phi Y,Z)\xi + g(\phi hY,Z)\xi - \eta(Z)[\phi Y + \phi hY] \quad (32)$$

$$\tilde{R}(\xi,Y)\xi = (f_1 - f_3 - 2)[\eta(Y)\xi - Y] + (f_6 - f_4)hY - \phi Y - \phi hY \quad (33)$$

$$\tilde{S}(X,\xi) = \left[2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)\right]\eta(X) \quad (34)$$

Proof: In the view of relations (22) ,(27) ,(14) and (17) we have obtain the above lemma.

4. RICCI SEMI-SYMMETRIC GENERALIZED (k,μ) SPACE-FORM

Theorem 3: *A Ricci semi-symmetric generalized* (k,μ) *space-form with the semi-symmetric metric connection is an* η*-Einstein manifold.*

Proof: Let a (2n+1)-dimensional generalized (k,μ) space-form M is said to be Ricci semi-symmetric with respect to semi-symmetric metric connection which satisfying the condition

$$\tilde{R}(X,Y)\cdot\tilde{S} = 0. \quad (35)$$

So we have

$$\tilde{S}(\tilde{R}(X,Y)U,V) + \tilde{S}(U,\tilde{R}(X,Y)V) = 0. \quad (36)$$

Put $X = U = \xi$ and using (32),(33) ,(34) in (36), we obtain

$$\tilde{S}(Y,V) = A_1 g(Y,V) + A_2 g(hY,V) - A_3 g(\phi Y,hV) + A_4 g(\phi Y,V) - \left[\frac{(2n-1)k}{(f_1 - f_3 - 2)}\right]\eta(Y)\eta(V), \quad (37)$$

where

$$A_1 = [\left(2n(f_1 - f_3) - \frac{(2n-1)}{2} - trace(\alpha)\right) + (2n-1)(f_4 - f_6 + k - 1)]$$

$$A_2 = \left[\frac{(f_4 - f_6)(4nf_1 - (2n+1)f_3 + 3f_2 - (n+1)) + 2(2n-1)}{(f_1 - f_3 - 2)}\right]$$

$$A_3 = \left[\frac{(4n-2)f_4 - 2nf_6 - 3f_2 + (1-2n)f_3 - (n-2)}{(f_1 - f_3 - 2)}\right]$$

$$A_4 = \left[\frac{(1-k)(2(2n-1)f_4 - 2nf_6) - 3f_2 + (1-2n)f_3 + 2n - 1}{(f_1 - f_3 - 2)}\right].$$

Take $Y=\phi Y$ and $V=hY$ and use (27) in (37), to get

$$\begin{aligned}\tilde{S}(Y,V)= &\left[A_1-(k-1)\frac{A_3(1-A_3)}{E}\right]g(Y,V)+\left[A_2-\frac{A_3(1-A_4)}{E}\right]g(hY,V)\\ &+\left[A_4-(k-1)\frac{A_3((2n-1)f_4-f_6-B)}{E}\right]g(\phi Y,V)\\ &-\left[(k-1)\frac{A_3(A_3-1)}{E}+\frac{(2n-1)k}{(f_1-f_3-2)}\right]\eta(Y)\eta(V).\end{aligned} \tag{38}$$

Replacing Y by hY and using (27) in (38), we have

$$\begin{aligned}\tilde{S}(Y,V)= &\Bigg[\left(A_1-\frac{(k-1)A_3(1-A_3)}{E}\right)+\Big((2n-1)f_4-f_6-A_2\\ &+\frac{A_3(1-A_4)-I_1(1-A_3)}{E}\Big)(k-1)\left(\frac{A_2}{I_2}-\frac{A_3(1-A_4)}{EI_2}\right)\Bigg]g(Y,V)\\ &+\Bigg[A_4-\left(\frac{(k-1)A_3[(2n-1)f_4-f_6-A_2]+I_1((2n-1)f_4-f_6-A_2)}{E}-1\right)\\ &(k-1)\left(\frac{A_2}{I_2}-\frac{A_3(1-A_4)}{EI_2}\right)\Bigg]g(\phi Y,V)+\Bigg[(k-1)\left(\frac{A_2}{I_2}-\frac{A_3(1-A_4)}{EI_2}\right)\\ &\left(A_2-(2n-1)f_4+f_6+\frac{I_1(A_3-1)-A_3(1-A_4)}{E}\right)\\ &-\left(\frac{(2n-1)k}{(f_1-f_3-2)}+\frac{(k-1)A_3(A_3-1)}{E}\right)\Bigg]\eta(Y)\eta(V),\end{aligned} \tag{39}$$

where

$$\begin{aligned}E&=(2nf_1+3f_2-f_3)-\frac{3}{2}+trace(\alpha)-A_1\\ I_1&=A_4-\frac{(k-1)A_3((2n-1)f_4-f_6-A_2)}{E}+1\\ I_2&=E+\frac{(k-1)A_3(1-A_3)+I_1(1-A_4)}{E}.\end{aligned}$$

Again replace Y by ϕY and use (27) in (39), to obtain

$$\tilde{S}(Y,V)=\lambda_1 g(Y,V)+\lambda_2\eta(Y)\eta(V), \tag{40}$$

where

$$\lambda_1 = \left((2n-1)f_4 - f_6 - A_2 + \frac{A_3(1-A_4) - I_1(1-A_3)}{E}\right)\left(\frac{A_2}{I_2} - \frac{A_3(1-A_4)}{EI_2}\right)$$
$$\left(\frac{A_4}{I_3} - \frac{(k-1)((2n-1)f_4 - f_6 - A_2)}{EI_3}\right) - (k-1)\left(\frac{A_2}{I_2I_3} - \frac{A_3(1-A_4)}{EI_2I_3}\right)$$
$$\left(\frac{I_1((2n-1)f_4 - f_6 - A_2)}{E} - 1\right)\left[1 - \left(A_4 - \frac{(k-1)A_3((2n-1)f_4 - f_6 - A_2)}{E}\right)\right.$$
$$\left.\left(\frac{A_2}{I_2} - \frac{A_3(1-A_4)}{EI_2}\right)\left(\frac{(k-1)I_1((2n-1)f_4 - f_6 - A_2)}{E} - 1\right)\right]$$
$$- \frac{(k-1)(1-A_3)((2n-1)f_4 - f_6)}{E} + \left(\frac{[1-((2n-1)f_4 - f_6)](k-1)}{EI_2}\right)$$
$$\left((2n-1)f_4 - f_6 - A_2 + \frac{A_3(1-A_4) - I_1(1-A_3)}{E}\right)$$

$$I_3 = 3f_2 + n - 2 + 2\,trace\,(\alpha) + \frac{(k-1)A_3(1-A_3)}{E} - (2n-1)(f_4 - f_6 + k - 1)$$
$$-\left(\frac{A_2}{I_2} - \frac{A_3(1-A_4)}{EI_2}\right)(k-1)\left((2n-1)f_4 - f_6 - A_2 + \frac{A_3(1-A_4) - I_1(1-A_3)}{E}\right)$$
$$-(k-1)\left(\frac{1}{I_2} - \frac{(2n-1)f_4 - f_6}{EI_2}\right)\left(\frac{I_2((2n-1)f_4 - f_6 - A_2)}{E} - 1\right)$$
$$+\frac{[(2n-1)f_4 - f_6](k-1)[(2n-1)f_4 - f_6 - A_2]}{E}$$

$$\lambda_2 = \left(\frac{A_2}{I_2} - \frac{A_3(1-A_4)}{EI_2}\right)(k-1)\left(A_2 - (2n-1)f_4 + f_6 + \frac{I_1(A_3-1) - A_3(1-A_4)}{E}\right)$$
$$-\left(\frac{(2n-1)k}{(f_1 - f_3 - 2)} + \frac{(k-1)A_3(A_3-1)}{E}\right) + \left(\left(\frac{(1-k)A_3[(2n-1)f_4 - f_6 - A_2]}{EI_3}\right.\right.$$
$$\left.+\frac{A_4}{I_3}\right) - \left(\frac{A_2}{I_2I_3} - \frac{A_3(1-A_4)}{EI_2I_3}\right)(k-1)\left(\left.\frac{I_1[(2n-1)f_4 - f_6 - A_2]}{E} - 1\right)\right)$$
$$\left(\left(A_4 - \frac{(k-1)A_3[(2n-1)f_4 - f_6 - A_2]}{E}\right) - \left(\frac{A_2}{I_2} - \frac{A_3(1-A_4)}{EI_2}\right)(k-1)\right.$$
$$\left.\left(\frac{I_1[(2n-1)f_4 - f_6 - A_2]}{E} - 1\right)\right) - 1 - \frac{(k-1)(A_3-1)[(2n-1)f_4 - f_6]}{E} + (k-1)$$
$$\left(A_2 - (2n-1)f_4 + f_6 + \frac{I_1(A_3-1) - A_3(1-A_4)}{E}\right)\frac{[1-((2n-1)f_4 - f_6)]}{EI_2}\Bigg).$$

This completes the proof.

5. PROJECTIVE CURVATURE TENSOR GENERALIZED (k,μ) SPACE-FORM WITH RESPECT TO SEMI-SYMMETRIC METRIC CONNECTION

Let M be a (2n+1)-dimension generalized (k,μ) space-form. The projective curvature tensor $\tilde{P}$ of type (1,3) of M with respect to semi-symmetric metric connection is defined by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2n}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \quad (41)$$

5.1 ϕ-PROJECTIVELY SEMI-SYMMETRIC GENERALIZED (k,μ) SPACE-FORM

A Riemannian manifold (M^{2n+1}, g), $n > 1$, is said to be ϕ-projectively semi-symmetric with respect to semi-symmetric metric connection if $\tilde{P}(X,Y)\cdot\phi = 0$ holds on M.

Theorem 4: *Let M be a ϕ-projectively semi-symmetric generalized (k,μ) space-form with respect to semi-symmetric metric connection. Then M is an η-Einstein manifold.*

Proof: Let M be an ϕ-projectively semi-symmetric generalized (k,μ) space-form with respect to semi-symmetric metric connection.The condition $\tilde{P}(X,Y)\cdot\phi = 0$ turns into

$$(\tilde{P}(X,Y)\cdot\phi)Z = \tilde{P}(X,Y)\cdot\phi Z - \phi\tilde{P}(X,Y)Z = 0, \quad (42)$$

for any vector fields X,Y,Z. In view of (41), we have

$$\tilde{R}(X,Y)\phi Z - \phi\tilde{R}(X,Y)Z + \frac{1}{2n}[\tilde{S}(X,\phi Z)Y - \tilde{S}(Y,\phi Z)X + \tilde{S}(Y,Z)\phi X - \tilde{S}(X,Z)\phi Y] = 0 \quad (43)$$

Use (22) and (27) in (43), to obtain

$$\begin{aligned}
&f_1[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y] + f_2[g(X,\phi Z)Y - g(Y,\phi Z)X \\
&+ g(Y,Z)\phi X - g(X,Z)\phi Y + \eta(X)(g(Y,\phi Z)\xi + \eta(Z)\phi Y) - \eta(Y)(g(X,\phi Z)\xi + \eta(Z)\phi X)] \\
&+ f_3[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X] + f_4[g(Y,\phi Z)hX \\
&- g(X,\phi Z)hY + g(hY,\phi Z)X - g(hX,\phi Z)Y - g(Y,Z)\phi hX + g(X,Z)\phi hY - g(hY,Z)\phi X \\
&+ g(hX,Z)\phi Y] + f_5[2g(hY,\phi Z)hX - 2g(hX,\phi Z)hY + 2g(hX,Z)\phi hY - 2g(hY,Z)\phi hX] \\
&+ f_6[g(hX,\phi Z)\eta(Y)\xi - g(hY,\phi Z)\eta(X)\xi - \eta(X)\eta(Z)\phi hY + \eta(Y)\eta(Z)\phi hX] \\
&+ 2g(Y,Z)X - \eta(Y)\eta(Z)X + g(hY,Z)X - 3g(Y,\phi Z)X - 2g(X,Z)Y + \eta(X)\eta(Z)Y \\
&- g(hX,Z)Y + 3g(X,\phi Z)Y - 2g(\phi Y,Z)\phi X + g(Y,\phi Z)\phi hX + \eta(X)g(Y,\phi Z)\xi \\
&+ 2g(\phi X,Z)\phi Y - g(X,\phi Z)\phi hY - \eta(Y)g(X,\phi Z)\xi + 2g(\phi X,Z)\phi Y - g(X,\phi Z)\phi hY \\
&- \eta(Y)g(X,\phi Z)\xi + g(\phi Y,hZ)\phi X + 3g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X - g(\phi X,hZ)\phi Y \\
&- 3g(X,Z)\phi Y + \eta(X)\eta(Z)\phi Y - \eta(X)g(Y,Z)\xi + g(Y,Z)hX + g(X,Z)\eta(Y)\xi \\
&- g(X,Z)hY + \frac{1}{2n}[\tilde{S}(X,\phi Z)Y - \tilde{S}(Y,\phi Z)X + \tilde{S}(Y,Z)\phi X - \tilde{S}(X,Z)\phi Y] = 0.
\end{aligned} \quad (44)$$

Replacing X by ϕX in (44) and taking the inner product with W and using (6) ,we obtain

$$
\begin{aligned}
&f_1[g(Y,\phi Z)g(\phi X,W)-g(X,Z)g(Y,W)+g(Y,Z)g(X,W)-\eta(X)g(Y,Z)\eta(W)\\
&+\eta(X)\eta(Z)g(Y,W)+g(\phi X,Z)g(\phi Y,W)]+f_2[-g(\phi X,Z)g(\phi Y,W)-g(Y,Z)g(X,W)\\
&+\eta(X)\eta(W)g(Y,Z)+\eta(X)\eta(Y)\eta(Z)\eta(W)-g(Y,\phi Z)g(\phi X,W)]+f_3[g(X,Z)\eta(Y)\eta(W)\\
&-g(X,W)\eta(Y)\eta(Z)]+f_4[g(Y,\phi Z)g(\phi X,hW)-g(X,Z)g(hY,W)+\eta(X)\eta(Z)g(hY,W)\\
&+g(hY,\phi Z)g(\phi X,W)+g(hX,Z)g(Y,W)-g(Y,Z)g(hX,W)+g(\phi X,Z)g(\phi hY,W)\\
&+g(hY,Z)g(X,W)-\eta(X)\eta(W)g(hY,Z)+g(\phi X,hZ)g(\phi Y,W)]\\
&+f_5[2g(hY,\phi Z)g(\phi X,hW)+2g(hX,Z)g(hY,W)+2g(\phi X,hZ)g(\phi hY,W)\\
&-2g(hY,Z)g(hX,W)]+f_6[g(hX,W)\eta(Y)\eta(Z)-g(hX,Z)\eta(Y)\eta(W)]+2g(Y,Z)g(\phi X,W)\\
&-\eta(Y)\eta(Z)g(\phi X,W)+g(hY,Z)g(\phi X,W)-3g(Y,\phi Z)g(\phi X,W)-2g(\phi X,Z)g(Y,W)\\
&-g(\phi X,hZ)g(Y,W)+3g(X,Z)g(Y,W)-3\eta(X)\eta(Z)g(Y,W)+2g(\phi Y,Z)g(X,W)\\
&-2\eta(X)g(\phi Y,Z)\eta(W)+g(Y,\phi Z)g(hX,W)-2g(X,Z)g(\phi Y,W)+2\eta(X)\eta(Z)g(\phi Y,W)\\
&-g(X,Z)g(\phi hY,W)+\eta(X)\eta(Z)g(\phi hY,W)-g(X,Z)\eta(Y)\eta(W)+g(\phi hY,Z)g(X,W)\\
&-g(\phi hY,Z)\eta(X)\eta(W)-3g(Y,Z)g(X,W)+3g(Y,Z)\eta(X)\eta(W)+g(X,W)\eta(Y)\eta(Z)\\
&+g(hX,Z)g(\phi Y,W)-3g(\phi X,Z)g(\phi Y,W)+g(Y,Z)g(\phi X,hW)+g(\phi X,Z)\eta(Y)\eta(W)\\
&-g(\phi X,Z)g(hY,W)+\frac{1}{2n}[\tilde{S}(\phi X,\phi Z)g(Y,W)-\tilde{S}(Y,\phi Z)g(\phi X,W)\\
&-\tilde{S}(Y,Z)g(X,W)+\tilde{S}(Y,Z)\eta(X)\eta(W)-\tilde{S}(\phi X,Z)g(\phi Y,W)]=0.
\end{aligned}
\tag{45}
$$

Let $\tilde{e}_i, i=1,2,....(2n+1)$ be an orthonormal ϕ-basis of vector fields in $M(f_1,...,f_6)$. If we put $X=W=\tilde{e}_i$ in (45) and summing up with respect to i and using (6),we obtain

$$\tilde{S}(Y,Z)=B_1 g(Y,Z)+B_2\eta(Y)\eta(Z)+B_3 g(hY,Z)+B_4 g(\phi Y,Z)+B_5 g(\phi Y,hZ), \tag{46}$$

where,

$$B_1 = 2nf_1+(1-2n)f_2+3-6n+\left(\frac{3f_2-f_3+\frac{3}{2}-tracr(\alpha)}{n}\right)$$

$$B_2 = 2n-32nf_3-\left(\frac{3f_2-f_3+\frac{3}{2}-trace(\alpha)}{n}\right)$$

$$B_3=(2n-2)f_4,\ B_4=\left(4n-2-\frac{1}{n}\right),\ B_5=\left(\frac{1}{n}-2n\right).$$

Replacing Y by ϕY and Z by hZ in (46) and using (27) , then we have

$$\tilde{S}(Y,Z) = \left(B_1 + \frac{B_5(k-1)(n-4n^2+1)}{nL_1}\right)g(Y,Z) + \left(B_4 + \frac{B_5(k-1)(f_4-f_6)}{L_1}\right)g(\phi Y,Z) + \left(B_2 + \frac{B_5(k-1)(n-4n^2+1)}{nL_1}\right)\eta(Y)\eta(Z) + \left(B_3 + \frac{B_5(n-2n^2+1)}{nL_1}\right)g(hY,Z), \tag{47}$$

where

$$L_1 = 2(n+1)f_2 - f_3 + 9n - \frac{5}{2} - \left(\frac{3f_2 - f_3 + \frac{3}{2} - trace\ (\alpha)}{n}\right).$$

Now replacement of Y by hY in (47) and use of (27) , gives

$$\begin{aligned}\tilde{S}(Y,Z) = &\left[B_1 + \left(\frac{B_5(k-1)(n-4n^2+1)}{nL_1}\right) + \frac{(1-k)L_4}{L_3}\left(\left(\frac{L_2(n-4n^2+1)}{nL_1}\right)\right.\right.\\ &\left.\left.- f_4 + f_6 + \left(\frac{B_5(n-2n^2+1)}{nL_1}\right)\right)\right]g(Y,Z)\\ &+\left[B_2 + \left(\frac{B_5(k-1)(4n^2-n-1)}{nL_1}\right) + \frac{(1-k)L_4}{L_3}\left(\left(\frac{L_2(4n^2-n-1)}{nL_1}\right)\right.\right.\\ &\left.\left.+ f_4 - f_6 - \left(\frac{B_5(n-2n^2+1)}{nL_1}\right)\right)\right]\eta(Y)\eta(Z)\\ &\left[B_4 + \left(\frac{B_5(k-1)(f_4-f_6)}{L_1}\right) + \frac{L_4}{L3}\left(\left(\frac{L_2(1-k)(f_4-f6)}{L_1}\right)\right.\right.\\ &\left.\left.+(k-1)(2n-1)\right)\right]g(\phi Y,Z),\end{aligned} \tag{48}$$

where

$$\begin{aligned}L_2 &= \frac{(2n^2-n-1)}{n} + \frac{(k-1)B_5(f_4-f_6)}{L_1}\\ L_3 &= \frac{1}{n}\left[(2n^2-3)f_2 + (1-n)f_3 + 4n^2 - \frac{3}{2} + trace\ (\alpha)\right.\\ &\left.-\left(\frac{(k-1)(1-2n^2)(n-4n^2+1)}{nL_1}\right) + \frac{n-n^2)L_2}{L_1}\right]\\ L_4 &= (2n-2)f_4 + \frac{(1-2n^2)(n-2n^2+1)}{n^2L_1}\\ L_5 &= \frac{((2n-1)f_4 - f_6)(n-2n^2+1)}{nL_1} + (2n-1)\\ L_6 &= B_4 + \frac{B_5(k-1)(f_4-f_6)}{L_1} + \frac{L_4}{L_3}\left(\frac{L_2(1-k)(f_4-f_6)}{L_1} + (k-1)(2n-1)\right)\\ L_7 &= 2nf_1 + 3f_2 - f_3 + trace\ (\alpha) - 3n - \frac{1}{2} + \frac{((2n-1)f_4 - f_6)(k-1)(f_4-f_6)}{L_1}\\ &- \frac{L_5(1-k)}{L_3}\left(\frac{L_2}{L_1}(f_4-f_6) - (2n-1)\right) - \left[B_1 + \frac{B_5(k-1)(n-4n^2+1)}{nL_1}\right.\\ &\left.+\frac{L_4(1-k)}{L_3}\left(\frac{L_2(n-4n^2+1)}{L_1} - f_4 + f_6 + \frac{B_5(n-2n^2+1)}{nL_1}\right)\right].\end{aligned}$$

Again replacing Y by ϕY in (48) and using (27), we get

$$\tilde{S}(Y,Z) = \lambda_3 g(Y,Z) + \lambda_4 \eta(Y)\eta(Z). \tag{49}$$

Where

$$\begin{aligned}\lambda_3 = \; & B_1 + \left(\frac{(k-1)B_5(n-4n^2+1)}{nL_1}\right) + \frac{(1-k)L_4}{L_3}\left(\left(\frac{L_2(n-4n^2+1)}{nL_1}\right)\right.\\ & \left.- f_4 + f_6 + \left(\frac{B_5(n-2n^2+1)}{L_1}\right)\right) + \frac{L_6}{L_7}\left[\frac{(1-k)((2n-1)f_4 - f_6)(n-4n^2+1)}{nL_1}\right.\\ & - \frac{L_5(1-k)}{L_3}\left(\frac{L_2}{nL_1}(n-4n^2+1) - f_4 + f_6 + B_5(n-2n^2+1)\right) + (2n-1) - B_4\\ & \left.+ \frac{B_5(1-k)(f_4 - f_6)}{L_1} + \frac{(1-k)L_4}{L_3}\left(\frac{L_2(f_4 - f_6)}{L_1} - (2n-1)\right)\right]\end{aligned}$$

$$\begin{aligned}\lambda_4 = \; & B_2 + \left(\frac{B_5(k-1)(4n_2-n-1)}{nL_1}\right) + \frac{L_4(1-k)}{L_3}\left(\left(\frac{L_2(4n^2-n-1)}{nL_1}\right) - f_6 + f_4\right.\\ & \left.- \frac{B_5(n-2n^2+1)}{nL_1}\right) + \frac{L_6}{L_7}\left[B_4 + \frac{B_5(k-1)(f_4 - f_6)}{L_1}\right.\\ & + \frac{(1-k)L_4}{L_3}\left(\frac{L_2}{L_1}(f_4 - f_6) - (2n-1)\right) - \frac{((2n-1)f_4 - f_6)(k-1)(4n^2-n-1)}{nL_1}\\ & \left.+ (2n-1) + \frac{L_5(1-k)}{L_3}\left(\frac{L_2(4n^2-n-1)}{nL_1} - f_6 + f_4 - \frac{B_5(n-2n^2+1)}{nL_1}\right)\right]\end{aligned}$$

5.2 h-PROJECTIVELY SEMI-SYMMETRIC GENERALIZED (k,μ) SPACE-FORM

A Riemannian manifold (M^{2n+1}, g), $n > 1$, is said to be h-projectively semi-symmetric with respect to semi-symmetric metric connection if $\tilde{P}(X,Y)\cdot h = 0$ holds on M.

Theorem 5 *Let M be an h-projectively semi-symmetric generalized (k,μ) space-form with respect to semi-symmetric metric connection. Then M is an η-Einstein manifold.*

Proof: Let M be an h-projectively semi-symmetric generalized (k,μ) space-form with respect to semi-symmetric metric connection.The condition $\tilde{P}(X,Y)\cdot h = 0$ turns into

$$(\tilde{P}(X,Y)\cdot h)Z = \tilde{P}(X,Y)\cdot hZ - h\tilde{P}(X,Y)Z = 0, \tag{50}$$

for any vector fields X, Y, Z.

In view of (41), we have

$$\tilde{R}(X,Y)hZ - h\tilde{R}(X,Y)Z + \frac{1}{2n}\Big[\tilde{S}(X,hZ)Y - \tilde{S}(Y,hZ)X + \tilde{S}(Y,Z)hX - \tilde{S}(X,Z)hY\Big] = 0. \tag{51}$$

Use (22) and (27) in (51), to obtain

$$\begin{aligned}
& f_1[g(Y,hZ)X - g(X,hZ)Y - g(Y,Z)hX + g(X,Z)hY] + f_2\big[g(X,\phi hZ)\phi Y \\
& - g(Y,\phi hZ)\phi X + 4g(X,\phi Y)\phi hZ - g(X,\phi Z)h\phi Y + g(Y,\phi Z)h\phi X\big] \\
& + f_3\big[g(X,hZ)\eta(Y)\xi g(Y,hZ)\eta(X)\xi - \eta(X)\eta(Z)hY + \eta(Y)\eta(Z)hX\big] \\
& + (k-1) f_4\big[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi\big] \\
& + (k-1) f_5\big[- g(Y,Z)hX + \eta(Y)\eta(Z)hX + g(X,Z)hY - \eta(X)\eta(Z)hY + g(\phi X,Z)\phi hY \\
& - g(\phi Y,Z)\phi hX + g(hY,Z)X - g(hY,Z)\eta(X)\xi - g(hX,Z)Y + g(hX,Z)\eta(Y)\xi \\
& - g(\phi hX,Z)\phi Y + g(\phi hY,Z)\phi X\big] + (k-1) f_6\big[g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi \\
& + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\big] + g(\phi Y,hZ)X + (k-1)g(\phi Y,Z)X - 3g(Y,hZ)X \\
& - g(\phi Y,Z)hX - g(\phi hY,Z)hX + \frac{3}{2}g(Y,Z)hX - \eta(Y)\eta(Z)hX - g(\phi X,hZ)Y \\
& - (k-1)g(\phi X,Z)Y + 3g(X,hZ)Y + g(\phi X,Z)hY + g(\phi hX,Z)hY - \frac{3}{2}g(X,Z)hY \\
& + \eta(X)\eta(Z)hY + g(Y,hZ)\phi X + g(Y,Z)\phi hX + \eta(X)g(Y,hZ)\xi - g(Y,Z)h\phi X \\
& - (k-1)g(Y,Z)\phi X + \frac{3}{2}g(Y,Z)hX - g(X,hZ)\phi Y + g(X,hZ)h\phi Y \\
& - \eta(Y)g(X,hZ)\xi + (k-1)g(X,Z)\phi Y - \frac{3}{2}g(X,Z)hY + g(X,Z)h\phi Y \\
& + \frac{1}{2n}\Big[\tilde{S}(X,hZ)Y - \tilde{S}(Y,hZ)X + \tilde{S}(Y,Z)hX - \tilde{S}(X,Z)hY\Big] = 0.
\end{aligned} \tag{52}$$

Replacing X by hX in (52) and taking the inner product with W and using (6),(9), we obtain

$$\begin{aligned}
& f_1\big[g(Y,hZ)g(hX,W) + g(hX,Z)g(hY,W) + (k-1)\big(g(X,Z)g(Y,W) - \eta(X)\eta(Z) \\
& g(Y,W) + g(Y,Z)g(X,W) - \eta(X)\eta(W)g(Y,Z)\big)\big] + f_2\big[g(\phi hX,W) + 4g(hX,\phi Y) \\
& g(\phi hZ,W) - g(hX,\phi Z)g(h\phi Y,W) + (k-1)\big(g(X,\phi Z)g(\phi Y,W) + g(Y,\phi Z) \\
& g(\phi X,W)\big)\big] + (k-1) f_3\big[2\eta(X)\eta(Y)\eta(Z)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\
& - g(X,W)\eta(Y)\eta(Z)\big] + (k-1) f_4\big[\eta(Y)\eta(Z)g(hX,W) + g(hX,Z)\eta(Y)\eta(W)\big] \\
& + (k-1) f_5[g(hX,Z)g(hY,W) + g(\phi hX,Z)g(\phi hY,W) + g(hY,Z)g(hX,W) \\
& + g(\phi hY,Z)g(\phi hX,W) + (k-1)(g(Y,Z)g(X,W) - g(Y,Z)\eta(X)\eta(W) \\
& - g(X,W)\eta(Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z)\eta(W) + g(\phi Y,Z)g(\phi X,W) + g(X,Z)g(Y,Z) \\
& - g(Y,W)\eta(X)\eta(Z) - g(X,Z)\eta(Y)\eta(W) + g(\phi X,Z)g(\phi Y,W))] \\
& + (k-1) f_6[g(hX,Z)\eta(Y)\eta(W) - g(hX,W)\eta(Y)\eta(Z)] + g(\phi Y,hZ)g(hX,W) \\
& - 3g(Y,hZ)g(hX,W) + g(\phi hX,Z)g(hY,W) - \frac{3}{2}g(hX,Z)g(hY,W) + g(Y,hZ)
\end{aligned}$$

$$g(\phi hX,W)+(k-1)[g(\phi Y,Z)g(X,W)-\eta(X)\eta(W)g(\phi Y,Z)+g(\phi hY,Z)g(X,W)$$

$$-\eta(X)\eta(W)g(\phi hY,Z)-\frac{3}{2}g(Y,Z)g(X,W)+\frac{3}{2}\eta(X)\eta(W)g(Y,Z)+\eta(Y)\eta(Z)g(X,W)$$

$$-\eta(X)\eta(Y)\eta(Z)\eta(W)+g(\phi X,Z)g(Y,W)-g(\phi hX,Z)g(Y,W)-3g(X,Z)g(Y,W)$$

$$+3\eta(X)\eta(Z)g(Y,W)-g(\phi X,Z)g(hY,W)-2g(Y,Z)g(\phi X,W)-g(Y,Z)g(\phi hX,W)$$

$$-\frac{3}{2}g(Y,Z)g(X,W)+\frac{3}{2}g(Y,Z)\eta(X)\eta(W)+g(X,Z)g(\phi Y,W)-g(\phi Y,W)\eta(X)\eta(Z)$$

$$+g(hX,Z)g(h\phi Y,W)+g(h\phi Y,W)\eta(X)\eta(Z)+g(X,Z)\eta(Y)\eta(W)+\eta(X)\eta(Y)$$

$$\eta(Z)\eta(W)-\frac{3}{2}g(hX,Z)g(hY,W)+g(hX,Z)g(\phi Y,W)]+g(hX,Z)g(h\phi Y,W)$$

$$+\frac{1}{2n}[\bar{S}(hX,hZ)g(Y,W)-\bar{S}(Y,hZ)g(hX,W)-\bar{S}(hX,Z)g(hY,W)$$

$$+(k-1)\ (\bar{S}(Y,Z)\eta(X)\eta(W)-\bar{S}(Y,Z)g(X,W))]=0 \tag{53}$$

Let $\tilde{e}_i, i=1,2,....(2n+1)$ be an orthonormal ϕ-basis of vector fields in $M(f_1,..., f_6)$. If we put $X=W=\tilde{e}_i$ in (53) and summing up with respect to i and using (6),we obtain

$$\tilde{S}(Y,Z)=C_1g(Y,Z)+C_2\eta(Y)\eta(Z)+4ng(\phi Y,Z)-C_3g(\phi Y,hZ)-C_4g(hY,Z), \tag{54}$$

where

$$C_1=2nf_1+6f_2+(k-1)(2n-2)f_5-6n-\frac{(2nf_1+3f_2-f_3)}{n}+\frac{3(2n-1)}{4}$$
$$C_2=2nf_3-6f_2-(2n-2)f_5(k-1)+\frac{13}{2}+2n+\frac{(2nf_1+3f_2-f_3)}{n}$$

$$C_3=\frac{3}{2}+n+k \quad , \quad C_4=\frac{(2n-1)f_4-f_6}{n}$$

Replacing Y by ϕY and Z by hZ in (54) and using (27) , then we have

$$\tilde{S}(Y,Z) = \left[C_1 - \frac{(k-1)C_3(C_3-2n+1)}{M_1}\right]g(Y,Z) + \left[C_2 + \frac{(k-1)C_3(2n-1+C_3)}{M_1}\right]$$
$$\eta(Y)\eta(Z) + \left[4n - \frac{C_3(k-1)((2n-1)f_4 - f_6 + C_4)}{M_1}\right]g(\phi Y,Z) \tag{55}$$
$$+\left[\frac{C_3(1+2n)}{M_1} - C_4\right]g(hY,Z),$$

where

$$M_1 = 2nf_1 + 3f_2 - f_3 - \frac{3(2n-1)}{2} - trace(\alpha) - C_1.$$

Now replace Y by hY in (55) and use (27) , to obtain

$$\tilde{S}(Y,Z) = \left[C_1 - \frac{(k-1)C_3(C_3-2n+1)}{M_1} + (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)\right.$$
$$\left.\left(nC_4 - \frac{C_3(1+2n)}{M_1} + C_4 + \frac{M_2(C_3-2n+1)}{M_1}\right)\right]g(Y,Z)$$
$$+\left[C_2 + \frac{(k-1)C_3(2n-1+C_3)}{M_1} + (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)\right. \tag{56}$$
$$\left.\left(\frac{C_3(1+2n)}{M_1} - C_4 - nC_4 - \frac{M_2(2n-1+C_3)}{M_1}\right)\right]\eta(Y)\eta(Z)$$
$$+\left[4n - \frac{(k-1)C_3(nC_4+C_4)}{M_1} + (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)\right.$$
$$\left.\left(2n-1+\frac{M_2(nC_4+C_4)}{M_1}\right)\right]g(\phi Y,Z).$$

Where

$$M_2 = \frac{C_3(k-1)(nC_4+C_4)}{M_1} - 2n - 1$$
$$M_3 = M_1 + \frac{(k-1)C_3(C_3-2n+1) - M_2(2n+1)}{M_1}.$$

Again replace Y by ϕY in (56) and use (27), to get

$$\tilde{S}(Y,Z) = \lambda_5 g(Y,Z) + \lambda_6\eta(Y)\eta(Z), \tag{57}$$

where

$$\lambda_5 = C_1 - \frac{(k-1)C_3(C_3-2n+1)}{M_1} + (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)$$
$$\left(nC_4 + \frac{M_2(C_3-2n+1) - C_3(1+2n)}{M_1} + C_4\right) + \left[4n - \frac{(k-1)C_3(nC_4+C_4)}{M_1}\right.$$
$$\left. + (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)\left(2n-1+\frac{M_2(nC_4+C_4)}{M_1}\right)\right]$$
$$\frac{1}{M_4}\left[4n - \frac{(k-1)C_3(nC_4+C_4)}{M_1} + (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)\right.$$
$$\left(2n-1+\frac{M_2(nC_4+C_4)}{M_1}\right) + (2n-1) + \frac{(k-1)(2n-1)}{M_3}\left(nC_4 - \frac{C_3(1+2n)}{M_1}\right.$$
$$\left.\left. + C_4 + \frac{M_2(C_3-2n+1)}{M_1}\right)\right]$$

$$\lambda_6 = C_2 - \frac{(k-1)C_3(2n-1+C_3)}{M_1} + (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)$$
$$\left(\frac{C_3(1+2n) - M_2(2n-1+C_3)}{M_1} - nC_4 - C_4\right) + \left[4n - \frac{(k-1)C_3(nC_4+C_4)}{M_1}\right.$$
$$\left. - (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)\left(2n-1+\frac{M_2(nC_4+C_4)}{M_1}\right)\right]$$
$$\frac{1}{M_4}\left[4n - \frac{(k-1)C_3(nC_4+C_4)}{M_1} - (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)\right.$$
$$\left(2n-1+\frac{M_2(nC_4+C_4)}{M_1}\right) + (2n-1) + \frac{(k-1)(2n-1)}{M_3}\left(nC_4 - \frac{C_3(1+2n)}{M_1}\right.$$
$$\left.\left. + C_4 + \frac{M_2(C_3-2n+1)}{M_1}\right)\right]$$

$$M_4 = M_1 + \frac{(k-1)C_3(C_3-2n+1)}{M_1} - (k-1)\left(\frac{C_3(2n+1)}{M_1M_3} - \frac{C_4}{M_3}\right)$$
$$\left(nC_4 - \frac{C_3(2n+1)}{M_1} + C_4 + \frac{M_2(C_3-2n+1)}{M_1}\right).$$

Hence the proof.

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